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TERNARY HALFGROUPOIDS AND COORDINATIZATION

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In Section 1 we find the form of geometric systems corresponding to general ternary halfgroupoids in a similar way as there correspond affine planes to planar ternary groupoids. In Section 2 we describe some relations between autotopies of ternary (half)groupoids and the „coordinate" automorphisms of corresponding geometric systems. In Section 3 we characterize one type of geometric systems which are closely related to Sandler's pseudo planes.

1. TERNARY HALFGROUPOIDS AND POINT-LINE-SYSTEMS WITH PARALLELISM

We introduce the following concepts: geometry over ternary halfgroupoid, presystem with generalized parallelism and system with generalized parallelism. We shall show that these three concepts express essentially the same object and so we obtain a (possibly) large generalization of the well-known Hall's coordination scheme. The definitions are as follows:

Definition 1.1. A *ternary halfgroupoid* is a couple (S, τ) where S is a set with card $S \geqq 2$ and τ is a mapping of some nonempty set Domain $\tau \subseteqq S \times S \times S$ into S. For the case of Domain $\tau = S \times S \times S$ we get a *ternary groupoid*.

Definition 1.1a. Let $\boldsymbol{T} = (S, \tau)$ and $\boldsymbol{T}' = (S', \tau')$ be ternary halfgroupoids. An *isotopy* $\sigma : \boldsymbol{T} \to \boldsymbol{T}'$ is a quadruple $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ such that $\sigma_i : S \to S'$ $(i = 1, 2, 3, 4)$ is a bijection, $\{(a^{\sigma_1}, b^{\sigma_2}, c^{\sigma_3}) \mid (a, b, c) \in \text{Domain } \tau\} = \text{Domain } \tau'$ and $\tau'(a^{\sigma_1}, b^{\sigma_2}, c^{\sigma_3}) = (\tau(a, b, c))^{\sigma_4}$ for all $(a, b, c) \in$ Domain τ.(¹) For $\boldsymbol{T} = \boldsymbol{T}'$ we get an *autotopy*. For $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$ we obtain an *isomorphism* which becomes an *automorphism* if $\boldsymbol{T} = \boldsymbol{T}'$.

Definition 1.2. A *g. p. presystem*(²) is a quadruple $(\mathscr{P}, \mathscr{L}, \mathrm{I}, /\!/)$ where

(¹) Hence it follows that $\sigma^{-1} = (\sigma_1^{-1}, \sigma_2^{-1}, \sigma_3^{-1}, \sigma_4^{-1})$ is also an isotopy.

(²) This means: a *presystem with generalized parallelism*; similarly for a g. p. system.

(i) $\mathscr{P}$ and $\mathscr{L}$ are nonempty sets of elements called the *points* and the *lines* respectively, (ii) I is a binary relation between $\mathscr{P}$, $\mathscr{L}$ such that for each $p \in \mathscr{P}(l \in \mathscr{L})$ there exists a line l (a point p) with $p \mathrm{I} l$ and (iii) $/\!/$ a is decomposition(3) of $\mathscr{L}$ with members $L \subseteqq \mathscr{L}$ such that for each $p \in \mathscr{P}$ and each $L \in \mathscr{L}$ there is at most one line $l \in L$ with $p \mathrm{I} l$.

Definition 1.2a. Let $\mathbf{P} = (\mathscr{P}, \mathscr{L}, \mathrm{I}, /\!/)$ and $\mathbf{P}' = (\mathscr{P}', \mathscr{L}', \mathrm{I}', /\!/')$ be g. p. presystems. An *isomorphism* $\varrho : \mathbf{P} \to \mathbf{P}'$ is a pair (ϱ_1, ϱ_2) of bijections $\varrho_1 : \mathscr{P} \to \mathscr{P}'$, $\varrho_2 : \mathscr{L} \to \mathscr{L}'$ satisfying the following two properties: (i) $p \mathrm{I} l \Leftrightarrow p^{\sigma_1} \mathrm{I}' l^{\sigma_2}$ and (ii) l^{σ_1}, m^{σ_2} belong to a common member of $/\!/'$ if l, m belong to a common member of $/\!/$. If $\mathbf{P} = \mathbf{P}'$ then we get an *automorphism.*

Definition 1.3. A *g. p. system* is a triple $(\mathscr{P}, \mathscr{L}, /\!/)$ where $\mathscr{P}$ is a nonempty set of elements called the *points*, $\mathscr{L}$ is a nonempty set of certain nonempty subsets of $\mathscr{P}$ called the *lines* and $/\!/ = (L_\iota)_{\iota \in \mathrm{Domain}/\!/}$ is a family of nonempty subsets in $\mathscr{L}$ such that $\cup L_\iota = \mathscr{L}$ and that each member of $/\!/$ is a decomposition in $\mathscr{P}$. If $L_\alpha \cap L_\beta = \emptyset$ whenever $\alpha \neq \beta$ we get a *parallel system.*(4)

