Matematický časopis

Frank Harary; Bennet Manvel

On the number of cycles in a graph

Matematický časopis, Vol. 21 (1971), No. 1, 55--63

Persistent URL: http://dml.cz/dmlcz/126802

Matematický časopis 21 (1971), No. 1

ON THE NUMBER OF CYCLES IN A GRAPH

FRANK HARARY, BENNET MANVEL, Ann Arbor, Michigan, U.S.A.

A (p, q) *graph* consists of p points and q lines, with each line on exactly two points and each pair of points on at most one line. We define the n-cycles of a graph G, and show how they may be counted for $n = 3$, 4, and 5 in terms of the adjacency matrix of G. Corresponding formulas are derived for directed graphs. For complete graphs and bigraphs, explicit formulas are stated for the number of cycles of each length. In two appendices, the underlying graphs and digraphs are displayed for the closed n-walks with $n \leq 7$.

The number of cycles of given length in a graph or digraph does not appear to have been investigated by matrix algebra. On the other hand, there have been several works on the determination of the number of (non-redundant) paths of given length from one point to another in a digraph.

These include Harary and Ross [5] where the matrix manipulations are quite analogous to those developed in this paper, Parthasarathy [4], and Cartwright and Gleason [1] where the methods involve a careful consideration of the iterated „line digraphs" of a directed graph. We will use the basic lemmas on the interpretation of the powers of the adjacency matrix of a graph to obtain explicit formulas for the number of cycles of small length.

CYCLES IN GRAPHS

Relevant graph theoretic definitions not included here may be found in Harary [2]. An *n-walk* in a graph G is an alternating sequence of points and lines, $v_0, x_1, v_1, x_2, \ldots, x_n, v_n$ such that each line is incident with the two points just before and after it, and containing n (not necessarily distinct) lines. It is *closed* if $v_0 = v_n$, and a closed n-walk in which $n \geq 3$ and all points except the first and last are distinct is called an *n-cycle*.

In order to distinguish a walk from its reverse walk, we may say it is *rooted* at its initial point v_0. If the walk is closed, we may also specify a direction on it from its root, and then we say it is *oriented*. Every walk may be considered as rooted and oriented because it is a sequence (not a subgraph!). However, we can also speak about rooted or oriented shapes of walks as the subgraphs formed by the members of the walks.

If G has p points labeled $v_1, \dots v_p$, then the *adjacency matrix* $A = A(G)$ is the $p \times p$ matrix with $a_{ij} = 1$ if $v_i v_j$ is a line of G and 0 otherwise. Notice that this is a symmetric matrix for any graph. The next statement is of basic importance, and is given in Harary, Norman, and Cartwright [3, p. 112]

Graph Lemma. *The i, j entry, $a_{ij}^{(n)}$, of A^n is the number of n-walks in G between v_i and v_j which are rooted at v_i; in particular, $a_{ii}^{(n)}$ is the number of closed oriented n-walks in G rooted at v_i.*

Since every closed 3-walk is a triangle (i. e., 3-cycle), the number of triangles in G is $\frac{1}{6} \sum_{i=1}^{p} a_{ii}^{(3)}$ or $\frac{1}{6}$ $\mathrm{tr}(A^3)$, where the $\frac{1}{6}$ eliminates duplication to allow for three roots and two orientations of each triangle. Thus if we denote by $c_n(G)$ the number of n-cycles in G, we have

$$c_3(G) = \tfrac{1}{6}\, \mathrm{tr}(A^3). \tag{1}$$

It is natural to ask for a similar formula for $c_n(G)$ when $n > 3$. The following theorem gives an answer for the case $n = 4$.

