

Matematický časopis

Ján Jakubík

Center of a bounded lattice

Matematický časopis, Vol. 25 (1975), No. 4, 339--343

Persistent URL: <http://dml.cz/dmlcz/126677>

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1975

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

CENTER OF A BOUNDED LATTICE

JÁN JAKUBÍK

Let L be a complete lattice and let $C(L)$ be the center of L . Several authors have found sufficient conditions for $C(L)$ to be a closed sublattice of L . Each of the following conditions (a) — (d) is sufficient for $C(L)$ to be closed:

- (a) L is a continuous geometry (von Neumann [10]).
- (b) L is orthocomplemented modular (Kaplansky [8]).
- (c) L is orthomodular (Foulis [2] Holland [3]).
- (d) L is relatively semi-orthocomplemented (Maeda [9]).

All the above results were generalized by Janowitz [7] who proved that a sufficient condition is

- (e) L is relatively complemented.

In this note it will be shown that a sufficient condition for the closedness of $C(L)$ can be expressed by using properties of congruence relations on L ; namely the following condition is sufficient:

- (f) Any two congruence relations on L are permutable and if $a, b, c, d \in L$, $a \leq b < c \leq d$, $b R c$ for some congruence relation R on L , then there are elements $b_1, c_1 \in L$ such that $a < b_1 \leq d$, $a \leq c_1 < d$, $a R b_1$, $c_1 R d$.

It is easy to verify that (e) implies (f) and that (e), (f) are not equivalent (cf. Lemma 9); hence our result is a generalization of that of Janowitz.

Janowitz (loc. cit.) remarks that it is not known if the centre of a complete lattice must be a closed sublattice (cf. also Birkhoff [1], Problem 34).

In [5] it was proved that the centre of each infinitely distributive complete lattice is a closed sublattice and that there exists a nondistributive complete lattice L such that $C(L)$ is not closed in L . In [6] it was shown that there exists a distributive complete lattice whose centre fails to be closed and that the following condition is necessary and sufficient for $C(L)$ to be a closed sublattice of L :

- (g) If $x, y \in L$, $x \geq y$, $\{a_\alpha\} \subset C(L)$, then

(1) $y \vee (x \wedge (\bigwedge a_\alpha)) = \bigwedge (y \vee (x \wedge a_\alpha))$,

(2) $x \wedge (y \vee (\bigvee a_\alpha)) = \bigvee (x \wedge (y \vee a_\alpha))$.

In Thm. 3 of [6] this result was applied for the investigation of direct factors of a conditionally complete lattice.

For the basic notions and denotations cf. Birkhoff [1].

Let L be a bounded lattice that need not be complete, $L = [u, v]$. We denote by $\mathcal{O}(L)$ the set of all congruence relations on L . The least and the greatest element of $\mathcal{O}(L)$ will be denoted by $\bar{0}$ and $\bar{1}$, respectively. For any $x \in L$ and any $R \in \mathcal{O}(L)$ we denote by $x(R)$ the set of all $y \in L$ with $y R x$.

Let I be a nonempty set and for each $i \in I$ let R_i and R'_i be congruence relations on L such that

- (i) R_i and R'_i are permutable,
- (ii) $R_i \wedge R'_i = \bar{0}$,
- (iii) $R_i \vee R'_i = \bar{1}$.

From (i), (ii) and (iii) we get (cf. [1], p. 164, Thm. 5) that the correspondence

$$(3) \quad x \rightarrow (x(R_i), x(R'_i)) \quad (x \in L)$$

is an isomorphism of L onto the direct product $A \times B$, where $A = L/R_i$, $B = L/R'_i$.

From (i) and (iii) it follows that there exist elements $c_i, c'_i \in [u, v]$ such that

$$\begin{aligned} uR_i c_i, \quad c_i R'_i v, \\ uR'_i c'_i, \quad c'_i R_i v. \end{aligned}$$

Moreover, from (ii) we obtain that the elements c_i and c'_i are uniquely determined and that

$$c_i \wedge c'_i = u, \quad c_i \vee c'_i = v.$$

Assume that the elements

$$\bigwedge c_i = c, \quad \bigvee c'_i = d$$

do exist in L . Hence $c, d \in [u, v]$. Denote $R = \bigwedge R_i$ and let R' be the least congruence on L in which c and v belong to the same class.

