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Vítězslav Novák

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TERNARY STRUCTURES AND PARTIAL SEMIGROUPS

VÍTĚZSLAV NOVÁK, Brno

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Transitive ternary structures and, especially, cyclically ordered sets can be transformed into other structures: into quasi-ordered sets ([3]), double binary structures ([4]), E-systems ([5]) etc. In this paper we describe a relation between transitive ternary structures and partial semigroups.

1. C-SEMIGROUPS

1.1. Let $G \neq \emptyset$ be a set, let $\cdot$ be a partial binary operation on G which has the following property:

let $x, y, z \in G$; if one of products $(x \cdot y) \cdot z$, $x \cdot (y \cdot z)$ or both products $x \cdot y$, $y \cdot z$ are defined then both products $(x \cdot y) \cdot z$, $x \cdot (y \cdot z)$ are defined and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Then the structure $\mathbf{G} = (G, \cdot)$ is called a *partial semigroup.*

1.2. A *homomorphism* of partial semigroups is defined in the obvious way. Thus, if $\mathbf{G} = (G, \cdot)$, $\mathbf{H} = (H, \cdot)$ are partial semigroups and $f\colon G \to H$, then f is a homomorphism of $\mathbf{G}$ into $\mathbf{H}$ if

$x, y \in G$ and $x \cdot y$ is defined $\Longrightarrow f(x) \cdot f(y)$ is defined in $\mathbf{H}$ and $f(x \cdot y) = f(x) \cdot f(y)$

A bijective homomorphism of $\mathbf{G}$ onto $\mathbf{H}$ such that f^{-1} is a homomorphism of $\mathbf{H}$ onto $\mathbf{G}$ is an *isomorphism* of $\mathbf{G}$ onto $\mathbf{H}$; $\mathbf{G}$ and $\mathbf{H}$ are *isomorphic* if there exists an isomorphism of $\mathbf{G}$ onto $\mathbf{H}$.

Let us note that a bijective homomorphism f of $\mathbf{G}$ onto $\mathbf{H}$ is an isomorphism iff $x, y \in G$, $f(x) \cdot f(y)$ is defined $\Longrightarrow x \cdot y$ is defined.

1.3. Let $\mathbf{G} = (G, \cdot)$ be a partial semigroup, $e \in G$. The element e is a *unit* in $\mathbf{G}$ if the following is satisfied:

if $e \cdot x$ is defined for some $x \in G$ then $e \cdot x = x$, if $y \cdot e$ is defined for some $y \in G$ then $y \cdot e = y$.

Let us denote by $E(\mathbf{G})$ the set of all units of a partial semigroup $\mathbf{G}$. In the sequel we shall deal with partial semigroups $\mathbf{G} = (G, \cdot)$ with the following property:

($*$) for any $x \in G$ there are units $e, e' \in E(\mathbf{G})$ such that $e \cdot x$ is defined and $x \cdot e'$ is defined.

We shall need some trivial and well known properties of partial semigroups; we present them with proofs as the proofs are very simple.

1.4. Lemma. *Let $\mathbf{G} = (G, \cdot)$ be a partial semigroup satisfying ($*$). Then for any $x \in G$ there exists just one unit $e \in E(\mathbf{G})$ such that $e \cdot x$ is defined and there exists just one unit $e' \in E(\mathbf{G})$ such that $x \cdot e'$ is defined.*

Proof. Let $e_1, e_2 \in E(\mathbf{G})$ and $e_1 \cdot x$, $e_2 \cdot x$ be defined. Then $e_2 \cdot x = x$ so that $e_1 \cdot (e_2 \cdot x)$ is defined. Hence $(e_1 \cdot e_2) \cdot x$, thus $e_1 \cdot e_2$ is defined and then $e_1 \cdot e_2 = e_1 = e_2$. Similarly the second assertion. □

1.5. Let $\mathbf{G} = (G, \cdot)$ be a partial semigroup satisfying (*) and $x \in G$. We denote by $e_L(x)$ the unit $e \in E(\mathbf{G})$ for which $e \cdot x$ is defined and by $e_R(x)$ the unit $e' \in E(\mathbf{G})$ for which $x \cdot e'$ is defined. $e_L(x)$ will be called the *left unit of x*, $e_R(x)$ the *right unit of x*.