Theorem 1. *If G is a (p, q) graph with adjacency matrix A, then the number of 4-cycles in G is:*

$$c_4(G) = \tfrac{1}{8}\, [\mathrm{tr}(A^4) - 2q - 2 \sum_{i \neq j} a_{ij}^{(2)}]. \tag{2}$$

Proof. The trace $\mathrm{tr}(A^4)$ counts all closed rooted oriented 4-walks in G. If such a 4-walk is not a cycle then it is either (a) a line, traversed twice in each direction, or (b) a pair of adjacent lines, each traversed once in each direction. We proceed to count these:

(a) there are q lines in G and each is a rooted 4-walk in two ways, one way for each of its endpoints. Hence these contribute $2q$ to $\mathrm{tr}(A^4)$; see Figure 1(a).

(b) There are $\frac{1}{2} \sum_{i \neq j} a_{ij}^{(2)}$ pairs of adjacent lines in G, where the $\frac{1}{2}$ cancels the duplication due to taking a walk from each of its two endpoints. On the other hand, each pair of adjacent lines is a closed rooted oriented 4-walk in G in 4 ways, once for each endpoint and twice (two orientations) for the middle point as in Figure 1(b). Thus we must take $2 \sum_{i \neq j} a_{ij}^{(2)}$ from $\mathrm{tr}(A^4)$.

After subtracting these from $\mathrm{tr}(A^4)$, we are left with only those walks which are 4-cycles, but each one is being counted at four different roots with two different orientations, so we divide by 8 to obtain the number of distinct 4-cycles of G.

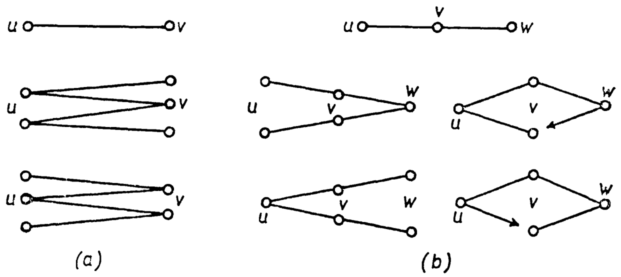


Fig. 1. Shapes of non-cyclic closed 4-walks in a graph.

Example 1. The utilities graph $K_{3,3}$ shown in Figure 2, may be interpreted as displaying the connections between three houses $\{1, 2, 3\}$ and three utilities $\{4, 5, 6\}$. We use the notation O_n for the $n \times n$ matrix with all zero entries

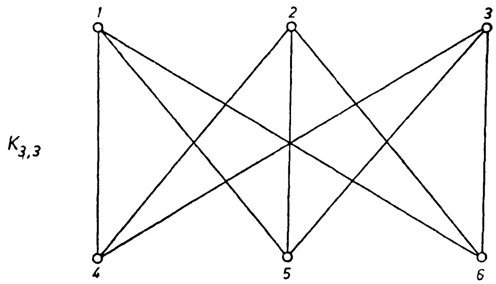


Fig. 2. The utilities graph.

and J_n for the $n \times n$ matrix with all entries 1. Now for $K_{3,3}$ we can write the adjacency matrix A and its powers as

$$A \quad \begin{pmatrix} O_3 & J_3 \\ J_3 & O_3 \end{pmatrix}, \; A^2 = \begin{pmatrix} 3J_3 & O_3 \\ O_3 & 3J_3 \end{pmatrix}, \; A^3 = \begin{pmatrix} O_3 & 9J_3 \\ 9J_3 & O_3 \end{pmatrix}, \; A^4 = \begin{pmatrix} 27J_3 & O_3 \\ O_3 & 27J_3 \end{pmatrix}.$$

From these we compute

$$\operatorname{tr}(A^3) = 0; \quad \operatorname{tr}(A^4) = 162; \quad 2q = \sum_{i,j} a_{ij} = 18; \quad \text{and} \quad 2\sum_{i \neq j} a_{ij}^{(2)} = 72.$$

Putting these into formulas (1) and (2), we find that G has no 3-cycles and 9 4-cycles.

The expression for $c_5(G)$ is only slightly more complicated.