Definition 1.3a. Let $\mathbf{P} = (\mathscr{P}, \mathscr{L}, /\!/)$ and $\mathbf{P}' = (\mathscr{P}', \mathscr{L}', /\!/')$ be g. p. systems An *isomorphism* between $\mathbf{P}$, $\mathbf{P}'$ is a bijection $\varrho : \mathscr{P} \to \mathscr{P}'$ having the following properties: (i) if $l \in \mathscr{L}'$ then $l \in \mathscr{L}'$ and if $l' \in \mathscr{L}'$ then there is a line $l \in \mathscr{L}$ with $l^\varrho = l'$ and (ii) l^ϱ, m^ϱ belong to a common member of $/\!/'$ if l, m belong to a common member of $/\!/$. If $\mathbf{P} = \mathbf{P}'$ we get an *automorphism.*

Construction 1.1. Let $\mathbf{T} = (S, \tau)$ be a ternary halfgroupoid. First we introduce some denotations: $\mathrm{Domain}_{i,j}\, \tau$ ($\mathrm{Domain}_k\, \tau$) is the projection of Domain τ which arises by leaving only the components with the indices $i, j = 1, 2, 3$ or $k = 1, 2, 3$, respectively. $\mathrm{Range}_u\, \tau$ is the set of all $\tau(x, y, u)$ for all $(x, y, u) \in \mathrm{Domain}\, \tau$ with a fixed $u \in \mathrm{Domain}_3\, \tau$. Λ_τ is the set of all $(u, v) \in S \times S$ with $u \in \mathrm{Domain}_3\, \tau$ and $v \in \mathrm{Range}_u\, \tau$. Now put $\mathscr{P} = \mathrm{Domain}_{1,2}\, \tau$, $\mathscr{L} = \Lambda_\tau$ and define $\mathrm{I} \subseteq \mathscr{P} \times \mathscr{L}$ by $(x, y) \mathrm{I} (u, v) \Leftrightarrow \tau(x, y, u) = v$ for all admissible $(x, y, u) \in \mathrm{Domain}\, \tau$ and $v \in \mathrm{Range}_u\, \tau$. Further set $L_u = \{(u, v) \in \Lambda_\tau \mid v \in \mathrm{Range}_u\, \tau\}$ for every $u \in \mathrm{Domain}_3\, \tau$ and $/\!/ = \{L_u \mid u \in \mathrm{Domain}_3\, \tau\}$ Then $(\mathscr{P}, \mathscr{L}, \mathrm{I}, /\!/)$ is a g. p. presystem which is canonically determined by $\mathbf{T}$ and will be denoted by $\overleftarrow{\mathbf{P}}(\mathbf{T})$.

Construction 1.2. Let a ternary halfgroupoid $\mathbf{T} = (S, \tau)$ be given. Put $\mathscr{P} = \mathrm{Domain}_{1,2}\, \tau$, $l_{u,v} = \{(x, y) \in \mathrm{Domain}_{1,2}\, \tau \mid \tau(x, y, u) = v\}$ for each $(u, v) \in \Lambda_\tau$, $\mathscr{L} = \{l_{u,v} \mid (u, v) \in \Lambda_\tau\}$, $L_u = \{l_{u,v} \mid v \in \mathrm{Range}_u\, \tau\}$ for each $u \in \mathrm{Domain}_3\, \tau$,

(3) A *decomposition of* (or *on*) a set $S \neq \emptyset$ is a nonempty set of nonempty subsets in S which cover S. A *decomposition in* a set $S \neq \emptyset$ is a nonempty set of nonempty subsets in S.

(4) Here more generally as in André's paper [1], pp. 89—102.

$/\!/ = (L)_{\text{Domain}_3 \tau}$. Then $(\mathscr{P}, \mathscr{L}, /\!/)$ is a g. p. system which is canonically determined by $\boldsymbol{T}$. This g. p. system will be denoted by $\overline{\mathbf{P}}(\boldsymbol{T})$.