Thus e_L, e_R are mappings $G \to E(\mathbf{G})$.

1.6. Lemma. *Let $\mathbf{G}$ be a partial semigroup satisfying ($*$) and $e \in E(\mathbf{G})$. Then $e_L(e) = e_R(e) = e$.*

Proof. We have $e_L(e) \cdot e = e = e_L(e)$ and similarly $e = e_R(e)$. □

1.7. Lemma. *Let $\mathbf{G} = (G, \cdot)$ be a partial semigroup satisfying ($*$), let $x, y \in G$ and let $x \cdot y$ be defined. Then $e_L(x \cdot y) = e_L(x)$, $e_R(x \cdot y) = e_R(y)$.*

Proof. Denote $e_L(x \cdot y) = e$. As $e \cdot (x \cdot y)$ is defined, $(e \cdot x) \cdot y$ and therefore $e \cdot x$ is defined. Then $e = e_L(x)$. Similarly for the right unit. □

1.8. Lemma. *Let $\mathbf{G} = (G, \cdot)$ be a partial semigroup satisfying ($*$) and $x, y \in G$. Then $x \cdot y$ is defined iff $e_R(x) = e_L(y)$.*

Proof. If $x \cdot y$ is defined then $(x \cdot e_R(x)) \cdot y$ is defined, thus $x \cdot (e_R(x) \cdot y)$ and also $e_R(x) \cdot y$ is defined which implies $e_R(x) = e_L(y)$. Conversely, let $e_R(x) = e_L(y) = e$. Then

both $x \cdot e$ and $e \cdot y$ are defined, thus $(x \cdot e) \cdot y = x \cdot y$ is defined. □

We shall study partial semigroups $\mathbf{G} = (G, \cdot)$ satisfying $(*)$ with the further property:

$(**)$ the pair of mappings $\{e_L, e_R\}$ distinguishes elements of G, i.e.

$$x, y \in G,\ e_L(x) = e_L(y),\ e_R(x) = e_R(y) \Longrightarrow x = y.$$

Partial semigroups in which $(*)$, $(**)$ hold will be called *c-semigroups.*

2. Ternary structures

2.1. Let $G \neq \emptyset$ be a set, let t be a ternary relation on G. The pair $\mathbf{G} = (G, t)$ will be called a *ternary structure.* A ternary relation t on G (and the structure (G, t)) is called *transitive* if

$$x, y, z, u \in G,\ (x, y, z) \in t,\ (z, y, u) \in t \Longrightarrow (x, y, u) \in t.$$

Let (G, t) be a ternary structure and $x \in G$. We say that x is an *isolated element* if neither $(x, y, z) \in t$ nor $(y, x, z) \in t$ nor $(y, z, x) \in t$ for any $y, z \in G$.

2.2. Let $\mathbf{G} = (G, t)$, $\mathbf{H} = (H, t')$ be ternary structures and $f\colon G \to H$. f is a *homomorphism* of $\mathbf{G}$ into $\mathbf{H}$ if

$$x, y, z \in G,\ (x, y, z) \in t \Longrightarrow (f(x), f(y), f(z)) \in t'.$$

A homomorphism f of $\mathbf{G}$ into $\mathbf{H}$ is *strong* if it is surjective and

$$u, v, w \in H,\ (u, v, w) \in t' \Longrightarrow \text{ there exist } x \in f^{-1}(u),\ y \in f^{-1}(v),\ z \in f^{-1}(w)$$

with $(x, y, z) \in t$.

A bijective strong homomorphism of $\mathbf{G}$ onto $\mathbf{H}$ is an *isomorphism.* Ternary structures $\mathbf{G}, \mathbf{H}$ are *isomorphic* if there is an isomorphism of $\mathbf{G}$ onto $\mathbf{H}$.

