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DARBOUX PROPERTY OF REGULAR MEASURES

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In this paper we investigate the relationship between the regularity and the Darboux property of measures. We give a necessary and sufficient condition in order that a regular measure in abstract space may have the Darboux property (Theorem 2). Then it implies a necessary and sufficient condition for the Darboux property of the regular Borel measure (Corollary 1), the Baire measure (Corollary 2) and the regular weakly Borel measure (Corollary 3). Further we show that the Darboux property remains unchanged under an arbitrary extension of a Baire measure to a Borel measure (Theorem 3) and also under a restriction of a regular Borel measure to a Baire measure (Corollary 4) but it need not be the case under a restriction of a non-regular Borel measure to a Baire measure (Example). The used terminology is in accordance with [1].

Definition 1. *We say that a measure* μ *defined on a ring* $\mathscr{R}$ *has the Darboux property if for every set* $E \in \mathscr{R}$ *and for every number* $a \in \langle 0, \mu(E)\rangle$ *there exists a set* $A \in \mathscr{R}$ *with* $A \subset E$ *such that* $\mu(A) = a$.

Definition 2. *We say that a set* $E \in \mathscr{R}$ *is an atom with respect to a measure* μ *if* $\mu(E) > 0$ *and if for every set* $A \in \mathscr{R}$ *with* $A \subset E$ *we have either* $\mu(A) = 0$ *or* $\mu(A) = \mu(E)$.

It has been proved ([2, §2, Prop. 7]) that every σ-finite non-atomic measure (i.e. a measure having no atom) on a δ-ring has the Darboux property. It is evident that the assumption of „non-atomicity" of μ is also the necessary condition of this assertion. Summarizing we have:

Theorem 1. *A* σ*-finite measure defined on a* δ*-ring has the Darboux property if and only if it is non-atomic.*

It is easy to see that a regular Borel measure need not have the Darboux property. In the following we show what condition must be fulfilled in order that a regular measure may have the Darboux property.

Let X be an arbitrary non-empty set. Let $\mathscr{C}$, $\mathscr{U}$ be nonempty classes of subsets of X such that

(i) *every set* $U \in \mathscr{U}$ *belongs to the* σ*-ring generated by* $\mathscr{C}$ *i.e.* $\mathscr{S}(\mathscr{C}) \supset \mathscr{U}$,

(ii) *every cover* $\{U_\alpha\}_{\alpha \in I}$ *of an arbitrary set* $C \in \mathscr{C}$ *consisting of sets* $U_\alpha \in \mathscr{U}$ *has a countable subcover.*

Definition 3. *Let* $\mathscr{C}$, $\mathscr{U}$ *be non-empty classes of subsets of a non-empty set* X. *Let for* $\mathscr{C}$, $\mathscr{U}$ *the conditions* (i) *and* (ii) *hold. A measure* μ *defined on the* σ*-ring* $\mathscr{S}(\mathscr{C})$ *is said to be* $(\mathscr{C}, \mathscr{U})$*-regular if for every set* $A \in \mathscr{S}(\mathscr{C})$

$$\mu(A) = \sup\{\mu(C) : C \in \mathscr{C},\ C \subset A\} = \inf\{\mu(U) : U \in \mathscr{U},\ U \supset A\}.$$

Theorem 2. *Let* $\mathscr{C}$, $\mathscr{U}$ *be non-empty classes of subsets of a non-empty set* X. *Let for* $\mathscr{C}$, $\mathscr{U}$ *the conditions* (i) *and* (ii) *hold. A* $(\mathscr{C}, \mathscr{U})$*-regular measure* μ *defined on* $\mathscr{S}(\mathscr{C})$ *and finite on* $\mathscr{C}$ *has the Darboux property if and only if for every* $x \in \cup \mathscr{S}(\mathscr{C})$ *there exists a set* $A \in \mathscr{S}(\mathscr{C})$ *with* $x \in A$ *and* $\mu(A) = 0$.

Proof. Let μ have the Darboux property. Since μ is finite on $\mathscr{C}$ it is σ-finite on $\mathscr{S}(\mathscr{C})$. Hence for every $x \in \cup \mathscr{S}(\mathscr{C})$ there exists a set $A \in \mathscr{S}(\mathscr{C})$ such that $x \in A$ and $\mu(A) < \infty$. Since μ has the Darboux property, there exists a set $A_1 \in \mathscr{S}(\mathscr{C})$ such that $A_1 \subset A$ and $\mu(A_1) = 2^{-1}\mu(A)$. Denote by B_1 one of the sets A_1, $A - A_1$ to which x belongs. For B_1 we have $\mu(B_1) = 2^{-1}\mu(A)$. Suppose that we have found the first n members of a non-increasing sequence of sets $A \supset B_1 \supset B_2 \supset \ldots \supset B_{n-1}$ with $x \in B_i \in \mathscr{S}(\mathscr{C})$ and $\mu(B_i) = 2^{-i}\mu(A)$ for $i = 1, 2, \ldots, n-1$. From the Darboux property of μ there follows the existence of a set $A_n \in \mathscr{S}(\mathscr{C})$ such that $A_n \subset B_{n-1}$ and $\mu(A_n) = 2^{-1}\mu(B_{n-1}) = 2^{-n}\mu(A)$. Denote by B_n one of the sets A_n, $B_{n-1} - A_n$ to which x belongs. For B_n we have $\mu(B_n) = 2^{-n}\mu(A)$. Put $B = \bigcap_{n=1}^{\infty} B_n$. Then evidently $x \in B$, $B \in \mathscr{S}(\mathscr{C})$ and

$$\mu(B) = \mu\Big(\bigcap_{n=1}^{\infty} B_n\Big) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} 2^{-n}\mu(A) = 0.$$