Construction 1.3. Let a g. p. presystem $\boldsymbol{P} = (\mathscr{P}, \mathscr{L}, \mathrm{I}, /\!/)$ be given where $\mathscr{P} \subseteq S \times S$ for a sufficiently large set S. Then we can choose injections $\alpha : /\!/ \to S$ and $\beta_L : L \to S$ (for each $L \in /\!/$) and define τ by $\tau(x, y, u) = v \Leftrightarrow \Leftrightarrow (x, y) \mathrm{I} \beta^{-1}_{\alpha^{-1}u} v$ for all admissible $(x, y) \in \mathscr{P}$, $u \in \alpha /\!/$, $v \in \beta(\alpha^{-1}_{\alpha^{-1}u} u)$. This τ is well-defined on a certain subset of $S \times S \times S$ so that a ternary halfgroupoid (S, τ) is obtained. This is canonically determined by $\boldsymbol{P}$, α and $(\beta_L)_{L \in /\!/}$ and will be denoted by $\mathbf{T}(\boldsymbol{P}, \alpha, (\beta_L)_{L \in /\!/})$.

Remark. Clearly $\overline{\mathbf{P}}(\mathbf{T}(\boldsymbol{P}, \alpha, (\beta_L)_{L \in /\!/})$ is isomorphic to $\boldsymbol{P}$.

Construction 1.4. Let a g. p. system $\boldsymbol{P} = (\mathscr{P}, \mathscr{L}, /\!/)$ be given with $\mathscr{P} \subseteq S \times S$ where S is a sufficiently large set. Then we can choose injections $\alpha : \text{Domain } /\!/ \to S$ and $\beta_\iota : L_\iota \to S$ (for each $\iota \in \text{Domain } /\!/$) and define τ by $\tau(x, y, u) = v \Leftrightarrow (x, y) \in \beta^{-1}_{\alpha^{-1}u} v$ for all admissible $(x, y) \in \mathscr{P}$, $u \in \alpha /\!/, v \in \beta_{\alpha^{-1}u}(\alpha^{-1}u)$. We obtain similarly as in Construction 1.3 a ternary halfgroupoid (S, τ) which is canonically determined by $\boldsymbol{P}, \alpha, (\beta_\iota)_{\iota \in \text{Domain} /\!/}$ and will be denoted by $\mathrm{T}(\boldsymbol{P}, \alpha, (\beta_\iota)_{\iota \in \text{Domain} /\!/})$.

Remark. Clearly $\overline{\mathbf{P}}(\mathbf{T}(\boldsymbol{P}, \alpha, (\beta_\iota)_{\iota \in \text{Domain} /\!/})) = \boldsymbol{P}$.

Construction 1.5. Let $\boldsymbol{P} = (\mathscr{P}, \mathscr{L}, \mathrm{I}, /\!/)$ be a g. p. presystem. Put $\bar{l} = \{p \in \mathscr{P} \mid p \mathrm{I} l\}$ for each $l \in \mathscr{L}$. Define $\overline{\mathscr{L}}$ as the set $\{\bar{l} \mid l \in \mathscr{L}\}$. Further choose a bijection $\alpha : J \to /\!/$ where J is a convenient index set. Now let $\overline{/\!/}$ be the family $(\overline{\alpha\iota})_{\iota \in J}$ where $\overline{\alpha\iota} = \{\bar{l} \mid l \in \alpha\iota\}$ for all $\iota \in J$. Then $(\mathscr{P}, \overline{\mathscr{L}}, \overline{/\!/})$ is a g. p. system canonically determined by $\boldsymbol{P}$ and α. This g. p. system will be denoted by $\widehat{\mathbf{P}}(\boldsymbol{P})$.

Remark. If $\boldsymbol{P}, \boldsymbol{P}'$ are isomorphic g. p. presystems them also $\widehat{\mathbf{P}}(\boldsymbol{P}), \widehat{\mathbf{P}}(\boldsymbol{P})'$ are isomorphic.

Construction 1.6. Let $\boldsymbol{T} = (S, \tau)$ be a ternary halfgroupoid satisfying the middle cancellation law: if $\tau(x, y_1, u) = \tau(x, y_2, u)$ for some (x, y_1, u) $(x, y_2, u) \in \text{Domain } \tau$ then $y_1 = y_2$. Define $\tau^\bullet$ by $\tau^\bullet(x, u, v) = y \Leftrightarrow \tau(x, y, u) = v$ for all $(x, y, u) \in \text{Domain } \tau$. Then $\tau^\bullet$ is well-defined on some uniquely determined subset of $S \times S \times S$ and $\boldsymbol{T}^\bullet = (S, \tau^\bullet)$ is a ternary halfgroupoid satisfying the right cancellation law: if $\tau^\bullet(x, u, v_1) = \tau^\bullet(x, u, v_2)$ for some $(x, u, v_1), (x, u, v_2) \in \in \text{Domain } \tau^\bullet$ then $v_1 = v_2$. Conversely, if $\boldsymbol{T} = (S, \tau)$ is a ternary halfgroupoid satisfying the right cancellation law then we may define $\hat{\tau}$ by $\hat{\tau}(x, y, u) = = v \Leftrightarrow \tau(x, u, v) = y$ for all $(x, u, v) \in \text{Domain } \tau$. Such $\hat{\tau}$ is well-defined on some subset of $S \times S \times S$ and the obtained ternary halfgroupoid $\hat{\boldsymbol{T}} = (S, \tau)$ satisfies the middle cancellation law.