2.3. Let (G, t) be a ternary structure. We put

$$r(t) = \{(x, y, x) \in G^3;\ \text{there is } z \in G \text{ with } (x, y, z) \in t \text{ or } (z, y, x) \in t\}$$

and denote $c(t) = t \cup r(t)$

2.4. Lemma. *Let (G, t) be a transitive ternary structure. Then the structure $(G, c(t))$ is transitive, as well.*

Proof. Let $(x, y, z) \in c(t)$, $(z, y, u) \in c(t)$. If $(x, y, z) \in t$, $(z, y, u) \in t$ then $(x, y, u) \in t \subset c(t)$. If $(x, y, z) \in c(t) - t$ then $z = x$ and thus $(x, y, u) \in c(t)$. Similarly in the case $(z, y, u) \in c(t) - t$. Hence $c(t)$ is a transitive relation. □

2.5. Let (G, t) be a transitive ternary structure. We define a partial binary operation $\cdot$ on the set $c(t)$ as follows:

for $m_1 = (x_1, y_1, z_1) \in c(t)$, $m_2 = (x_2, y_2, z_2) \in c(t)$ the product $m_1 \cdot m_2$ is defined iff $x_2 = z_1$, $y_2 = y_1$; in that case $m_1 \cdot m_2 = (x_1, y_1, z_2)$.

In other words, we put

$$(x, y, z) \cdot (z, y, u) = (x, y, u).$$

2.6. Theorem. *Let (G, t) be a transitive ternary structure. Then $\mathbf{G} = (c(t), \cdot)$ is a c-semigroup in which $E(\mathbf{G}) = r(t)$ and $e_L(m) = (x, y, x)$, $e_R(m) = (z, y, z)$ for any $m = (x, y, z) \in c(t)$.*

Proof. Let $m_1, m_2, m_3 \in c(t)$ and suppose that $(m_1 \cdot m_2) \cdot m_3$ is defined. Then $m_1 = (x, y, z)$, $m_2 = (z, y, u)$ for suitable $x, y, z, u \in G$ and $m_1 \cdot m_2 = (x, y, u)$. Thus $m_3 = (u, y, v)$ for a suitable $v \in G$ so that $(m_1 \cdot m_2) \cdot m_3 = (x, y, v)$. We see that $m_2 \cdot m_3$ is defined and $m_2 \cdot m_3 = (z, y, v)$ so that $m_1 \cdot (m_2 \cdot m_3)$ is defined and $m_1 \cdot (m_2 \cdot m_3) = (x, y, v) = (m_1 \cdot m_2) \cdot m_3$. Similarly in the case when $m_1 \cdot (m_2 \cdot m_3)$ is defined. Let both $m_1 \cdot m_2$ and $m_2 \cdot m_3$ be defined. Then $m_1 = (x, y, z)$, $m_2 = (z, y, u)$, $m_3 = (u, y, v)$; thus $m_1 \cdot m_2 = (x, y, u)$ and $(m_1 \cdot m_2) \cdot m_3$ is defined. Hence $(c(t), \cdot)$ is a partial semigroup. If $e \in r(t)$ then $e = (x, y, x)$ so that if $e \cdot m$ is defined for some $m \in c(t)$ then $m = (x, y, z)$ and $e \cdot m = (x, y, z) = m$. Similarly if $m \cdot e$ is defined for some $m \in c(t)$. Thus $e \in E(\mathbf{G})$ and $r(t) \subset E(\mathbf{G})$.

Let $m = (x, y, z) \in c(t)$. Then $e = (x, y, x) \in r(t)$, thus $e \in E(\mathbf{G})$ and $e \cdot m = (x, y, x) \cdot (x, y, z) = (x, y, z) = m$. We see that $e = e_L(m)$; similarly $e' = (z, y, z) = e_R(m)$. Thus the partial semigroup $\mathbf{G} = (c(t), \cdot)$ satisfies (*) and $e_L(m) = (x, y, x)$, $e_R(m) = (z, y, z)$ for any $m = (x, y, z) \in c(t)$.

We show $E(\mathbf{G}) = r(t)$. If $e \in E(\mathbf{G})$ then $e_L(e) = e$ by 1.6 so that $e \cdot e$ is defined and $e \cdot e = e$. If $e = (x, y, z)$ then necessarily $e = (z, y, u)$ so that $z = x$ and $e = (x, y, x) \in r(t)$. Thus $E(\mathbf{G}) \subset r(t)$, which implies $E(\mathbf{G}) = r(t)$.

