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THE MAXIMAL SEMILATTICE DECOMPOSITION OF A SEMIGROUP, RADICALS AND NILPOTENCY

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In paper [7] M. Petrich dealt with the maximal semilattice decomposition of a semigroup and he studied the classes of this decomposition. In the present paper a description of these classes and their products is given in terms of Luh Jiang completely prime radicals and faces of a semigroup S. Also the case of the commutative semigroup is discussed.

The last, 5th section is self-contained. Here a characterization of the class of all periodic semigroups with period 1, a characterization of the class of all periodic semigroups with index 1 and some characterizations of the class of all bands are given. We accomplished this using the mappings $M \to N_i(M)$ ($i = 1, 2, 3$), where $N_1(M)$ ($N_2(M)$) [$N_3(M)$] is the set of all strongly (weakly) [almost] nilpotent elements with respect to the subset M of the semigroup S (see [9]).

1. On an equivalence relation.

Let S be a non-empty set and Γ a family of subsets of S with the property $(\alpha): S \in \Gamma$, $\square \notin \Gamma$.

On S the following relation can be introduced (see [5] p. 203): $x \underset{\sim}{\Gamma} y$ if and only if for every $A \in \Gamma$, either $x, y \in A$ or $x, y \notin A$. This relation is an equivalence relation on S. If $x \underset{\sim}{\Gamma} y$ holds, we say that x and y are Γ-equivalent and the equivalence relation is called Γ-equivalence relation.

Let $\Delta = \{S\} \cup \{B \mid B = S \setminus A, A \in \Gamma, A \neq S\}$. Evidently the Δ-equivalence relation is equal to the Γ-equivalence relation.

If M is an arbitrary (non-empty) subset of S, then the intersection of all sets of Γ, which contain the set M, will be called a Γ-hull of the set M.

The following Lemmas are evident.

Lemma 1. *The elements x and y of S are Γ-equivalent if and only if they have the same Γ-hull (Δ-hull).*

Lemma 2. *The Γ-equivalence relation is the intersection of the universal equi-*

valence relation and of all equivalence relations on S that have only two classes: A and $S \setminus A$, $A \in \Gamma$, $A \neq S$.

Corollary. *Every class $M_x (x \in M_x)$ of the Γ-equivalence relation (Δ-equivalence relation) is the intersection of all sets of $\Gamma \cup \Delta$ which contain the element x.*

Remark. If S is a semigroup and Γ the family of all right (left) [two-sided] ideals of S, then the classes of this Γ-equivalence relation are the $\mathfrak{r}$ —($\mathfrak{l}$ —) [$\mathfrak{f}$ —] classes, introduced by J. A. Green in his paper [4]. In section 3 we shall show that the foregoing construction is useful also in the case if Γ is the family of all completely prime ideals of a semigroup S.

2. Completely prime ideals and faces

A (two-sided) ideal P of a semigroup S will be called a completely prime ideal, if $xy \in P$ $(x, y \in S)$ implies that either $x \in P$ or $y \in P$.

A subset $M \subseteqq S$ will be called a face of S, if $S \setminus M$ is a completely prime ideal or $M = S$ (the empty set will be considered to be a face).

It is known that a subset M of S is a face of S if and only if $x, y \in M$ is equivalent to $xy \in M$.

We now establish some Lemmas about completely prime ideals and faces.

Lemma 3. *Let M_1 be a face of S and M_2 a face of M_1. Then M_2 is a face of S.*

Corollary 1. *If P_1 is a completely prime ideal of S and P_2 a completely prime ideal of $S \setminus P_1$, then $P_1 \cup P_2$ is a completely prime ideal of S.*

Corollary 2. *Let M be a face of S and P' a completely prime ideal of M. Then $P' \cup (S \setminus M)$ is a completely prime ideal of S.*

Lemma 4. *Let S be a semigroup, S' a subsemigroup and M a face of S. Then $M \cap S'$ is a face of S'.*

Corollary. *Let S be a semigroup, S' a subsemigroup and P a completely prime ideal of S. Then $P \cap S'$ is a completely prime ideal of S'.*

It is evident that the intersection of an arbitrary number of faces of a semigroup S is a face of S. Moreover the union of an arbitrary number of completely prime ideals of a semigroup S is a completely prime ideal of S.