Theorem 2. *The number of 5-cycles in a (p, q) graph G with adjacency matrix A is*

$$c_5(G) = \frac{1}{10} \{\mathrm{tr}(A^5) - 5\,\mathrm{tr}(A^3) - 5 \sum_{i=1}^{p} (\sum_{j=1}^{p} a_{ij} - 2)\, a_{ii}^{(3)}\}. \tag{3}$$

Proof. By the Graph Lemma, $\mathrm{tr}(A^5)$ counts all closed rooted oriented 5-walks in G. If such a 5-walk is not a cycle then it must have one of the two shapes given in Fig. 3.

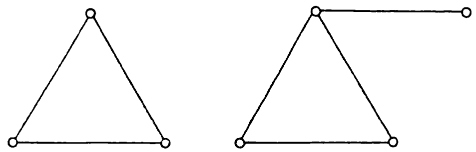


Fig. 3. The two shapes of closed 5-walks which are not 5-cycles.

A triangle can be a closed 5-walk in 30 ways. The 5 closed 5-walks starting in one direction from one point v are shown in Figure 4.

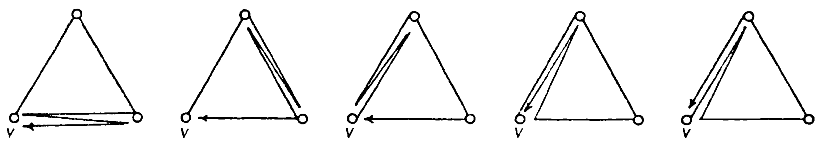


Fig. 4. Closed 5-walks on a triangle rooted at v with clockwise initial direction.

Since $\mathrm{tr}(A^3)$ counts each triangle 6 times, to remove this shape from the number of closed 5-walks, we subtract $5\,\mathrm{tr}(A^3)$. The other shape gives a closed 5-walk in 10 ways, and is counted twice in the last term of (3). We divide by 10 because each 5-cycle is counted ten times in $\mathrm{tr}(A^5)$.

The situation gets rather out of hand for higher n, as shown by the listing of shapes of closed 6-walks in Appendix 1.

CYCLES IN DIGRAPHS

Similar results may be proved for digraphs. A *digraph* D consists of a set of points and a collection of ordered pairs of those points called *arcs* or *(directed) lines*. Any definitions omitted here may be found in Harary, Norman, and Cartwright [3].

A *(directed) walk* in a digraph is an alternating sequence of points and arcs, $v_0, x_1, v_1, \ldots, x_n, v_n$, in which each arc x_i is $v_{i-1}v_i$ (the direction is from v_{i-1} to v_i). The *length* of such a walk is n, the number of occurrences of arcs in it.

A *closed* walk has the same first and last points and a *cycle* is a nontrivial closed walk with all points distinct (except the first and last).

If D has p points $v_1, \ldots, v_p$, then the *adjacency matrix* $A(D)$ of D is the $p \quad p$ matrix with $a_{ij} = 1$ if there is an arc in D from v_i to v_j, and 0 otherwise. Note that this matrix need not be symmetric. As in the undirected case, we have a basic lemma (Harary, Norman, and Cartwright [3, p. 112]).

Digraph Lemma. *The i, j entry of A^n, $a_{ij}^{(n)}$, is the number of n-walks in D from v_i to v_j, and, in particular, $a_{ii}^{(n)}$ is the number of closed n-walks in D rooted at v_i.*

We do not need to specify an orientation for walks in a digraph, since every walk is directed.