Let $m_1 = (x_1, y_1, z_1) \in c(t)$, $m_2 = (x_2, y_2, z_2) \in c(t)$ and $e_L(m_1) = e_L(m_2)$, $e_R(m_1) = e_R(m_2)$. Then $(x_1, y_1, x_1) = (x_2, y_2, x_2)$ so that $x_1 = x_2$, $y_1 = y_2$ and $(z_1, y_1, z_1) = (z_2, y_2, z_2)$ so that $z_1 = z_2$. Hence $m_1 = m_2$ and the pair of mappings $\{e_L, e_R\}$ distinguishes elements of $c(t)$, i.e. $(c(t), \cdot)$ is a c-semigroup. □

3.1. Let $\mathbf{G} = (G, t)$ be a transitive ternary structure. Denote by $S(\mathbf{G}) = (c(t), \cdot)$ the c-semigroup constructed in 2.5. If $\mathcal{T}$ is the class of all ternary structures and $\mathcal{C}$ is the class of all c-semigroups then S is a mapping of $\mathcal{T}$ into $\mathcal{C}$:

$$S\colon \mathcal{T} \to \mathcal{C}.$$

3.2. Let $\mathbf{M} = (M, \cdot)$ be a c-semigroup. Let us define a binary relation $\varrho(\mathbf{M})$ on the set $E(\mathbf{M})$ as follows:

$$(e, e') \in \varrho(\mathbf{M}) \Leftrightarrow \text{ there is } m \in M \text{ with } e = e_L(m),\ e' = e_R(m).$$

3.3. Lemma. *Let $\mathbf{M} = (M, \cdot)$ be a c-semigroup. Then the relation $\varrho(\mathbf{M})$ on $E(\mathbf{M})$ is reflexive and transitive.*

Proof. If $e \in E(\mathbf{M})$ then $e_L(e) = e_R(e) = e$ by 1.6 and $(e, e) \in \varrho(\mathbf{M})$ by definition. Let $e_1, e_2, e_3 \in E(\mathbf{M})$, $(e_1, e_2) \in \varrho(\mathbf{M})$, $(e_2, e_3) \in \varrho(\mathbf{M})$. Then there exist $m, n \in M$ with $e_1 = e_L(m)$, $e_2 = e_R(m)$, $e_2 = e_L(n)$, $e_3 = e_R(n)$. By 1.8 the product $m \cdot n$ is defined and by 1.7 $e_L(m \cdot n) = e_L(m) = e_1$, $e_R(m \cdot n) = e_R(n) = e_3$. Thus $(e_1, e_3) \in \varrho(\mathbf{M})$. □

3.4. The relation $\varrho(\mathbf{M})$ on $E(\mathbf{M})$ need not be symmetric so that it is not an equivalence relation in general. Let $\Theta(\mathbf{M})$ be the equivalence relation on $E(\mathbf{M})$ generated by $\varrho(\mathbf{M})$. Thus $(e, e') \in \Theta(\mathbf{M})$ iff there exist a positive integer n and elements $e_1, \dots, e_n \in E(\mathbf{M})$ such that $e_1 = e$, $e_n = e'$ and $(e_i, e_{i+1}) \in \varrho(\mathbf{M}) \cup \varrho(\mathbf{M})^{-1}$ for all $i = 1, \dots, n-1$.

3.5. Let $\mathbf{M} = (M, \cdot)$ be a c-semigroup, $\varrho(\mathbf{M})$ the binary relation on $E(\mathbf{M})$ defined in 3.2 and $\Theta(\mathbf{M})$ the equivalence relation on $E(\mathbf{M})$ generated by $\varrho(\mathbf{M})$. Put

$$G = E(\mathbf{M}) \cup E(\mathbf{M})|_{\Theta(\mathbf{M})}$$

and define a ternary relation t on G:

$$(x, y, z) \in t \Leftrightarrow x, z \in E(\mathbf{M}), y \in E(\mathbf{M})|_{\Theta(\mathbf{M})}, (x, z) \in \varrho(\mathbf{M}) \text{ and } x, z \in y.$$

We denote by $T(\mathbf{M})$ the ternary structure (G, t).