Remark. Let $\boldsymbol{T} = (S, \tau)$ be a ternary halfgroupoid satisfying the middle cancellation law. Define τ^* by $\tau^*(u, v, x) = y \Leftrightarrow \tau^\bullet(x, u, v) = y$ for all $(x, u, v) \in \in \text{Domain } \tau^\bullet$. The obtained halfgroupoid $\boldsymbol{T}^* = (S, \tau^*)$ is said to be *dual* to $\boldsymbol{T}$

(and also $\breve{\overline{\mathbf{P}}}(\mathbf{T}^*)$ or $\overline{\mathbf{P}}(\mathbf{T}^*)$ can be said to be *dual* to $\breve{\overline{\mathbf{P}}}(\mathbf{T})$ or to $\overline{\mathbf{P}}(\mathbf{T})$, respectively) Clearly $(\mathbf{T}^*)^* = \mathbf{T}$.

2. GEOMETRIC SIGNIFICANCE OF AUTOTOPISMS

Proposition 2.1. *Let σ be an autotopy of a given ternary halfgroupoid $\mathbf{T} = (S, \tau)$. Then the rule $(x, y) \to (x^{\sigma_1}, y^{\sigma_2})$ for $(x, y) \in \mathrm{Domain}_{1,2}\, \tau$ and $(u, v) \to (u^{\sigma_3}, v^{\sigma_4})$ for $(u, v) \in \Lambda_\tau$ defines an automorphism of $\breve{\overline{\mathbf{P}}}(\mathbf{T})$.*

Proof. From $(x, y)\ \mathrm{I}\ (u, v)$ it follows successively $\tau(x, y, u) = v$, $\tau(x^{\sigma_1}, y^{\sigma_2}, u^{\sigma_3}) = v^{\sigma_4}$ and $(x^{\sigma_1}, y^{\sigma_2})\ \mathrm{I}\ (u^{\sigma_3}, v^{\sigma_4})$. This may be also reversed (on the whole we have condition (i) from Definition 1.2a). From $\tau(x, y, u) = v \Leftrightarrow \tau(x^{\sigma_1} y^{\sigma_2}\, u^{\sigma_3}) = v^{\sigma_4}$ also condition (ii) from Definition 1.2a follows.

Convention. Let S_1, S_2 be nonempty sets. Denote by X the set of all $x(b) = \{(x, y) \in S_1 \times S_2 \mid y = b\}$, $b \in S_2$ and by Y the set of all $y(a) = \{(x, y) \in S_1 \times S_2 \mid x = a\}$, $a \in S_1$.

Proposition 2.2. *Let there be given a g. p. presystem $\mathbf{P} = (\mathscr{P}, \mathscr{L}, \mathrm{I}, /\!/)$ with $\mathscr{P} \subseteq S_1 \times S_2$ for some at least two-element sets S_1 and S_2. Let S_3, S_4 be arbitrary sets such that there is a bijection $\alpha : /\!/ \to S_3$ and that there are injections $\beta_L : L \to S_4$ (for $L \in /\!/$) with $\bigcup_{L \in /\!/} \beta_L L = S$ and with $\beta_L L \cap \beta_M M = \emptyset$ whenever L, M are distinct members of $/\!/$. Then each coordinate automorphism*(6) *$\varrho = (\varrho_1, \varrho_2)$ of $\mathbf{P}$ induces an autotopy of $\mathbf{T}(\mathbf{P}, \alpha, (\beta_L)_{L \in /\!/})$.*