Since every closed 3-walk of D is a directed triangle, the number of rooted triangles, or 3-cycles, of D is $\operatorname{tr}(A^3)$. Since each such cycle can have 3 roots, the number of distinct 3-cycles in D is

$$c_3(D) = \tfrac{1}{3} \operatorname{tr}(A^3). \tag{4}$$

In order to state the result for $c_4(D)$, we define a new matrix $\bar{A} - [\bar{a}_{ij}]$ derived from the adjacency matrix A of D by the following rule:

$$a_{ij} = \bar{a}_{ji} - a_{ij}a_{ji}.$$

Then $\bar{A}$ is the adjacency matrix of the undirected graph derived from D by calling two points adjacent if and only if they form a symmetric pair (2-cycle) in D. With this definition of $\bar{A}$ and its entries $\bar{a}_{ij}$, and the notational definition $\sigma(A) \quad \sum_{i,j} a_{ij}$, we can state the following theorem.

Theorem 3. *For any digraph D with adjacency matrix A, the number of 4-cycles is*

$$c_4(D) = \tfrac{1}{4} [\operatorname{tr}(A^4) - \sigma(\bar{A}) - 2 \sum_{i \neq j} \bar{a}_{ij}^{(2)}]. \tag{5}$$

Proof. The $\sigma(\bar{A})$ counts the 4-walks with shape of Fig. 5(a) and the $\sum_{i \neq j} a_{ij}^{(2)}$, counts those with shape of Fig. 5(b) twice. Since these latter are 4-walks in four ways, we need the factor 2. The rest of the proof is like that of Theorem 1.

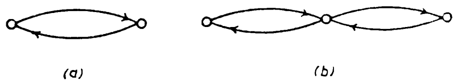


Fig. 5. The two shapes of closed 4-walks in a digraph.

The result for 5-cycles in D is:

Theorem 4. *The number of 5-cycles in a digraph D, with adjacency matrix A, is*

$$c_5(D) = \tfrac{1}{5} \{\operatorname{tr}(A^5) - 5 [\sum_i (a_{ii}^{(3)} \cdot \sum_j \bar{a}_{ij}) - \sum_{i \neq j} (a_{ij}^{(2)} \cdot \bar{a}_{ij})]\}. \tag{6}$$

Proof. There are only two shapes a closed 5-walk can have, besides a 5-cycle, and those are shown in Figure 6.

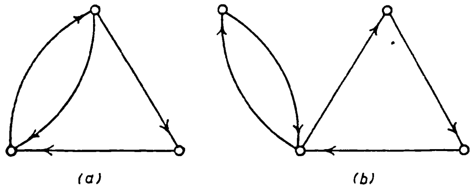


Fig. 6. Shapes of non-cyclic closed 5-walks in a digraph.

Each of these is a 5-walk in 5 ways, and they are counted in

(7) $$\sum_i (a_{ii}^{(3)} \cdot \sum_j a_{ij})$$

but shapes of type (a) (see Figure 6) are counted twice in (7), so we take away the number of shapes of type (a), which are counted by

(8) $$\sum_{i \neq j} (a_{ij}^{(2)} \cdot a_{ij}).$$

Then the difference of (7) and (8) counts each shape exactly once, and we obtain equation (6).

SPECIAL GRAPHS AND BIGRAPHS

For certain special graphs these formulas are not needed. For example, in the *complete graph*, K_p, on p points, which has every pair of points adjacent, the number of rooted, oriented n-cycles is clearly

(9) $$p(p-1)(p-2)\ldots(p-(n-1)) = \frac{p!}{(p-n)!}$$

since the root may be chosen p ways, the next point $p - 1$ ways, and so on. Thus the number of distinct n-cycles is

(10) $$c_n(K_p) = \frac{p!}{2n(p-n)!}.$$

The *complete bigraph* $K_{r,s}$ has a set of $r + s$ points divided into two disjoint sets V_1 and V_2 of r and s points, with two points adjacent if and only if one is in V_1 and the other in V_2. It is easily seen and well-known that $K_{r,s}$ has no odd cycles. The number of rooted oriented $2n$-cycles for $n = 2$ to min $\{r, s\}$ in $K_{r,s}$ is