3.6. Theorem. *Let $\mathbf{M} = (M, \cdot)$ be a c-semigroup. Then $T(\mathbf{M}) = (G, t)$ is a transitive ternary structure in which $t = c(t)$.*

Proof. Let $x, y, z, u \in G$, $(x, y, z) \in t$, $(z, y, u) \in t$. Then $x, z, u \in E(\mathbf{M})$, $y \in E(\mathbf{M})|_{\Theta(\mathbf{M})}$, $(x, z) \in \varrho(\mathbf{M})$, $x, z \in y$ and $(z, u) \in \varrho(\mathbf{M})$, $z, u \in y$. By 3.3 $(x, u) \in \varrho(\mathbf{M})$ and $x, u \in y$. Thus $(x, y, u) \in t$ and t is transitive. Let $x, y, z \in G$, $(x, y, z) \in t$ so that $x, z \in E(\mathbf{M})$, $y \in E(\mathbf{M})|_{\Theta(\mathbf{M})}$, $(x, z) \in \varrho(\mathbf{M})$, $x, z \in y$. By 3.3 $(x, x) \in \varrho(\mathbf{M})$ and thus $(x, y, x) \in t$; similarly $(z, y, z) \in t$. Hence $c(t) = t$. □

3.6 implies that T is a mapping of $\mathcal{C}$ into $\mathcal{T}$, i.e.

$$T \colon \mathcal{C} \to \mathcal{T}.$$

3.7. Theorem. *Let $\mathbf{M} = (M, \cdot)$ be a c-semigroup. Then $\mathbf{M}$ is isomorphic to $(S \circ T)(\mathbf{M})$.*

Proof. Denote $T(\mathbf{M}) = (G, t)$ where $G = E(\mathbf{M}) \cup E(\mathbf{M})|_{\Theta(\mathbf{M})}$ and $(S \circ T)(\mathbf{M}) = S(G, t) = (c(t), \cdot)$. By 3.6 we have $c(t) = t$. Let us define a mapping $f \colon M \to c(t)$: $m \in M \Longrightarrow f(m) = (e_L(m), y, e_R(m))$ where $y \in E(\mathbf{M})|_{\Theta(\mathbf{M})}$ is such an element that $e_L(m) \in y$, $e_R(m) \in y$. By the definition of the relation t we have $f(m) \in t = c(t)$ so that f is really a mapping of M into $c(t)$. Let $(x, y, z) \in c(t)$. Then $x, z \in E(\mathbf{M})$, $y \in E(\mathbf{M})|_{\Theta(\mathbf{M})}$, $x, z \in y$ and $(x, z) \in \varrho(\mathbf{M})$, which means that there exists $m \in M$ with $x = e_L(m)$, $z = e_R(m)$. Then by definition $(x, y, z) = f(m)$ and the mapping f is surjective.

Let $m, n \in M$ and $f(m) = f(n)$. Then $(e_L(m), y, e_R(m)) = (e_L(n), z, e_R(n))$ where $e_L(m) \in y$, $e_L(n) \in z$, thus $e_L(m) = e_L(n)$, $e_R(m) = e_R(n)$. Hence $m = n$, $\mathbf{M}$ being a c-semigroup. Thus $f \colon M \to c(t)$ is injective and also bijective.

Let $m, n \in M$ and let $m \cdot n$ be defined. By definition $f(m) = (e_L(m), y, e_R(m))$ where $e_L(m), e_R(m) \in y$ and $f(n) = (e_L(n), z, e_R(n))$ where $e_L(n), e_R(n) \in z$. As $m \cdot n$ is defined, by 1.8 we have $e_R(m) = e_L(n)$. This implies $y = z$ so that $f(n) = (e_R(m), y, e_R(n))$. Hence the product $f(m) \cdot f(n)$ is defined in $(c(t), \cdot)$ and $f(m) \cdot f(n) = (e_L(m), y, e_R(n))$. By 1.7 we have $e_L(m \cdot n) = e_L(m)$, $e_R(m \cdot n) = e_R(n)$ and further $e_L(m \cdot n) = e_L(m) \in y$, $e_R(m \cdot n) = e_R(n) \in z = y$. Thus $f(m \cdot n) = (e_L(m \cdot n), y, e_R(m \cdot n)) = (e_L(m), y, e_R(n)) = f(m) \cdot f(n)$ and f is a homomorphism of $\mathbf{M}$ onto $(S \circ T)(\mathbf{M})$.