Let S' be a subsemigroup of a semigroup S and M a non-empty subset of S'. The intersection of all completely prime ideals of S', which contain M, will be denoted by $C(M, S')$. If $M = \{x\}$ is a one-element set, we shall write $C(x, S')$ instead of $C(\{x\}, S')$. If M is a (two-sided) ideal of S', then $C(M, S')$ is called a Luh Jiang radical of the semigroup S' with respect to M. For $S' = S$ we shall write $C(M)$ and $C(x)$ instead of $C(M, S)$ and $C(x, S)$. The principal two-sided (right) [left] ideal of S', generated by the element x, will be denoted by $J(x, S')$ $(R(x, S'))$ $[L(x, S')]$. If $S' = S$, we shall write $J(x)$, $R(x)$, $L(x)$ instead of $J(x, S)$, $R(x, S)$ and $L(x, S)$.

It can be shown (see [8]) that $C(M)$ is the set of all such elements $r \in S$ that every face, which contains r, has a non-empty intersection with M.

Lemma 5. *For every two elements $x, y \in S$, $C(xy) = C(J(xy)) = C(J(x)J(y)) = C(J(x) \cap J(y)) = C(J(x)) \cap C(J(y)) = C(x) \cap C(y)$ holds.*

Proof. For every semigroup S, $J(xy) \subseteqq J(x) \cap J(y)$ holds, therefore $C(J(xy)) \subseteqq C(J(x) \cap J(y)) = C(J(x)) \cap C(J(y))$ (see [8]).

Let $r \in C(J(x) \cap J(y))$. Then each face of S, which contains r, contains an element of $J(x) \cap J(y)$. Hence each face, which contains r, contains x and y. Thus it contains xy and it has a nonempty intersection with $J(xy)$, which implies $r \in C(J(xy))$. This means that $C(J(x) \cap J(y)) \subseteqq C(J(xy))$ and therefore $C(J(xy)) = C(J(x)) \cap C(J(y))$.

Evidently in every semigroup S, $J(xy) \subseteqq J(x)\,J(y) \subseteqq J(x) \cap J(y)$ holds. Thus $C(J(xy)) = C(J(x)\,J(y)) = C(J(x) \cap J(y)) = C(J(x)) \cap C(J(y))$.

The rest of Lemma 5 follows from the fact that for every $x \in S$, $C(x) = C(J(x))$ holds.

3. The maximal semilattice decomposition of a semigroup S.

Let $\mathscr{T}$ be the family of all completely prime ideals of a semigroup S. The $\mathscr{T}$-equivalence relation is a congruence. This was shown by M. Petrich (see [7]) and follows immediately from Lemma 2, because every equivalence relation of Lemma 2 is a congruence. Moreover all factor semigroups of S modulo these congruences are semilattices, therefore the factor semigroup modulo $\mathscr{T}$-equivalence relation is a semilattice too. (This holds also for an $\mathscr{A}$-equivalence relation, where $\mathscr{A}$ is a subfamily of the family $\mathscr{T}$.)

A decomposition of a semigroup S will be called (in agreement with [7]) a semilattice decomposition of S if this decomposition belongs to a congruence of S and the factor semigroup modulo this congruence is a semilattice. M. Petrich has shown (see [7]) that the decomposition belonging to the $\mathscr{T}$-equivalence relation is the maximal semilattice decomposition of the semigroup S in the sense that every homomorphic image of S, which is a semilattice, is a homomorphic image of the factor semigroup of S modulo the $\mathscr{T}$-equivalence relation.

M. Petrich [7] has shown some properties of the maximal semilattice decomposition of a semigroup S. On the base of the preceding two sections we can establish some other properties of this decomposition.

Let $\mathfrak{M}$ be the set of all faces of a semigroup S without $\square$. Let N_x be the class of the maximal semilattice decomposition of S which contains the element x. If S' is a subsemigroup of the semigroup S, we denote by $N(x, S')$ the intersection of all faces of S' that contain x (it is the minimal face of S' which contains x). Instead of $N(x, S)$ we shall write simply $N(x)$.

From the preceding sections the following Theorems follow:

Theorem 1. *The fulfilment of the following conditions for elements x, y of a semigroup S is equivalent:*

a) $x\widetilde{\mathscr{T}}y$

b) $x\widetilde{\mathfrak{M}}y$

c) $N(x) = N(y)$ *(see [7])*

d) $C(x) = C(y)$

e) $C(J(x)) = C(J(y))$.