Proof. Since ϱ is a coordinate automorphism, $(x, y)^{\varrho_1} = (x^{\sigma_1}, y^{\sigma_2})$ for $(x, y) \in S_1 \times S_2$ defines bijections $\sigma_1 : S_1 \to S_1$, $\sigma_2 : S_2 \to S_2$. By the above choice of $(\beta_L)_{L \in /\!/}$, $(u, v)^{\varrho_i} = (u^{\sigma_3}, v^{\sigma_4})$ for $(u, v) \in \Lambda_\tau$ defines bijections $\sigma_3 : S_3 \to S_3$, $\sigma_4 : S_4 \to S_4$ and $(x, y)\ \mathrm{I}\ (u, v) \Rightarrow (x^{\sigma_1}, y^{\sigma_2})\ \mathrm{I}\ (u^{\sigma_3}, v^{\sigma_4})$ is equivalent to $\tau(x, y, u) = v \Rightarrow \tau(x^{\sigma_1}, y^{\sigma_1}, u^{\sigma_3}) = v^{\sigma_4}$. The properties of an automorphism of $\mathbf{P}$ guarantee that $\{(x^{\sigma_1}, y^{\sigma_2}, u^{\sigma_3}) \mid (x, y, u) \in \mathrm{Domain}\, \tau\} = \mathrm{Domain}\, \tau$.

Supplement. *If moreover $X \in /\!/$ with $\beta_X x(b) = b$, $b \in S_2$ then $\sigma_4 \mid_{S_2} = \sigma_2$ and $0^{\sigma_3} = 0$ for $0 = \alpha X$.*

Proof. By the present assumptions $\tau(x, y, 0) = y$ holds for all $(x, y) \in S_1 \times S_2$; and as ϱ is a coordinate automorphism, $\tau(x, y, 0) = y$ implies $\tau(x^{\sigma_1}, y^{\sigma_2}, 0^{\sigma_3}) = y^{\sigma_4}$ where necessarily $0^{\sigma_3} = 0$ and $y^{\sigma_2} = y^{\sigma_4}$ for all $y \in S_2$.

Proposition 2.3. *Let $\mathbf{P} = (\mathscr{P}, \mathscr{L}, /\!/)$, $/\!/ = (L_\iota)_{\iota \in S}$ be a parallel system with $\mathscr{P} = S \times S$ for a certain set S, $\mathrm{card}\, S \geqq 2$ and let $X = L_0$ for some element $0 \in S$ and $\mathrm{card}\, (y(0) \cap l) = 1$ for all $l \in \mathscr{P}$. Then there is a $\mathbf{T} = \mathbf{T}(\mathbf{P}, \mathrm{id}, (\beta_\iota)_{\iota \in S})$*

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(5) This may be compared with [2], pp. 39—42.

(6) i. e. an automorphism of $\mathbf{P}$ preserving as X as Y

such that every coordinate automorphism $\varrho = (\varrho_1, \varrho_2)$ *of* **P** *induces an autotopy* $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ *of* **T** *with* $0^{\sigma_3} = 0$ *and* $\sigma_2 = \sigma_4$. *Conversely, each autotopy* $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ *of* **T** *with* $0^{\sigma_3} = 0$ *induces a coordinate automorphism of* **P**.

Proof. Choose β_ι, $\iota \in S$ in such a way that $\beta_\iota l = v$ where $\{(0,v)\} = l \cap y\,(0)$ for each $l \in L_\iota$. Then $\tau(a, b, 0) = \tau(0, b, a) = b$ for all $a, b \in S$ and $\tau(x, y, u_1) = v_1 \Leftrightarrow \tau(x, y, u_2) = v_2$ for fixed (u_1, v_1), $(u_2, v_2) \in S \times S$ implies $u_1 = u_2$, $v_1 = v_2$. Let $\varrho = (\varrho_1, \varrho_2)$ be a coordinate automorphism of **P**. Then by $(x, y)^{\varrho_1} = (x^{\sigma_1}, y^{\sigma_2})$ for $(x, y) \in S \times S$ and $l^{\varrho_2}_{u,v} = l_{u^{\sigma_3}, v^{\sigma_4}}$ for $(u, v) \in S \times S$ the bijections $\sigma_i : S \to S$ $(i = 1, 2, 3, 4)$ with $0^{\sigma_3} = 0$ (this expresses the preserving of X) and with $\sigma_2 = \sigma_4$ are well defined. (This follows already from $\tau(a, b, 0) = b$ and from the preserving of X whereas $\tau(0, b, a) = b$ guarantees the necessary consistence.) The rest of Proposition 2.3 follows from the reversing of the preceding investigations.