Let $m, n \in M$ and let the product $f(m) \cdot f(n)$ be defined in $(S \circ T)(\mathbf{M}) = (c(t), \cdot)$. As $f(m) = (e_L(m), y, e_R(m))$ with $e_L(m), e_R(m) \in y$, $f(n) = (e_L(n), z, e_R(n))$ with $e_L(n), e_R(n) \in z$, we necessarily have $y = z$, $e_R(m) = e_L(n)$. By 1.8 we see that $m \cdot n$ is defined in $\mathbf{M}$ and thus $f \colon M \to c(t)$ is an isomorphism of $\mathbf{M}$ onto $(S \circ T)(\mathbf{M})$. □

3.8. Theorem. *Let $\mathbf{G} = (G, t)$ be a transitive ternary structure without isolated elements and such that $c(t) = t$. Then there exists a strong homomorphism of the structure $(T \circ S)(\mathbf{G})$ onto the structure $\mathbf{G}$.*

Proof. By definition we have $S(\mathbf{G}) = (c(t), \cdot) = (t, \cdot)$; let us denote by $\mathbf{M}$ this c-semigroup. Then $(T \circ S)(\mathbf{G}) = T(\mathbf{M}) = (E(\mathbf{M}) \cup E(\mathbf{M})|_{\Theta(\mathbf{M})}, t')$ where $(u, v, w) \in t' \Leftrightarrow u, w \in E(\mathbf{M})$, $v \in E(\mathbf{M})|_{\Theta(\mathbf{M})}$, $u, w \in v$ and there exists $m \in t$ with $u = e_L(m)$, $w = e_R(m)$. If $m = (x, y, z)$ then by 2.6 we have $e_L(m) = u = (x, y, x)$, $e_R(m) = w = (z, y, z)$. Let us define a mapping $f\colon E(\mathbf{M}) \cup E(\mathbf{M})|_{\Theta(\mathbf{M})} \to G$:

if $u \in E(\mathbf{M})$, $u = (x, y, x)$ then $f(u) = x$
if $u \in E(\mathbf{M})|_{\Theta(\mathbf{M})}$ and if $(x, y, x) \in u$ for some $(x, y, x) \in E(\mathbf{M})$
then $f(u) = y$.

We must show that the definition of f is correct, i.e. the following implication holds:

$$\text{if } u \in E(\mathbf{M})|_{\Theta(\mathbf{M})},\ (x_1, y_1, x_1) \in u,\ (x_2, y_2, x_2) \in u \text{ then } y_1 = y_2.$$

Assume $(x_1, y_1, x_1) \in u$, $(x_2, y_2, x_2) \in u$. Then either $(x_1, y_1, x_1) = (x_2, y_2, x_2)$ which implies $y_1 = y_2$ or there exists a finite sequence $(p_1, q_1, p_1), (p_2, q_2, p_2), \ldots, (p_n, q_n, p_n)$ of elements in $E(\mathbf{M})$ such that $(p_1, q_1, p_1) = (x_1, y_1, x_1)$, $(p_n, q_n, p_n) = (x_2, y_2, x_2)$ and $((p_i, q_i, p_i), (p_{i+1}, q_{i+1}, p_{i+1})) \in \varrho(\mathbf{M}) \cup \varrho(\mathbf{M})^{-1}$ for $i = 1, \ldots, n-1$. It suffices to show that in this case $q_i = q_{i+1}$ for $i = 1, \ldots, n-1$. If $((p_i, q_i, p_i), (p_{i+1}, q_{i+1}, p_{i+1})) \in \varrho(\mathbf{M})$ then there exists $m = (p, q, r) \in t$ with $(p_i, q_i, p_i) = e_L(m)$, $(p_{i+1}, q_{i+1}, p_{i+1}) = e_R(m)$. Then by 2.6 $(p_i, q_i, p_i) = (p, q, p)$, $(p_{i+1}, q_{i+1}, p_{i+1}) = (r, q, r)$ and $q_i = q = q_{i+1}$.

If $((p_i, q_i, p_i)), (p_{i+1}, q_{i+1}, p_{i+1})) \in \varrho(\mathbf{M})^{-1}$ then $((p_{i+1}, q_{i+1}, p_{i+1}), (p_i, q_i, p_i)) \in \varrho(\mathbf{M})$ and $q_{i+1} = q_i$ as well. Thus $q_1 = \ldots = q_n$, i.e. $y_1 = y_2$.