Now let there for an arbitrary $x \in \cup \mathscr{S}(\mathscr{C})$ exist a set $A \in \mathscr{S}(\mathscr{C})$ with $x \in A$ and $\mu(A) = 0$. Let B be an arbitrary set of positive measure. Since μ is a regular measure there exists a set $C \in \mathscr{C}$ such that $C \subset B$ and $0 < \mu(C)$. Further to every $x \in C$ there exists a set $A_x \in \mathscr{S}(\mathscr{C})$ such that $x \in A_x$ and $\mu(A_x) = 0$. Now we use the regularity of μ and find to every set A_x a set $U_x \in \mathscr{U}$ with $U_x \supset A_x$ and $\mu(U_x) < \mu(C)$. Since the class of sets $\{U_x\}_{x \in C}$ covers the set C by the assumption of this theorem there exists a countable subclass $\{U_n\}_{n=1}^{\infty}$ covering C. Then

$$0 < \mu(C) = \mu\Big(\bigcup_{n=1}^{\infty} (C \cap U_n)\Big) \leqslant \sum_{n=1}^{\infty} \mu(C \cap U_n),$$

whence we obtain that there exists an integer n_0 such that $\mu(C \cap U_{n_0}) > 0$. We have $C \cap U_{n_0} \subset B$ and

$$0 < \mu(C \cap U_{n_0}) \leqslant \mu(U_{n_0}) < \mu(C) \leqslant \mu(B).$$

Consequently the set B is not an atom and, since B was chosen arbitrarily, μ is non-atomic on $\mathscr{S}(\mathscr{C})$. It follows that the measure μ has the Darboux property by Theorem 1.

Corollary 1. *A regular Borel measure μ on a locally compact Hausdorff topological space X has the Darboux property if and only if $\mu(\{x\}) = 0$ for every $x \in X$.*

Corollary 2. *A Baire measure ν on a locally compact Hausdorff topological space X has the Darboux property if and only if for every $x \in X$ there exists a Baire set $A \subset X$ such that $x \in A$ and $\nu(A) = 0$.*

Corollary 3. *Let X be a Hausdorff topological space. Denote by $\mathscr{C}$ the class of closed and by $\mathscr{U}$ the class of open subsets of X. Let each open cover of an arbitrary set $F \in \mathscr{C}$ have a countable subcover. Let μ be a $(\mathscr{C}, \mathscr{U})$-regular weakly Borel measure* ([1,57. Ex. 21]) *which is finite on $\mathscr{C}$. Then in order that the measure μ may have the Darboux property, it is necessary and sufficient that $\mu(\{x\}) = 0$ for every $x \in X$.* (Cf. [4], 11, 44.)

Corollary 4. *If a regular Borel measure μ on a locally compact Hausdorff space has the Darboux property, then the restriction of μ on the class of Baire sets has the Darboux property.*

Proof. Let μ be a regular Borel measure with the Darboux property. By [1, 56. Th. 1] we deduce for an arbitrary $x \in X$

$$0 = \mu(\{x\}) = \inf\{\mu(U) : x \in U,\ U \text{ is an open Baire set}\}.$$

Then we can find a non-increasing sequence $\{U_n\}_{n=1}^{\infty}$ of open Baire sets containing x such that $\mu(U_n) < n^{-1}$ for $n = 1, 2, \ldots$ Denote $A = \bigcap_{n=1}^{\infty} U_n$. Since A is a Baire set and $x \in A$ we have

$$0 = \mu(\{x\}) \leqslant \mu(A) = \lim_{n\to\infty} \mu(U_n) \leqslant \lim_{n\to\infty} n^{-1} = 0$$

and by Corollary 2 the assertion is proved.

We shall give now a sufficient condition for the Darboux property of an arbitrary Borel measure.

