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ON SOME VARIETIES OF WEAKLY ASSOCIATIVE LATTICE GROUPS

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1. Basic notions

A *weakly associative lattice (wa-lattice)* is an algebra $A = (A, \vee, \wedge)$ with two binary operations satisfying the identities

(I)	$a \vee a = a;$	$a \wedge a = a.$
(C)	$a \vee b = b \vee a;$	$a \wedge b = b \wedge a.$
(Abs)	$a \vee (a \wedge b) = a;$	$a \wedge (a \vee b) = a.$
(WA)	$((a \wedge c) \vee (b \wedge c)) \vee c = c;$	$((a \vee c) \wedge (b \vee c)) \wedge c = 0.$

This notion has been introduced by E. Fried in [3] and by H. L. Skala in [11] and [12]. The notion of a *wa*-lattice is then a generalization of that of a lattice because the identities of associativity of the operations $\vee$ and $\wedge$ are replaced by weaker conditions of weak associativity (WA). Nevertheless, similarly as for lattices, we can define also for *wa*-lattices a binary relation $\leqslant$ on A such that

$$\forall a, b \in A;\, a \leqslant b \iff_{df} a \wedge b = a.$$

This relation is reflexive and antisymmetric and every two-element subset $\{a, b\} \subseteq A$ has the join $\sup\{a, b\} = a \vee b$ and the meet $\inf\{a, b\} = a \wedge b$ in A. Moreover (also as for lattices), each such binary relation defines on A a structure of a *wa*-lattice. (So, we can equivalently view any *wa*-lattice as a set with a binary relation.)

A *tournament*, i.e. a set $A \neq \emptyset$ with a reflexive and antisymmetric binary relation $\leqslant$ such that

$$\forall a, b \in A;\ a \leqslant b \quad \text{or} \quad b \leqslant a,$$

is a special case of a *wa*-lattice.

If $(G,+)$ is a group and $(G,\vee,\wedge)=(G,\leqslant)$ is a *wa*-lattice and if for any $a,b,c,d\in A$

$$(D_\vee)\qquad a+(b\vee c)+d=(a+b+d)\vee(a+c+d),$$

then the system $G=(G,+,\vee,\wedge)$ is called a *weakly associative lattice group* (*wal-group*). (See [8], [9], [10]. In [12] a *wal*-group is called a *trellis*-group.)

It is evident that in a *wal*-group the conditions

$$(D_\wedge)\qquad a+(b\wedge c)+d=(a+b+d)\wedge(a+c+d),$$

$$(M)\qquad a\leqslant b \implies c+a+d\leqslant c+b+d$$

are satisfied for any $a,b,c,d\in A$ and that every of these conditions is equivalent to $(D_\vee)$.

If for a *wal*-group G the *wa*-lattice $(G,\leqslant)$ is a tournament, then G is called a *totally semi-ordered group* (a to-group).

In contrast to the situation for lattice ordered groups (l-groups) and linearly ordered groups (o-groups) that are torsion free, there are many non-trivial finite *wal*-groups and *to*-groups.

2. The lattice of *wal*-ideals

The kernels of homomorphisms of *wal*-groups (i.e. *wal*-homomorphisms) will be called *wal-ideals*. The *wal*-ideals are special cases of *wal*-subgroups, that means of such subgroups which are both subgroups and *wa*-sublattices.

More precisely:

Proposition 1. ([8, Theorems 9 and 11], [10, Lemma 2.1].) *For a normal convex wal-subgroup H of a wal-group G the following conditions are equivalent:*

(a) *H is a wal-ideal of G.*

(b) $\forall a,b,\in H,\ x,y\in G\ (x\leqslant a,\ y\leqslant b\implies\ \exists c\in H,\ x\vee y\leqslant c)$.

(c) $\forall a,b,c\in H,\ x,y\in G;\ x\leqslant a,\ y\leqslant b\implies(x\vee y)\vee c\in H$.

Denote by $\mathcal{L}(G)$ the set of *wal*-ideals of a *wal*-group G. It is evident that $\mathcal{L}(G)$ ordered by set inclusion forms a complete lattice with the least element $\{0\}$ and the greatest element G.

Proposition 2. *If G is a wal-group, then $\mathcal{L}(G)$ is a complete sublattice of the lattice of subgroups of the group G.*

P r o o f. It is obvious that the intersection of any system of *wal*-ideals of G is also a *wal*-ideal of G. Moreover, *wal*-groups are Ω-groups in the sense of Kurosch, hence by [6, III.2.4], the *wal*-ideal generated by a system of *wal*-ideals of a *wal*-group G coincides with the subgroup of the additive group of G generated by these ideals as subgroups. □

Let us show that the lattice $\mathcal{L}(G)$ is distributive. For this, we will use known properties of varieties of algebras. The class of all *wal*-groups is by definition a variety of algebras of type $\langle +, 0, -(\cdot), \vee, \wedge \rangle$ of signature $\langle 2, 0, 1, 2, 2 \rangle$. Recall that a variety of algebras is called *arithmetical* if it is congruence distributive and permutable. (See [2].)