Let $x \in G$. As $\mathbf{G}$ has no isolated elements there are $y, z \in G$ such that $(x, y, z) \in t$ or $(z, y, x) \in t$ or $(y, x, z) \in t$. In the first and second cases we have $(x, y, x) \in r(t) \subset t$ and by 2.6 $(x, y, x) \in E(\mathbf{M})$. Then by definition $f(x, y, x) = x$. In the third case $(y, x, y) \in r(t) \subset t$ and $(y, x, y) \in E(\mathbf{M})$. If $u \in E(\mathbf{M})|_{\Theta(\mathbf{M})}$ is such an element that $(y, x, y) \in u$ then $f(u) = x$ by the definition of f. Thus $f\colon E(\mathbf{M}) \cup E(\mathbf{M})|_{\Theta(\mathbf{M})} \to G$ is surjective.

Let $u, v, w \in E(\mathbf{M}) \cup E(\mathbf{M})|_{\Theta(\mathbf{M})}$, $(u, v, w) \in t'$. Then $u, w \in E(\mathbf{M})$, $v \in E(\mathbf{M})|\Theta(\mathbf{M})$, $u, w \in v$ and there exists $m = (x, y, z) \in t$ such that $u = e_L(m)$, $w = e_R(m)$. Thus $u = (x, y, x)$, $w = (z, y, z)$ and $f(u) = x$, $f(w) = z$, $f(v) = y$ by definition of f. Hence $(f(u), f(v), f(w)) \in t$ and f is a surjective homomorphism of $(T \circ S)(\mathbf{G})$ onto $\mathbf{G}$.

Let $x, y, z \in G$, $(x, y, z) \in t$. Then $(x, y, x) \in t$, $(z, y, z) \in t$ and $(x, y, x) \in E(\mathbf{M})$, $(z, y, z) \in E(\mathbf{M})$. If we denote $(x, y, z) = m$, $(x, y, x) = u$, $(z, y, z) = w$ and if $v \in E(\mathbf{M})|_{\Theta(\mathbf{M})}$ is such an element that $u \in v$ then $u = e_L(m)$, $w = e_R(m)$ and $(u, w) \in \varrho(\mathbf{M})$, $u, w \in v$. Then $(u, v, w) \in t'$ by the definition of t' and at the same time $f(u) = x$, $f(v) = y$, $f(w) = z$. Hence the homomorphism f of $(T \circ S)(\mathbf{G})$ onto $\mathbf{G}$ is strong. □

4. Examples

4.1 Let $G = \{x, y, z, u\}$, $t = \{(x, y, z), (z, y, u), (x, y, u), (x, y, x), (z, y, z), (u, y, u)\}$, $\mathbf{G} = (G, t)$. We construct $(T \circ S)(\mathbf{G})$.

Clearly $c(t) = t$ and $\mathbf{G}$ contains no isolated elements. Let us denote $m_1 = (x, y, z)$, $m_2 = (z, y, u)$, $m_3 = (x, y, u)$, $e_1 = (x, y, x)$, $e_2 = (z, y, z)$, $e_3 = (u, y, u)$. By 2.5 and 2.6 in the c-semigroup $S(\mathbf{G}) = \mathbf{M}$ we have:

$$\begin{aligned} m_1 \cdot m_2 = m_3, & \\ e_1 &= e_L(m_1) = e_L(m_3), \\ e_2 &= e_R(m_1) = e_L(m_2), \\ e_3 &= e_R(m_2) = e_R(m_3). \end{aligned}$$

Thus $E(\mathbf{M}) = \{e_1, e_2, e_3\}$ and by 3.2 $(e_1, e_2) \in \varrho(\mathbf{M})$, $(e_2, e_3) \in \varrho(\mathbf{M})$, $(e_1, e_3) \in \varrho(\mathbf{M})$ so that $\Theta(\mathbf{M}) = E(\mathbf{M})^2$, $E(\mathbf{M})|_{\Theta(\mathbf{M})} = \{\{e_1, e_2, e_3\}\}$ and $(T \circ S)(\mathbf{G}) = (\{e_1, e_2, e_3, \{e_1, e_2, e_3\}\}, t')$, where by 3.5

$$\begin{aligned} &(e_1, \{e_1, e_2, e_3\}, e_2) \in t', \\ &(e_2, \{e_1, e_2, e_3\}, e_3) \in t', \\ &(e_1, \{e_1, e_2, e_3\}, e_3) \in t', \\ &(e_1, \{e_1, e_2, e_3\}, e_1) \in t', \\ &(e_2, \{e_1, e_2, e_3\}, e_2) \in t', \\ &(e_3, \{e_1, e_2, e_3\}, e_3) \in t'. \end{aligned}$$