(11) $$2rs(r-1)(s-1)\ldots(r-n+1)(s-n+1) = 2\frac{r!}{(r-n)!}\frac{s!}{(s-n)!}.$$

Each unoriented $2n$-cycle of $K_{r,s}$ is counted twice in (11) for orientation and $2n$ times for its roots, so we divide by $4n$ to get

$$c_{2n}(K_{r,s}) = \frac{n!^2}{2n}\binom{r}{n}\binom{s}{n}. \tag{12}$$

For the special case $r = s$, we have

$$c_{2n}(K_{r,r}) = \frac{1}{2n}\left[n!\binom{r}{n}\right]^2. \tag{13}$$

For the utilities graph $K_{3,3}$ of Example 1, the results can be checked against this formula.

Appendix 1

Shapes of closed n-walks in graphs, $n \leqslant 7$.

The number of shapes for given n is not known. However, all shapes for n are shapes foı $+ 2$ and for every multiple of n. (Fig. 7.)

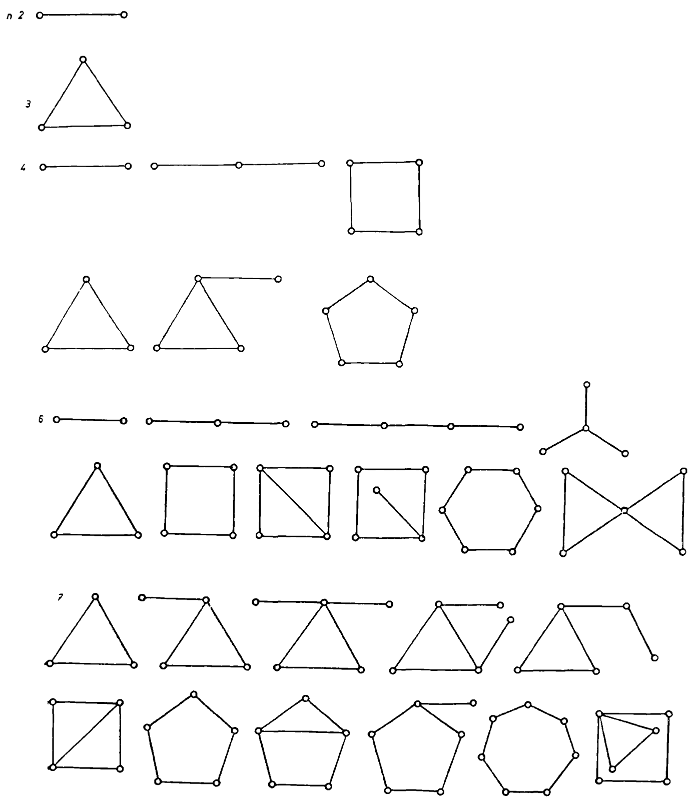
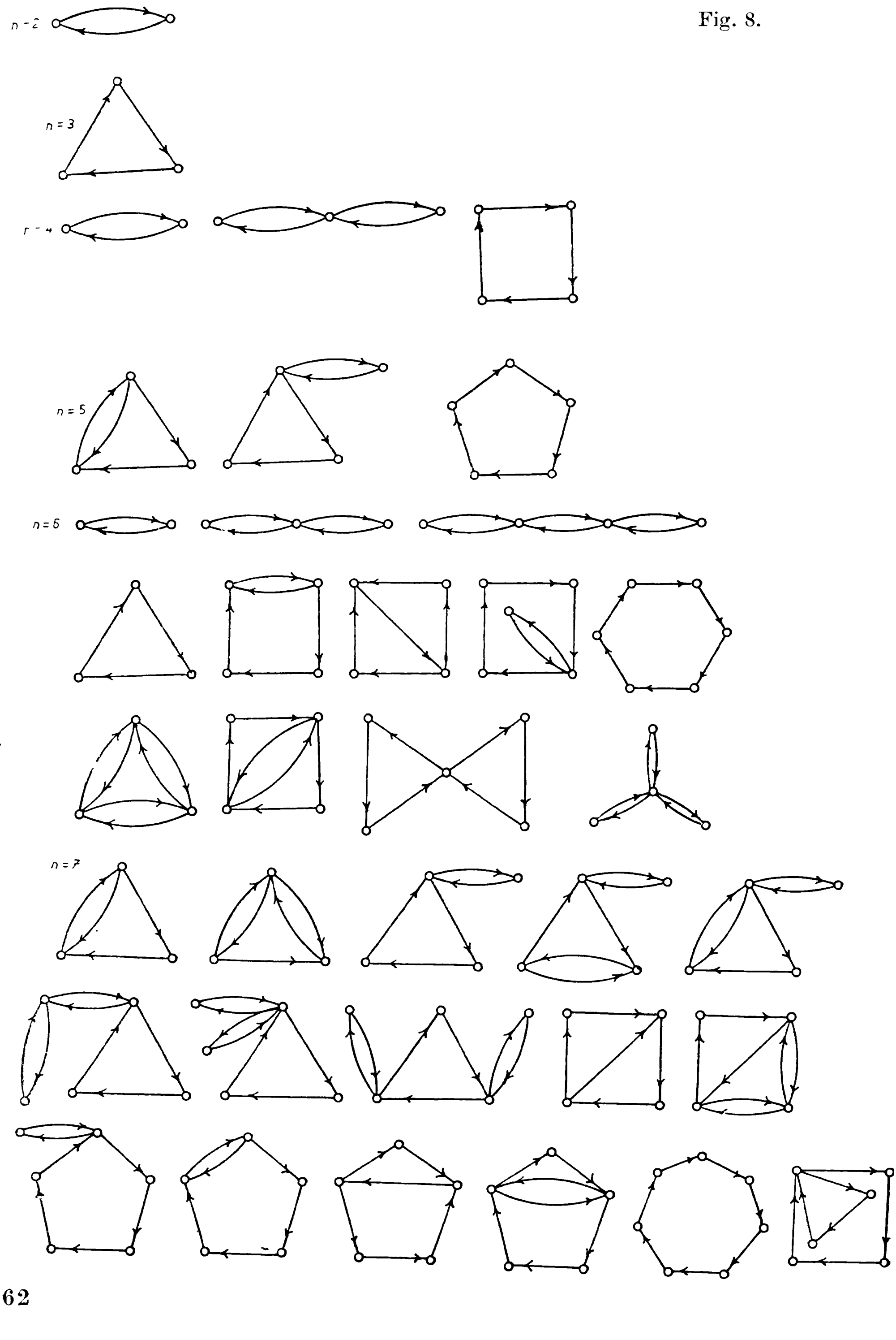


Fig. 7.

Fig. 8.



Appendix 2

Shapes of closed n-walks in digraphs, $n \leqslant 7$.

As in the undirected case, no method of enumerating these shapes is known. For each n, the underlying graph of each shape is the shape of a closed n-walk in an undirected graph. Note that a shape for n is also a shape for each multiple of n, but not necessarily for n 2. (Fig. 8.)

REFERENCES

[1] Cartwright D., Gleason T. C., *The number of paths and cycles in a digraph*, Psychometrika *31* (1966), 179–199.

[2] Harary F., *Graph Theory*, Addison-Wesley, Reading, Mass. 1969.

[3] Harary F., Norman R. Z., Cartwright D., *Structural models: an introduction to the theory of directed graphs*, Wiley, New York, 1965.

[4] Parthasarathy K., *Enumeration of paths in digraphs*, Psychometrika *29* (1964), 153 165.

[5] Ross I. C., Harary F., *On the determination of redundancies in sociometric chains*, Psychometrika *17* (1952), 195 208.

Received February 28, 1969.

Research Center for Group Dynamics
University of Michigan