Theorem 3. *The variety of all wal-groups is arithmetical.*

Proof. By [2, Theorem II.12.5], the variety $\mathcal{V}$ is arithmetical if and only if there is a ternary Mal'cev term $m(x, y, z)$ such that

$$m(x, y, x) = m(x, y, y) = m(y, y, x) = x$$

in $\mathcal{V}$.

For the variety of *wal*-groups we can use the term

$$m(x, y, z) = x - (((x \vee y) \wedge (x \vee z)) \wedge (y \vee z)) + z,$$

which is in the case of l-groups equivalent to the term used in [4, p. 231]. □

We get, as a direct corollary, the following theorem.

Theorem 4. *The lattice of wal-ideals of any wal-group is distributive.*

3. The lattice of varieties of *wal*-groups

It is well known (see e.g. [4], [5], [7]) that the varieties of l-groups (considered in the language $\mathcal{L} = (+, 0, -(\cdot), \vee, \wedge))$ form a complete dually Brouwerian lattice $\mathbf{L}$ in which the variety of abelian l-groups $\mathcal{A}b_l$ is the least non-zero element. The variety of representable l-groups $\mathcal{R}_l$ is another important element of $\mathbf{L}$ because it is the variety generated by all linearly ordered groups. (The elements of $\mathcal{R}_l$ are precisely all subdirect sums of o-groups.) Recall that $\mathcal{R}_l$ is characterized by any of the following identities:

$$(x \wedge (-y - x + y)) \vee 0 = 0, \tag{1}$$

$$2(x \wedge y) = 2x \wedge 2y. \tag{2}$$

It is clear that the varieties of *wal*-groups considered also in the language $\mathcal{L}$ form a complete lattice **WAL**, too.

Theorem 5. *The lattice* **WAL** *is distributive and contains the lattice* **L** *as a complete* $\wedge$*-subsemilattice.*

Proof. In general, if $\mathcal{V}$ is an arbitrary variety of algebras, X is an infinite countable set, and F_X is the free algebra on X in the variety $\mathcal{V}$, then the lattice of subvarieties of $\mathcal{V}$ is anti-isomorphic to the lattice $FI(F_X)$ of fully invariant congruences on F_X. Since, by Theorem 3, the lattice $\mathcal{C}on(F_X)$ of all congruences on F_X is distributive and since the fully invariant congruences form a (complete) sublattice of $\mathcal{C}on(F_X)$, the lattice **WAL** is distributive.

Further, by [9, Proposition 1.10] a *wal*-group G is an l-group if and only if the identity

$$(L) \qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

is satisfied in G, therefore $\mathbf{L} \subseteq \mathbf{WAL}$. □

Denote by $\mathcal{A}b_{wal}$ the variety of abelian *wal*-groups and by $\mathcal{R}_{wal}$ the class of representable *wal*-groups (i.e. the class of *wal*-groups that are subdirect sums of *to*-groups). Recall that in the case of l-groups the representable l-groups form the variety $\mathcal{R}_l$ and that $\mathcal{A}b_l \subset \mathcal{R}_l$. Hence there are questions what are the relations between the classes $\mathcal{A}b_{wal}$ and $\mathcal{R}_{wal}$ and whether $\mathcal{R}_{wal}$ forms a variety of *wal*-groups.

To answer the first question, we will recall some notions and results of [9]. A convex *wal*-subgroup H of a *wal*-group G is called *solid* if it satisfies the condition (c) from Proposition 1. If H is a convex *wal*-subgroup, then we can define the structure of a *wa*-lattice on the set $G/_lH$ of left cosets of G by H by

$$x + H \leqslant y + H \iff_{df} \exists\, a \in H;\, x + a \leqslant y.$$

A solid subgroup H is called *straightening*, if it satisfies the following mutually equivalent conditions:

(a) $x, y \in G,\ 0 \leqslant x \wedge y \in H \implies x \in H$ or $y \in H$.

(b) $x, y \in G,\ x \wedge y = 0 \implies x \in H$ or $y \in H$.

(c) $G/_lH$ is a tournament.

Note that for *wal*-groups, the notion of a straightening subgroup is not equivalent, in contrast to l-groups, to the notion of a prime subgroup, i.e. finitely irreducible element of the lattice of solid subgroups of G. (See [9, Remark 2.2].)

By ([9, Theorem 2.6]) we have that a *wal*-group is representable if and only if the intersection of all its straightening *wal*-ideals is equal to $\{0\}$.

Theorem 6. *The classes $\mathcal{A}b_{wal}$ and $\mathcal{R}_{wal}$ are non-comparable.*

Proof. It is obvious that if G is an l-group, then $G \in \mathcal{R}_l$ if and only if $G \in \mathcal{R}_{wal}$, hence $\mathcal{R}_{wal} \nsubseteq \mathcal{A}b_{wal}$.