Theorem 3. *If a Baire measure ν has the Darboux property, then an arbitrary extension of ν to a Borel measure has the Darboux property.*

Proof. Let a Borel measure μ be an extension of a Baire measure ν having the Darboux property. Let E be a Borel set with $\mu(E) > 0$. Since E is σ-bounded ([1, 57. Th. 1]), there exists a compact set C such that $\mu(C \cap E) > 0$. By Corollary 2 for every $x \in X$ there exists a Baire set A_x with $x \in A_x$ and $\mu(A_x) = 0$. From the regularity of the Baire measure we deduce that for every $x \in C$ there exists an open Baire set U_x containing A_x with $\mu(U_x) < < \mu(C \cap E)$. Since the class $\{U_x\}_{x \in C}$ is an open cover of the set C we can find a finite subcover $\{U_i\}_{i=1}^{n}$ covering C. For this subcover we have

$$0 < \mu(C \cap E) = \mu(\bigcup_{i=1}^{n} (C \cap E \cap U_i)) \leqslant \sum_{i=1}^{n} \mu(C \cap E \cap U_i)$$

and therefore there is an integer $i_0 \leqslant n$ such that $\mu(C \cap E \cap U_{i_0}) > 0$. Now we have $C \cap E \cap U_{i_0} \subset E$ and

$$0 < \mu(C \cap E \cap U_{i_0}) \leqslant \mu(U_{i_0}) < \mu(C \cap E) \leqslant \mu(E).$$

Consequently the set E is not an atom and the measure μ has the Darboux property.

Remark. Let ν be a Baire restriction of a Borel measure μ. From Theorem 3 it follows that if ν has the Darboux property then μ has the Darboux property. We shall show that the converse is not true, i.e. there exists a (non-regular) Borel measure having the Darboux property such that its restriction to the Baire sets has not the Darboux property.

Example. Let X be the set of ordinals less than or equal to the first uncountable ordinal Ω. With the order topology, X is a compact Hausdorff space ([3, 52. Ex. 10a]). Let μ be the Dieudonné's non-regular Borel measure on X ([3, 52. Ex. 10c]). Further let ν be the Borel measure on the unit interval $\langle 0, 1\rangle = Y$ of the real line with the usual topology, the completion of which is the Lebesque measure. The space $X \times Y$ with the product topology $\mathscr{U}$ is a compact Hausdorff space ([6, 3. Th. 5 and 5. Th. 13]) and since ν is a regular measure, we can extend the product $\mu \times \nu$ of the measures into a Borel measure ϱ ([5, Th. 4. 4]).

For an arbitrary element (x_0, y_0) of $X \times Y$ there exists a Baire set, namely $X \times \{y_0\}$, such that $(x_0, y_0) \in X \times \{y_0\}$ and $\varrho(X \times \{y_0\}) = 0$. Therefore ϱ has the Darboux property (Corollary 2, Theorem 3).

Now we define a transformation $T\colon X \times Y \to X \times Y$

$$T(x, y) = \begin{cases} (x, y) & \text{if } x \neq \Omega \\ (x, 0) & \text{if } x = \Omega \end{cases}.$$

Consider the space $Z = T(X \times Y)$ with the topology $\mathscr{V}$ induced by the transformation T, i.e. $\mathscr{V} = \{V \subset Z\colon T^{-1}(V) \in \mathscr{U}\}$. Evidently Z is a compact Hausdorff space. Denote by $\mathscr{B}(X \times Y)$ and $\mathscr{B}(Z)$ the classes of all Borel subsets of the space $X \times Y$ and Z, respectively. Since T is a continuous transformation we have $T^{-1}(B) \in \mathscr{B}(X \times Y)$ for an arbitrary set $B \in \mathscr{B}(Z)$. It follows that we can define a Borel measure τ on Z such that $\tau(B) = \varrho(T^{-1}(B))$ for every set $B \in \mathscr{B}(Z)$.

Let us note that the measure τ is non-regular. (For every neighbourhood V of $(\Omega, 0)$ we have $\tau(V) = 1$.)

Now we show that the measure τ has the Darboux property but the restriction of τ on the Baire sets has not the Darboux property.

Put $Z_1 = (X \times Y) - (\{\Omega\} \times Y) = Z - \{(\Omega, 0)\}$. Evidently $Z_1 \in \mathscr{V} \subset \mathscr{B}(Z)$ and the restriction of T on Z_1 is an identical transformation.

Let $E \in \mathscr{B}(Z)$ be such that $\tau(E) > 0$. Put $E_1 = E \cap Z_1 = E - \{(\Omega, 0)\}$. Then $\tau(E) = \tau(E_1) = \varrho(T^{-1}(E_1)) = \varrho(E_1)$. Since ϱ has the Darboux property for each number a which is non-negative and less than or equal to $\varrho(E_1)$, there exists a set $F \in \mathscr{B}(X \times Y)$ such that $F \subset E_1$ and $\varrho(F) = a$. Since T is a continuous transformation and the restriction of T on Z_1 is identical we have

$$F \in \mathscr{S}(\mathscr{U}) \cap Z_1 = \mathscr{S}(\mathscr{U} \cap Z_1) = \mathscr{S}(\mathscr{V} \cap Z_1),$$

whence $F \in \mathscr{B}(Z)$. Further

$$\tau(F) = \varrho(T^{-1}(F)) = \varrho(F) = a;$$

thus τ has the Darboux property.

Finally assume that the restriction of τ on the Baire sets has the Darboux property. Then there exists a Baire set A such that $\tau(A) = 0$ and $(\Omega, 0) \in A$ (Corollary 2). From the regularity of a Baire measure we deduce that there exists an open Baire set V such that $(\Omega, 0) \in A \subset V$ and $\tau(V) < 1$. This is a contradiction.

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