The mapping $f\colon E(\mathbf{M}) \cup E(\mathbf{M})|_{\Theta(\mathbf{M})} \to G$ constructed in the proof of Theorem 3.8 is

$$f(e_1) = x,\ f(e_2) = z,\ f(e_3) = u,\ f(\{e_1, e_2, e_3\}) = y$$

and it is an isomorphism of $(T \circ S)(\mathbf{G})$ onto $\mathbf{G}$.

4.2. Let $G = \{x, y, z\}$, $t = \{(x, y, z), (y, z, x), (z, x, y), (x, y, x), (z, y, z), (y, z, y), (x, z, x), (z, x, z), (y, x, y)\}$, $\mathbf{G} = (G, t)$; we find $(T \circ S)(\mathbf{G})$.

As in 4.1, we have $c(t) = t$ and $\mathbf{G}$ contains no isolated elements. Put $m_1 = (x, y, z)$, $m_2 = (y, z, x)$, $m_3 = (z, x, y)$, $e_1 = (x, y, x)$, $e_2 = (z, y, z)$, $e_3 = (y, z, y)$, $e_4 = (x, z, x)$, $e_5 = (z, x, z)$, $e_6 = (y, x, y)$.

In the c-semigroup $S(\mathbf{G}) = \mathbf{M}$ we have

$$e_1 = e_L(m_1),\ e_2 = e_R(m_1),\ e_3 = e_L(m_2),\ e_4 = e_R(m_2),\ e_5 = e_L(m_3),\ e_6 = e_R(m_3)$$

and the product in $\mathbf{M}$ is defined only with the corresponding units. Further we have

$$(e_1, e_2) \in \varrho(\mathbf{M}),\ (e_3, e_4) \in \varrho(\mathbf{M}),\ (e_5, e_6) \in \varrho(\mathbf{M})$$

so that

$$E(\mathbf{M})|_{\Theta(\mathbf{M})} = \{\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}\}$$

and

$$(T \circ S)(\mathbf{G}) = T(\mathbf{M}) = (\{e_1, e_2, e_3, e_4, e_5, e_6, \{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}\}, t')$$

where

$$\begin{aligned}
&(e_1, \{e_1, e_2\}, e_2) \in t',\\
&(e_3, \{e_3, e_4\}, e_4) \in t',\\
&(e_5, \{e_5, e_6\}, e_6) \in t',\\
&(e_1, \{e_1, e_2\}, e_1) \in t',\\
&(e_2, \{e_1, e_2\}, e_2) \in t',\\
&(e_3, \{e_3, e_4\}, e_3) \in t',\\
&(e_4, \{e_3, e_4\}, e_4) \in t',\\
&(e_5, \{e_5, e_6\}, e_5) \in t',\\
&(e_6, \{e_5, e_6\}, e_6) \in t'.
\end{aligned}$$

As G has three elements and the carrier of the structure $(T \circ S)(\mathbf{G})$ has nine elements, the structures $\mathbf{G}$ and $(T \circ S)(\mathbf{G})$ cannot be isomorphic. The strong homomorphism f of $(T \circ S)(\mathbf{G})$ onto $\mathbf{G}$ constructed in the proof of Theorem 3.8 has the form

$$f(e_1) = x,\ f(e_2) = z,\ f(e_3) = y,\ f(e_4) = x,\ f(e_5) = z,\ f(e_6) = y,$$
$$f(\{e_1, e_2\}) = y,\ f(\{e_3, e_4\}) = z,\ f(\{e_5, e_6\}) = x.$$

4.3. Problem. Find necessary and sufficient conditions for a transitive ternary structure $\mathbf{G} = (G, t)$ to be isomorphic to $(T \circ S)(\mathbf{G})$.

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Author's address: Department of Mathematics, Masaryk University, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic.