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ON THE MAXIMUM VALUE OF A CLASS OF DETERMINANTS

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1. Result

For $m = 1, 2, 3, \ldots$ let $X(m)$ be the set of $m \times m$-matrices such that $x_{i,j} \in R^1$, $|x_{i,j}| \leqq 1$ for $i, j = 1, 2, \ldots m$.
Define

$$g(m) = \sup_{(x_{i,j}) \in X(m)} |\det(x_{i,j})|$$

The aim of this note is to prove that

$$\lim_{m \to \infty} (g(m))^{1/m} / m^{\frac{1}{2}} = 1 \,. \text{ (*)} \tag{1}$$

2. Preliminaries

Hadamard inequality reads

$$|\det (x_{i,j})| \leqq \prod_{i=1}^{m} \left(\sum_{j=1}^{m} x_{i,j}^2\right)^{1/2} \tag{2}$$

hence

$$g(m) \leqq m^{\frac{1}{2}m} \tag{3}$$

so that it remains to be proved that

$$\liminf_{m \to \infty} (g(m))^{1/m} / m^{\frac{1}{2}} \geqq 1 \,. \tag{4}$$

Equality in (2) holds, iff $(x_{i,j})$ is anorthogonal matrix. Hence $g(m) = m^{\frac{1}{2}m}$ iff there exists a matrix $(y_{i,j}) \in X(m)$ such that $|y_{i,j}| = 1$ for $i, j = 1, 2, \ldots m$ and $\sum_{j=1}^{m} y_{i,j}\, y_{k,j} = 0$ for $i \neq k$; such a matrix $(y_{i,j})$ is called a Hadamard matrix. $(x_{i,j}) \in X(m)$ is a Hadamard matrix, iff $|\det(x_{i,j})| = m^{\frac{1}{2}m}$.

(*) Formula (1) was needed in an investigation on functional differential equations, cf. [1].

Let $\mathscr{H}$ be the set of such $m = 1, 2, 3, \ldots$ that there exists a Hadamard matrix of order m. It is well known (cf. [2], Chapter 14) that

(5) $$\text{if } m, n \in \mathscr{H}, \text{ then } m\,n \in \mathscr{H},$$

(6) $$\text{if } m \in \mathscr{H},\ m > 2, \text{ then } m \equiv 0 \pmod 4,$$

(7) $$2^k \in \mathscr{H} \text{ for } k = 0, 1, 2, \ldots$$

(8) if $m = (p^k + 1) \equiv 0 \pmod 4$, p being an odd prime, $k = 1, 2, 3, \ldots$, then $m \in \mathscr{H}$,

(9) if $m = h(p^k + 1)$, $h \in \mathscr{H}$, $h > 2$, p being an odd prime, $k = 1, 2, 3, \ldots$, then $m \in \mathscr{H}$

and there are known several other sufficient conditions for $m \in \mathscr{H}$, but the conjecture that $\mathscr{H}$ contains all $m \equiv 0 \pmod 4$ remains undecided so far.

J. H. E. Cohn in [3] showed that for every $\varepsilon > 0$

(10) $$g(m) \geqq m^{(\frac{1}{2}-\varepsilon)m} \text{ for all sufficiently large } m.$$

G. F. Clements and B. Lindström in [4] obtained an estimate of $g(m)$ from below, from which it follows that

(11) $$\liminf_{m\to\infty} (g(m))^{1/m}/m^{\frac{1}{2}} \geqq \left(\frac{3}{4}\right)^{1/2}.$$

3. Lemmas.

Lemma 1. Let β be irrational $\beta > 0$. Let S be the set of all $u + v\beta$, u, v being nonnegative integers. Let $S = \{s_1, s_2, s_3, \ldots\}$, $s_1 < s_2 < s_3 < \ldots$. For every $\delta > 0$ there exists a $D > 0$ and to every $d \geqq D$ there exists a k, $K = 1, 2, 3, \ldots$ such that $s_k \leqq d < s_{k+1} < s_k + \delta$.

Proof. By the Dirichlet theorem (cf. [5], Chapter 2) for every $\delta > 0$ there exist integers p, q such that $0 < q \leqq \delta^{-1} + 1$, $p \geqq 0$, $|q\beta - p| < \delta$. Hence $p < \delta^{-1}\beta + \beta + \delta$. Let D be the least integer such that $D > (|q\beta - p|^{-1} + 1)(\delta^{-1}\beta + \beta + \delta)$ and let $r \geqq D$ be an integer. Let there be distinguished two cases: (i) $q\beta - p > 0$, (ii) $q\beta - p < 0$. In the case (i) define $w_i = r + i(q\beta - p)$, $i = 0, 1, 2, \ldots J$, J being the whole part of $|q\beta - p|^{-1} + 1$. It follows that $r - ip \geqq 0$, so that $w_i \in S$ for $i = 0, 1, \ldots J$, $r = w_0 < w_1 < \ldots < w_J$, $w_J > r + 1$, $w_{i+1} < w_i + \delta$. Therefore for every $d \in \langle r, r + 1)$ there exists an $i = 0, 1, \ldots J - 1$ such that $d \in \langle w_i, w_{i+1})$, so that $w_i \leqq d < w_{i+1} < w_i + \delta$. In the case (ii) it is defined $w_i = r + 1 + i(q\beta - p)$, $i = 0, 1, \ldots J$ and the argumentation is similar. The proof is complete.

Lemma 2. For every $\varepsilon > 0$ there exists a $C > 1$ and to every $m \geqq C$ there exist $a, b \in \mathscr{H}$ such that $a \leqq m < b < a(1 + \varepsilon)$.

Proof: Find $z \in \mathscr{H}$ such that $z \neq 2^k$, $k = 0, 1, 2, \ldots$ (cf. (8), (9)). By (5) and (7) $2^u . z^v = 2^{u+\beta v} \in \mathscr{H}$ for $u, v = 0, 1, 2, \ldots$ and $\beta = \lg z/\lg 2$ is irrational. Put $\delta = \lg (1 + \varepsilon)/\lg 2$, find D according to Lemma 1 and put $C = 2^D$. If $m \geqq C$, then $d = \lg m/\lg 2 \geqq D$ and by Lemma 1 there exist two pairs of nonnegative integers (u, v), $(\tilde{u}, \tilde{v})$ such that $u + \beta v \leqq d < \tilde{u} + \beta\tilde{v} < u + \beta v + \delta$. Hence $2^u . z^v \leqq m < 2^{\tilde{u}} . z^{\tilde{v}} < 2^u . z^v . 2^\delta = 2^u . z^v . (1 + \varepsilon)$. The proof is complete.

Lemma 3. $g(m + n) \geqq g(m) . g(n)$, $m, n = 1, 2, 3, \ldots$. This follows immediately from the definition of $g(m)$.

4. Proof of (1). By (11) there exists an α, $0 < \alpha < 1$ such that

$$g(n) \geqq \alpha^n n^{\frac{1}{2}n}, \qquad n = 1, 2, 3, \ldots . \tag{12}$$

Choose ε, $0 < \varepsilon < \mathrm{e}^{-1}$ and find $C > 1$ according to Lemma 2. Let $m \geqq C$. By Lemma 2 there exists an $a \in \mathscr{H}$ such that $a \leqq m < a(1 + \varepsilon)$. By Lemma 3 and (12) $g(m) \geqq g(a)\, g(m - a) \geqq a^{\frac{1}{2}a} \alpha^{m-a}(m - a)^{\frac{1}{2}(m-a)}$,

$$\frac{(g(m))^{1/m}}{m^{\frac{1}{2}}} \geqq \left(\frac{a}{m}\right)^{\frac{1}{2}\frac{a}{m}} \left(1 - \frac{a}{m}\right)^{\frac{1}{2}\left(1-\frac{a}{m}\right)} \alpha^{\left(1-\frac{a}{m}\right)} \geqq (1 - \varepsilon)^{\frac{1}{2}(1-\varepsilon)} . \varepsilon^{\frac{1}{2}\varepsilon} . \alpha^{\varepsilon}$$

and (4) holds, as ε is arbitrary. The proof of (1) is complete.

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