Proof. $N(x)$ is the $\mathfrak{M}$-hull of x, $C(x)$ is the $\mathscr{T}$-hull of x and $C(x) = C(J(x))$.

Remark. In d) of Theorem 1 we can replace the elements x and y by their principal right (left) ideals, since $C(x) = C(R(x)) = C(L(x))$.

M. Petrich has shown (see [7]) that a class N_x of the maximal semilattice decomposition of a semigroup S contains no proper completely prime ideals. From this follows

Lemma 6a. *Let S' be a subsemigroup of a semigroup S and let $N_x \subseteqq S'$. Then every completely prime ideal of S' is either disjoint with N_x or contains N_x.*

Proof. Suppose, by way of contradiction, that there exists a completely prime ideal P' of S', which has a non-empty intersection with N_x but P' does not contain N_x. Then, as a consequence of Lemma 4, $N_x \cap P'$ is a proper completely prime ideal of N_x. But this is a contradiction.

Evidently the following Lemma holds.

Lemma 6b. *Let S' be a subsemigroup of a semigroup S and let $N_x \subseteqq S'$. Then every face of S' is either disjoint with N_x or contains N_x.*

Now we can easilly prove

Theorem 2. *For each $x \in S$ we have:*

a) N_x is the intersection of all completely prime ideals and of all faces of the semigroup S that contain the element x.

b) $N_x = N(x) \cap C(x)$,

c) $N_x = C(x, N(x))$,

d) $N_x = N(x, C(x))$.

Proof. a) follows from the Corollary of Lemma 2. b) is equivalent to a). b) and the Corollary of Lemma 4 and Lemma 6a imply c). d) follows from b), Lemma 4 and Lemma 6a.

Remark 1. Evidently $N_x = N(x) \cap C(J(x))$, $N_x = C(J(x, N(x)), N(x))$ and $N_x = N(x, C(J(x)))$ holds.

Remark 2. From Lemma 4 and its Corollary, from Lemma 6a and Lemma

6b it is evident that if S' is a subsemigroup of a semigroup S and $N_x \subsetneqq S'$, then N_x is also a class of the maximal semilattice decomposition of the semigroup S'. Thus if $N_x \subsetneqq S'$, we have by Theorem 2:

$N_x = N(x, S') \cap C(x, S')$,
$N_x = C(x, N(x, S'))$ and
$N_x = N(x, C(x, S'))$.

We can take for S' an arbitrary face of S which contains x or an arbitrary completely prime ideal of S which contains x.

The set of all faces of a semigroup S is a lattice if for every two faces M_1 and M_2 of S, $M_1 \wedge M_2 = M_1 \cap M_2$ and $M_1 \vee M_2$ is the minimal face of S, which contains M_1 and M_2. Then we have

Theorem 3. *For every two elements x, y of S the following holds:*

a) $N_{xy} = (N(x) \vee N(y)) \cap C(x) \cap C(y)$
b) $N_{xy} = (N(x) \vee N(y)) \cap C(J(x)\, J(y))$.

Proof. By Theorem 2, $N_{xy} = N(xy) \cap C(xy)$. Evidently $N(xy) = N(x) \vee \vee N(y)$, but on the other hand $C(xy) = C(J(xy)) = C(J(x)\, J(y)) = C(J(x)) \cap \cap C(J(y))$ by Lemma 5 and this proves Theorem 3.

4. The case of a commutative semigroup.

There are other possibilities how to express the sets $C(M)$ of a commutative semigroup S. This leads to other expressions for the $\mathcal{T}$-equivalence relation and for the classes N_x.

Let J be a (two-sided) ideal of S.

a) Let x be such an element of S that for some positive integers n, $x^n \in J$ holds. Then x will be called a nilpotent element of the semigroup S with respect to the ideal J. The set of all nilpotent elements of the semigroup S with respect to J will be denoted by $\tilde{N}(J)$.

b) An ideal I of the semigroup S, each element of which is nilpotent with respect to J, will be called a nilideal of S with respect to J. The union $R^*(J)$ of all nilideals of S with respect to J is called the Clifford radical of S with respect to J.

c) An ideal (subsemigroup) I of the semigroup S, for which there exists such a positive integer n that $I^n \subsetneqq J$, is called a nilpotent ideal (subsemigroup) of S with respect to J. The union $R(J)$ of all nilpotent ideals of S with respect to J will be called the Schwarz radical of S with respect to J.

d) An ideal I of the semigroup S, with the property that each subsemigroup of I generated by a finite number of elements of I is nilpotent with respect to J, is called a locally nilpotent ideal of S with respect to J. The union $L(J)$

of all locally nilpotent ideals of S with respect to J will be called the Ševrin radical of S with respect to J.

e) An ideal P of the semigroup S is called a prime ideal of S, if for any two ideals A and B of S, $AB \subseteqq P$ implies either $A \subseteqq P$ or $B \subseteqq P$. The intersection $M(J)$ of all prime ideals of S that contain the ideal J is called the McCoy radical of S with respect to J.

Remark. It is known that in a commutative semigroup an ideal P is a prime ideal if and only if it is a completely prime ideal.

Then from Theorem 1 we obtain

Theorem 4. *The following conditions for the elements x, y of a commutative semigroup S are equivalent:*

a) $x\,(\underset{\sim}{\mathcal{T}})y$

b) $x\,(\underset{\sim}{\mathfrak{M}})y$

c) $C(x) = C(y)$

d) $M(x) = M(y)$

e) $C(J(x)) = C(J(y))$

f) $M(J(x)) = M(J(y))$

g) $\tilde{N}(J(x)) = \tilde{N}(J(y))$

h) $R^*(J(x)) = R^*(J(y))$

i) $R(J(x)) = R(J(y))$

j) $L(J(x)) = L(J(y))$.

The proof follows from the Remark preceding Theorem 4 and from the fact that in every commutative semigroup S, $C(J) = M(J) = \tilde{N}(J) = R^*(J) = R(J) = L(J)$ holds for each ideal J of S (see [8] and [1]).

From Theorem 2 we obtain

Theorem 5. *For every $x \in S$ we have:*

a) $N_x = N(x) \cap C(x) = N(x) \cap M(x)$

b) $N_x = N(x) \cap C(J(x)) = N(x) \cap M(J(x)) = N(x) \cap \tilde{N}(J(x)) = N(x) \cap R^*(J(x)) = N(x) \cap R(J(x)) = N(x) \cap L(J(x))$.

c) $N_x = C(x, N(x)) = M(x, N(x))$

d) $N_x = C(J(x, N(x)), N(x)) = M(J(x, N(x)), N(x)) = \tilde{N}(J(x, N(x)), N(x)) = R^*(J(x, N(x)), N(x)) = R(J(x, N(x)), N(x)) = L(J(x, N(x)), N(x))$

e) $N_x = N(x, C(x)) = N(x, M(x))$

f) $N_x = N(x, C(J(x))) = N(x, M(J(x))) = N(x, \tilde{N}(J(x))) = N(x, R^*(J(x))) = N(x, R(J(x))) = N(x, L(J(x)))$.

A similar adaptation of Theorem 3 for commutative semigroups is obvious.

5. *On nilpotency.*

In paper [9] the notions of strong nilpotency, weak nilpotency and almost nilpotency of an element of a semigroup S with respect to an arbitrary subset of S were introduced. We shall characterize three classes of periodic semigroups using these notions.

Let S be a semigroup and M a subset of S.

a) An element $x \in S$ will be called strongly nilpotent with respect to M if there exists such a positive integer N that for every integer $n \geqslant N$, $x^n \in M$ holds. The set of all strongly nilpotent elements of S with respect to M will be denoted by $N_1(M)$.

b) An element $x \in S$ will be called weakly nilpotent with respect to M if for infinitely many positive integers n, $x^n \in M$ holds. The set of all weakly nilpotent elements of S with respect to M will be denoted by $N_2(M)$.

c) An element $x \in S$ will be called almost nilpotent with respect to M, if for some positive integers n, $x^n \in M$ holds. The set of all almost nilpotent elements of S with respect to M will be denoted by $N_3(M)$.

In paper [9] the mappings $M \to N_i(M)$, $i = 1, 2, 3$ were studied. We shall show some other properties of these mappings.