Lemma 1. *Let $i \in I$. If $x \in [u, v]$, $x \in u(R_i)$, then $x \leq c_i$.*

Proof. From $x \in u(R_i)$ we obtain $x(R_i) = u(R_i)$. Because of $x \leq v$ we have $x(R'_i) \leq v(R'_i) = c_i(R'_i)$. Therefore, since (3) is an isomorphism and $c_i(R_i) = x(R_i)$, we infer that $x \leq c_i$.

The assertion dual to Lemma 1 can be proved analogously.

Lemma 2. *c is the greatest element of the set $u(R)$.*

Proof. We have $u \leq c \leq c_i$ for each $i \in I$, thus $uR_i c$ and hence $c \in u(R)$. Let $x \in u(R)$. Then $x \in u(R_i)$ and so by Lemma 1, $x \leq c_i$ for each c_i ; therefore $x \leq c$.

In a dual way we obtain

Lemma 2'. *d is the least element of the set $v(R)$.*

Lemma 3. *Assume that L fulfils (f). Then $c \wedge d = u$, $c \vee d = v$.*

Proof. Put $c \wedge d = \bar{u}$ and suppose that $u < \bar{u}$. Then $uR\bar{u}$ and hence it follows from (f) that there exists $u_1 \in L$ with $u \leq u_1 < d$, $u_1R\bar{u}$, but this contradicts Lemma 2'. Therefore $c \wedge d = u$. In a dual way we can verify that $c \vee d = v$.

Lemma 4. *Assume that L fulfils (f). Let $[p, x]$, $[y, q]$ be transposed intervals of L , $[p_1, x_1] \subset [p, x]$, $p_1 < x_1$. Let S be a congruence relation on L such that p_1Sx_1 . Then there is $t \in L$ with $y < t \leq q$ such that ySt .*

Proof. We may suppose that $x \wedge y = p$, $x \vee y = q$ (in the case $p \wedge q = y$, $p \vee q = x$ we proceed dually). From (f) it follows that there is $z \in L$ such that $p < z \leq x$, pSz . Denote $z \vee y = t$. Then $y < t \leq q$, ySt .

Analogously we can verify the dual assertion.

Let I and I' be intervals of L . The interval I is said to be weakly projective to I' if there are intervals I_0, I_1, \dots, I_n of L such that $I_0 = I$, $I_n = I'$ and I_{j+1} is a subinterval of an interval that is transposed to I_j ($j = 0, 1, \dots, n - 1$).

Lemma 5. *Let $x, y \in L$, $x < y$. Then $x R' y$ if and only if there are elements $x_0, x_1, \dots, x_n \in L$ such that $x_0 = x$, $x_n = y$, $x_j < x_{j+1}$ and $[c, v]$ weakly projective to $[x_j, x_{j+1}]$ for $j = 0, 1, \dots, n - 1$.*

This follows from [4], § 1.

By induction we obtain from Lemma 4 and Lemma 5:

Lemma 6. *Assume that L fulfils (f). Let $x, y \in L$, $x < y$, $x R' y$. Let R_2 be a congruence relation on L such that xR_2y . Then there is $t \in L$ such that $c < t \leq v$, cR_2t .*

Lemma 7. *If L fulfils (f), then $R \wedge R' = \bar{0}$.*

Proof. Suppose that $R \wedge R' \neq \bar{0}$. Then there are elements $x, y \in L$ with $x < y$, $x(R \wedge R') y$. Put $R \wedge R' = R_2$ and let t be as in Lemma 6.

We have $c < t \leq v$ and cRt , which is a contradiction (cf. Lemma 2).

We have $uRcR'v$, hence $u(R \vee R')v$ and therefore

$$(4) \quad R \vee R' = \bar{1}.$$

Lemma 8. *Assume that L fulfils (f). The correspondence*

$$x \rightarrow (x(R'), x(R)) \quad (x \in L)$$

is an isomorphism of L onto $(L/R') \times (L/R)$.

This follows from (4), and Lemma 7 from the fact that any two congruence relations on L are permutable.

The notion of the center $C([u, v])$ of the lattice $L = [u, v]$ is defined as follows. An element $c \in L$ belongs to $C([u, v])$ if and only if there are lat-

tices A, B and an isomorphism φ of L onto the direct product $A \times B$ such that, if we denote $\varphi(z) = (z_A, z_B)$ for each $z \in L$, then

$$c_A = v_A, \quad c_B = u_B.$$

Each element $c \in C([u, v])$ has exactly one relative complement in the interval $[u, v]$; this relative complement will be denoted c' .

Another way of defining the set $C([u, v])$ is as follows (it is a direct consequence of [1], p. 164, Thm. 5): An element $c_i \in L$ belongs to $C([u, v])$ if and only if there are $R(c_i), R'(c_i) \in \Theta(L)$ such that $R(c_i), R'(c_i)$ are permutable, $R(c_i) \wedge R'(c_i) = 0, R(c_i) \vee R'(c_i) = \bar{1}$ and $uR(c_i)c_i, c_iR'(c_i)v$.

Theorem 1. *Let $L = [u, v]$ be a lattice fulfilling (f), $\emptyset \neq \{c_i\} \subset C([u, v])$. Assume that the elements $\bigwedge c_i, \bigvee c_i$ exist in L . Then $\bigwedge c_i, \bigvee c_i \in C([u, v])$.*

Proof. Denote $\bigwedge c_i = c, \bigvee c_i = d$. Let $R(c_i)$ and $R'(c_i)$ be as in the definition of the center $C([u, v])$. Denote $R(c_i) = R_i, R'(c_i) = R'_i$. By means of the congruence relations R_i, R'_i we construct the congruence relations R and R' as above. Then we have

$$u R c, \quad c R' v, \quad u R' d, \quad d R v.$$

From this and from (4) and Lemmas 7, 8 we infer that c and d belong to $C([u, v])$.

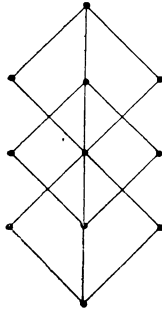


Fig. 1

Corollary 1. *Let L be a complete lattice fulfilling (f). Then $C(L)$ is a closed sublattice of L .*

Lemma 9. *For any lattice L , (e) implies (f). The conditions (e), (f) are not equivalent.*

Proof. Let L be a relatively complemented lattice. It is well known that any two congruence relations on L are permutable ([1], p. 163). Let $a, b, c, d \in L, a \leq b < c \leq d, bRc$ for some congruence relation R on L . Let b_1 be a relative

complement of b in the interval $[a, c]$. Then $a < b_1 \leq c$ and aRb_1 . In a dual way we can find $c_1 \in L$ such that $a \leq c_1 < d$, c_1Rd . Hence L fulfils (f).

Let L be the lattice in Fig. 1. The lattice L is modular and any two prime intervals of L are projective. Thus L has no nontrivial congruence relations and hence L fulfils (f). On the other hand, L is not relatively complemented.

From Lemma 9 and Corollary 1 we obtain:

Corollary 2. (Janowitz [7]) *Let L be a complete relatively complemented lattice. Then $C(L)$ is a closed sublattice of L .*

Let us remark that the condition (g) does not appear to be an immediate consequence of the condition (e). We can use (g) for getting the well-known result on the infinite distributivity of a complete Boolean algebra. In fact, if L is a complete Boolean algebra, then $C(L) = L$ and hence by Thm. 2, [6], the condition (g) is valid. By putting $x = 1$ in (1) and $y = 0$ in (2) we get the infinite distributive laws for L .

REFERENCES

- [1] BIRKHOFF, G.: Lattice theory, third edition. Amer. Math. Soc. Colloquium Publications Vol. XXV, Providence 1967.
- [2] FOULIS, D. J.: A note on orthomodular lattices. Portug. Math., 21, 1962, 65–72.
- [3] HOLLAND, Jr. S. S.: A Radon-Nikodym Theorem in dimension lattices. Trans. Amer. Math. Soc., 108, 1963, 66–87.
- [4] JAKUBÍK, J.: Relácie kongruentnosti a slabá projektívnosť vo zväzoch. Časop. pěstov. mat., 80, 1955, 206–216.
- [5] JAKUBÍK, J.: Centrum nekonečne distributívnych zväzov. Mat. fyz. čas. 8, 1957, 116–120.
- [6] JAKUBÍK, J.: Center of a complete lattice. Czechosl. Math. J. 23, 1973, 125 – 138.
- [7] JANOWITZ, M. F.: The center of a complete relatively complemented lattice is a complete sublattice. Proc. Amer. Math. Soc., 18, 1967, 189–190.
- [8] KAPLANSKY, J.: Any orthocomplemented complete modular lattice is a continuous geometry. Ann. Math., 61, 1955, 524–541.
- [9] MAEDA, S.: On relatively semi-orthocomplemented lattices. Hiroshima Univ. J. Sci. Ser. A 24, 1960, 155–161.
- [10] von NEUMANN, J.: Continuous geometry. Princeton Univ. Press, N. Y., 1960.

Received December 10, 1973

*Katedra matematiky
Strojnickej fakulty Vysokej školy technickej
Švermova 5
040 01 Košice*