Proposition 2.4. *Let* $\mathbf{P} = (\mathscr{P}, \mathscr{L}, /\!/)$, $/\!/ = (L_\iota)_{\iota \in S}$ *be a parallel system such that* (i) $\mathscr{P} = S \times S$ *for a set* S, card $S \geqq 2$, (ii) $X = L_0$ *for some element* $0 \in S$ (iii) card $(y(0) \cap l) = 1$ *for all* $l \in \mathscr{P}$, (iv) $d = \{(x, y) \in S \times S \mid x = y\} \in L_1$ *for some element* $1 \in S$ *and* (v) *each point of* y (1) *is contained in a unique line through* (0,0) *and each line through* (0,0) *intersects* $y(1)$ *in exactly one point. Then there is a* $\mathbf{T} = \mathbf{T}(\mathbf{P}, \alpha, (\beta_\iota)_{\iota \in S})$ *such that every coordinate automorphism of* **P** *fixing* (0,0) *and* (1,1) *induces an automorphism of* **T** *fixing* 0. *Conversely, every automorphism of* **T** *preserving* 0 *induces a coordinate automorphism of* **P** *fixing* (0,0) *and* (1,1).

Proof. For each $\iota \in S$ let $\alpha\iota = u$ where $\{(1, u)\} = l \cap y(1)$ for $(0,0) \in l \in L_\iota$. Further let $\beta_\iota m = v$ where $\{(0, v)\} = m \cap y\,(0)$ for each $m \in L_\iota$. By Proposition 2.3, to any coordinate automorphism ϱ of **P** preserving (0,0) there corresponds the autotopy $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ with $0^{\sigma_i} = 0$ $(i = 1, 2, 3, 4)$ and with $\sigma_2 = \sigma_4$. Condition (iv) is equivalent to $\tau(a, b, 0) = 1 \Leftrightarrow a = b$ and by our choice of α and $(\beta_\iota)_{\iota \in S}$ it follows that $\tau(1, a, b) = 0 \Leftrightarrow a = b$. By the properties of ϱ it must follow that $1^{\sigma_1} = 1^{\sigma_2} = 1$ and $\tau(1, a, a) = 0 \Rightarrow \tau(1, a^{\sigma_2}, a^{\sigma_3}) = 0 \Rightarrow \sigma_2 = \sigma_3$ whereas $\tau(a, a, 0) = 1 \Rightarrow \tau(a^{\sigma_1}, a^{\sigma_2}, 0) = 1 - \sigma_1 = \sigma_2$. Reversing these considerations we get the rest of Proposition 2.4.

Remark. The particular case of Proposition 2.3—4 for **P** to be an affine plane is studied in [3].

3. ON A TYPE OF PARALLEL SYSTEMS

Definition 3.1. A parallel system $\mathbf{P} = (\mathscr{P}, \mathscr{L}, /\!/)$ is said to be *natural* ([7]) if (a) $\mathscr{P} = S \times S$ for a set S, card $S \geqq 2$ (b) Domain $/\!/ = S$, i. e., $/\!/ = (L_\iota)_{\iota \in S}$, (c) $X = L_0$ for an element $0 \in S$, (d) card $(x(a) \cap l) =$ card $(y(a) \cap l) = 1$ for all $a \in S$ and $l \in \mathscr{L} \setminus X$ and (e) $d = \{(x, y) \in S \times S | x = y\} \in \mathscr{L}$.

Definition 3.2. A ternary groupoid $\mathbf{T} = (S, \tau)$ is said to be *natural*(7) if (1) for $u_1, u_2, v \in S$ with $u_1 \neq u_2$ there exist $x, y_1, y_2 \in S$; $y_1 \neq y_2$ such that $\tau(x, y_1, u_1) \neq \tau(x, y_2, u_2)$, (2) the equation $\tau(x, y, u) = v$ has a unique solution $x \in S$ ($y \in S$) for any given $y, u, v \in S$; $y \neq 0$ ($x, u, v \in S$), (3) there is a distinguished element $0 \in S$ with $\tau(a, b, 0) = \tau(0, b, a) = b$ for all $a, b \in S$ and (4) there is a distinguished element $1 \in S$ with $\tau(a, a, 1) = 0$ for all $a \in S$.

Proposition 3.1. *If* $\mathbf{T} = (S, \tau)$ *is a natural ternary groupoid then*: (A) $0 \neq 1$, (B) *from* $\tau(x, y, u_1) = v_1$, $\tau(x, y, u_2) = v_2$ *for fixed* (u_1, v_1), $(u_2, v_2) \in S \times S$ *it follows* $u_1 = u_2$, $v_1 = v_2$ *and* (C) $\mathbf{T}^\bullet$ *is characterized by the following conditions*: (1•) *for* $u_1, u_2, v \in S$; $u_1 \neq u_2$ *there exists* $x \in S$ *such that* $\tau^\bullet(x, u_1, v) \neq \tau^\bullet(x, u_2, v)$, (2•) *the equation* $\tau^\bullet(x, u, v) = y$ *has a unique solution* $x \in S$ ($v \in S$) *for any given* $u, v, y \in S$; $u \neq 0$ ($x, y, u \in S$) (3•) *there is a distinguished element* $0 \in S$ *such that* $\tau^\bullet(a, 0, b) = \tau^\bullet(0, a, b) = b$ *for all* $a, b \in S$ *and* (4•) *there is a distinguished element* $1 \in S$ *such that* $\tau^\bullet(a, 1, 0) = a$ *for all* $a \in S$.