Theorem 6. *The class of all periodic semigroups with the period 1 is equal to the class of all semigroups in which the mappings $M \to N_1(M)$ and $M \to N_2(M)$ are equal.*

Proof. Let $a \in S$. Let $A = \langle a \rangle$ (the cyclic semigroup generated by a) and let A' be the set of the elements of the sequence $\{a^{2k}\}_{k=1}^{\infty}$. Clearly $a \in N_2(A')$. If the mappings $M \to N_1(M)$ and $M \to N_2(M)$ are equal, then $a \in N_1(A')$. Hence there exists such a positive integer N_1 that for all integers $n > N_1$, $a^n \in A'$ holds. Therefore $\langle a \rangle$ is a cyclic semigroup of finite order (and S is a periodic semigroup). Let r be the index and m the period of $\langle a \rangle$. Then $a \in$ $\in N_2(a^r) = N_1(a^r)$. Thus there exists such a positive integer N_2 that for all integers $n' > N_2$, $a^{n'} = a^r$, i. e. the period $m = 1$. This means that the semigroup S is a periodic semigroup with the period 1.

If conversely S is a periodic semigroup with the period 1 and $a \in N_2(M)$, then there exist infinitely many positive integers n such that $a^n \in M$. But then there exists such a positive integer N that for all integers $n > N$, $a^n \in M$. Hence $a \in N_1(M)$. Therefore we have $N_2(M) = N_1(M)$ for every subset $M \subseteq S$ and the mappings $M \to N_1(M)$ and $M \to N_2(M)$ are equal.

Theorem 7. *The class $\mathfrak{I}$ of all bands is equal to the class of all semigroups in which the mappings $M \to N_1(M)$ and $M \to N_3(M)$ are equal. The class $\mathfrak{I}$ is also the class of all semigroups in which the mappings $M \to N_1(M)$, $M \to N_2(M)$ and $M \to N_3(M)$ are equal.*

Proof. Let $a \in S$. Then $a \in N_3(a)$. If $M \to N_1(M)$ and $M \to N_3(M)$ are equal, then $a \in N_1(a) = N_3(a)$. Hence a is strongly nilpotent with respect to $\{a\}$ and there exists such a positive integer N that for all integers $n > N$,

$a^n = a$. This holds for every $a \in S$ i. e. S is a periodic semigroup with the period 1 and the index 1. Thus S is a band.

If conversely S is a band, then $a \in N_3(M)$ implies $a \in N_1(M)$. Therefore $N_3(M) = N_1(M)$ for every subset M of S, i. e. the mappings $M \to N_1(M)$ and $M \to N_3(M)$ are equal.

The last statement of Theorem 7 follows immediately.

If S is a band, then the mappings $M \to N_i(M)$, $i = 1, 2, 3$ are clearly identity mappings. Moreover, we have

Theorem 8. *The class of all bands is equal to the class of*

a) all semigroups in which the mapping $M \to N_1(M)$ *is the identity mapping,*

b) all semigroups in which the mapping $M \to N_2(M)$ *is the identity mapping,*

c) all semigroups in which the mapping $M \to N_3(M)$ *is the identity mapping.*

Proof. a) If $M \to N_1(M)$ is the identity mapping, then $N_1(a) = a$, i. e. there exists such a positive integer N that for all integers $n > N$, $a^n = a$ holds. Hence $\langle a \rangle$ is a cyclic semigroup with the period 1 and the index 1, therefore a is an idempotent.

b) Let $M \to N_2(M)$ be the identity mapping. Then $N_2(a) = a$ and for some positive integers $n > 1$, $a^n = a$ holds. Therefore $\langle a \rangle$ is a cyclic group with the identity e. Hence there exists such a positive integer m, that $a^m = e$, which implies $a \in N_2(e) = e$. Thus a is an idempotent.

c) If $M \to N_3(M)$ is the identity mapping, then $a \in N_3(a^2) = a^2$ implies $a = a^2$ for every $a \in S$.

The converse statement is evident.

Theorem 9. *The class of all periodic semigroups with the index 1 is equal to the class of all semigroups in which the mappings* $M \to N_2(M)$ and $M \to N_3(M)$ *are equal.*

Proof. Let the mappings $M \to N_2(M)$ and $M \to N_3(M)$ be equal. Let $a \in S$. Then $a \in N_3(a) = N_2(a)$ and for infinitely many positive integers n, $a^n = a$ holds. Thus $\langle a \rangle$ is a finite cyclic group. Hence S is *a* periodic semigroup with the index 1.

Let $a \in N_3(M)$ and let S be a periodic semigroup with the index 1. Then for infinitely many positive integers n, $a^n \in M$ holds. Hence $a \in N_2(M)$ q. e. d.

Remark. Theorem 6 follows also from Theorem 7 and 9, because $N_1(M) \subseteq$ $\subseteq N_2(M) \subseteq N_3(M)$ for every subset M of S.

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