Proof. Part (A): If $0 = 1$ then $a = \tau(a, a, 0)$ by (3) and consequently $a = 0$ by (4). This is a contradiction to card $S \geqq 2$.
Part (B): If we choose $x = 0$ then the left side of the investigated implication gives $v_1 = v_2$ so that (1) is already equivalent to (b).
Part (C): Only a transcription according to $\tau(a, b, c) = d \Leftrightarrow \tau^\bullet(a, c, d) = b$.

Proposition 3.2. *If* $\mathbf{T} = (S, \tau)$ *is a natural ternary groupoid then* $\overline{\mathbf{P}}(\mathbf{T})$ *is a natural parallel system. Conversely, if* $\mathbf{P} = (\mathscr{P}, \mathscr{L}, /\!/)$ *is a natural parallel system then there exists a* $\mathbf{T} = \mathbf{P}(\mathbf{P}, \alpha, (\beta_\iota)_{\iota \in S})$ *which is natural (with elements* 0,1 *determined by* $X = L_0$ *and* $d \in L_1$).

Proof. If $\mathbf{T}$ is a natural ternary groupoid, then for $\overline{\mathbf{P}}(\mathbf{T})$, card $S \geqq 2 \Rightarrow$ (a), $\text{Domain}_3\, \tau = S \Rightarrow$ (b), (3) $\Rightarrow$ (c), (2) $\Rightarrow$ (d) and (2) & (3) $\Rightarrow$ (e). Conversely, if $\mathbf{P}$ is a natural parallel system then put $\alpha = \text{id}$ and define $\beta_\iota l = v$ where $\{(0, v)\} = y(0) \cap l$ for each $l \in L_\iota$. Then $L_\alpha \cap L_\beta = \emptyset$ for $\alpha \neq \beta \Rightarrow$ (1), (d) $\Rightarrow$ (2). (c) together with the required form of $(\beta_\iota)_{\iota \in S} \Rightarrow$ (3) and (e) & (d) $\Rightarrow$ (4).

Proposition 3.3. *Let* $\mathbf{T} = (S, \tau)$ *be a natural ternary groupoid. Define* $\underset{\tau}{+}$, $\underset{\tau}{.}$ *by* $a \underset{\tau}{+} b = \tau^\bullet(a, 1, b)$, $a \underset{\tau}{.} b = \tau^\bullet(a, b, 0)$. *Then* $(S, \underset{\tau}{+})$ *is a loop and* $(S \setminus \{0\}, \underset{\tau}{.})$ *is a groupoid having the right unity and admitting the division from left*; *further* $a \underset{\tau}{.} 0 = 0 \underset{\tau}{.} a = 0$ *holds for all* $a \in S$.

Proof. In fact, $(S, \underset{\tau}{+})$ is a loop because of (2•) and (3•). Further $a \underset{\tau}{.} 0 = 0 \underset{\tau}{.} a = 0$ holds by (3•) for $b = 0$. Finally, the required properties of $(S \setminus \{0\}, \underset{\tau}{.})$ follow by (4•) and (2•) for $v = 0$ and $u \neq 0$.

(7) only a working term

Proposition 3.4. *Let* $\mathbf{T} = (S, \tau)$ *be a ternary grupoid satisfying* $(3^\bullet)$ — $(4^\bullet)$. *Let the „linearity property" be fulfilled:* $(5^\bullet)$ $\tau^\bullet(a, b, c) = a \underset{\tau}{\cdot} b \underset{\tau}{+} c$ for all $a, b, c \in S$.

Then $\mathbf{T}$ *is natural if and only if* $(S, \underset{\tau}{+})$ *is a loop,* $(S \setminus \{0\}, \underset{\tau}{\cdot})$ *is a grupoid with right unity and admitting the left division and* $R_{u_1} : x \to x \underset{\tau}{\cdot} u_1$, $R_{u_2} : x \to x \underset{\tau}{\cdot} u_2$ *are distinct for* $u_1 \neq u_2$.

Proof. If $\mathbf{T}$ is natural then all three above condition may be readily verified. Conversely, from these conditions $(2^\bullet)$ follows at once whereas $(1^\bullet)$ is guaranteed by $x \underset{\tau}{\cdot} u_1 + v = x \underset{\tau}{\cdot} u_2 + v \Rightarrow x \underset{\tau}{\cdot} u_1 = x \underset{\tau}{\cdot} u_2$.

Proposition 3.5. *Let* $(S, +)$ *be a loop with* card $S \geqq 2$. *Then each natural ternary groupoid* $\mathbf{T} = (S, \tau)$ *with* $\underset{\tau}{+} = +$ *and satisfying the linearity property may be constructed as follows: Choose an injection* $f : S \to S^S$ *such that* $S^{f(0)} = \{0\}$, *that each* $f(a) : S \to S$, $a \in S \setminus \{0\}$ *is a bijection and that* $f(1) : S \to S$ *is the identity mapping. Define* . *by* $x \,.\, y = x^{f(y)}$ *for all* $x, y \in S$. *The required ternary groupoid* $\mathbf{T} = (S, \tau)$ *is determined by* $\underset{\tau}{\cdot} = .\,$.

Proof. The properties $(2^\bullet)$ to $(4^\bullet)$ can be easily verified so that it suffices to investigate condition $(1^\bullet)$. Since f is an injection, for $u_1 \neq u_2$ there is an element $x \in S$ with $x^{f(u_1)} \neq x^{f(u_2)} \Leftrightarrow x \,.\, u_1 \neq x \,.\, u_2$ and this is equivalent to $x \,.\, u_1 + v \neq x \,.\, u_2 + v$ so that $(1^\bullet)$ must hold. Each natural ternary groupoid $\mathbf{T}$ with the linearity property satisfies all desired conditions by Proposition 3.4.

Proposition 3.6. *Let* $\mathbf{T} = (S, \tau)$ *be a natural parallel system. Then the linearity property is equivalent to the Desargues closure-condition in* $\overline{\mathbf{P}}(\mathbf{T})$ *of Fig.* 1 *and*

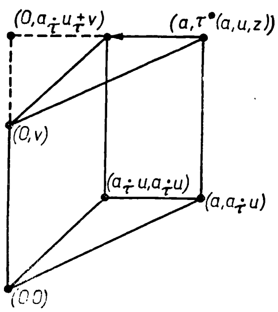

Fig. 1.

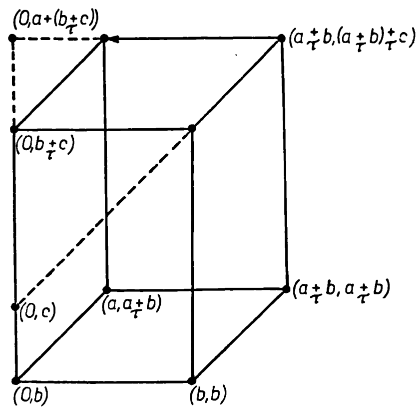

Fig. 2.

$\underset{\tau}{+}$ *is associative exactly if the Reidemeister closure-condition in* $\overline{\mathbf{P}}(\mathbf{T})$ *of Fig.* 2 *is fulfilled.* The proof can be derived from Fig. 1—2.

Proposition 3.7. *Let* $\mathbf{T} = (S, \tau)$ *be a natural ternary groupoid. Then it satisfies the linearity condition and its additive grupoid* $(S, \underset{\tau}{+})$ *is a group exactly if there is a group of coordinate translations*(8) *of* $\overline{\mathbf{P}} = \mathbf{P}(\mathbf{T})$ *acting transitively on* $y(0)$.

Proof. First let $\mathbf{T}$ satisfy the linearity condition and let $\underset{\tau}{+}$ be associative. Then the mappings given by $(x, y) \to (x, y \underset{\tau}{+} c)$, $c \in S$ form the desired group of coordinate translations. Conversely, if there is a group of coordinate translations acting transitively upon $y(0)$ then both closure-conditions of Fig. 1—2 are valid in $\mathbf{P}$ so that $\mathbf{T}$ satisfies the linearity condition and the derived operation $\underset{\tau}{+}$ is associative.

Remark. If a natural parallel system satisfies the further condition that each couple of points $p \in y(0)$, $q \notin y(0)$ is contained in precisely one line then we get a pseudo plane in the sense of Sandler.(9) By Proposition 3.5 natural parallel systems may easily be constructed different from pseudo planes.

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(8) i. e., of coordinate automorphisms of $\mathbf{P}$ preserving each $y(a)$, $a \in S$

(9) Cf. [4], p. 301.