Conversely, consider the abelian *wal*-group $G = (\mathbb{Z}, +, \leqslant)$, with the positive cone $G^+ = \{x \in G;\, 0 \leqslant x\} = \{0, 1, 2, 4, \ldots, 2n, \ldots\}$. Since G has no straightening subroup different from G, we have by [9, Theorem 2.6] that $G \notin \mathcal{R}_{wal}$, thus $\mathcal{A}b_{wal} \nsubseteq \mathcal{R}_{wal}$. □

However, neither of the identities (1) and (2) characterizing $\mathcal{R}_l$ in $\mathbf{L}$ gives an answer to the question whether $\mathcal{R}_{wal}$ is a variety of *wal*-groups.

For instance, let $G = (\mathbb{Z}_3, +, \leqslant)$, where $G^+ = \{0, 1\}$. Then

$$(2 \wedge -2) \vee 0 = 1 \neq 0,$$
$$2(1 \wedge 2) = 2 \neq 1 = 2 \cdot 1 \wedge 2 \cdot 2,$$

hence G satisfies neither (1) nor (2). But G is a *to*-group, therefore $G \in \mathcal{R}_{wal}$.

Nevertheless, we have

Proposition 7. *The class $\mathcal{R}_{wal}$ is a variety of wal-groups.*

Proof. We will use Birkhoff's characterization of varieties as classes of algebras of a given type closed under products, subalgebras and homomorphic images. For this, if H is a subgroup of a *wal*-group G, then $H \leqslant G$ will mean that H is a *wal*-subgroup of G, and $H \trianglelefteq G$ will mean that H is a normal *wal*-subgroup of G.

a) Obviously, the product (i.e. the cardinal sum) of representable *wal*-groups is a representable *wal*-group, too.

b) Let G be a subdirect sum of *to*-groups G_i, $i \in I$, and let $H \leqslant G$. Let us consider any straightening *wal*-ideal S_j of G and denote $H_j = H \cap S_j$. It is evident that $H_j \trianglelefteq H$. Let $a, b \in H_j$, $x \in H$, $a \leqslant x$, $x \leqslant b$ (in short: $a \leqslant x \leqslant b$, although $a \leqslant b$ need not be true). Because $a, b \in S_j$, we have $x \in S_j \cap H = H_j$. Hence H_j is convex.

Let $a, b, c \in H_j$, $x, y \in H$, $x \leqslant a$, $y \leqslant b$. Then $(x \vee y) \vee c \in S_j \cap H = H_j$, and thus H_j satisfies condition (*c*) of Proposition 1. That means H_j is a *wal*-ideal of H.

Let $x, y \in H$, $x \wedge y = 0$. Then $x \in S_j$ or $y \in S_j$, hence $x \in H_j$ or $y \in H_j$. Therefore H_j is straightening.

Now, let $(S_j\,;\, j \in J)$ be the system of all straightening *wal*-ideals of G. Then

$$\bigcap_{j \in J} H_j = \bigcap_{j \in J} (H \cap S_j) \subseteq \bigcap_{j \in J} S_j = \{0\},$$

and so, by [9, Theorem 2.6], H is a representable *wal*-group.

c) Let f be a *wal*-homomorphism of a *wal*-group G onto a *wal*-group G', let H_i be a straightening *wal*-ideal of G, and let $H_i' = f(H_i)$. Since *wal*-groups are Ω-groups, H_i' is, by [6, III.2.12], a *wal*-ideal of G'.

Consider $x' + H_i'$, $y' + H_i' \in G'/H_i'$. Let $x, y \in G$, $f(x) = x'$, $f(y) = y'$. We can suppose that $x + H_i \leqslant y + H_i$. Then there is $a \in H_i$ such that $x + a \leqslant y$, and hence $x' + f(a) \leqslant y'$. Because $f(a) \in H_i'$, we have $x' + H_i' \leqslant y' + H_i'$, therefore H_i' is straightening.

Suppose that G is representable and that $(H_i, i \in I)$ is the system of all straightening *wal*-ideals of G. If there is $j \in I$ such that $f(H_i) = \{0'\}$, then $\{0'\}$ is a straightening *wal*-ideal of G', hence G' is a *to*-group. Let $H_i' = f(H_i) \neq \{0'\}$ for each $i \in I$. Because f induces a bijection of the set of *wal*-ideals of G which are not contained in $\mathcal{K}er\ f$ onto the set of all *wal*-ideals of G', and because the *wa*-lattices G/H_i and $G'/f(H_i)$ are isomorphic, f induces also a bijection of the set of straightening *wal*-ideals of G onto the set of straightening *wal*-ideals of G'. If $H' = \bigcap_{i \in I} H_i' \neq \{0'\}$, then $H = f^{-1}(H')$ is a *wal*-ideal of G which is contained in all straightening *wal*-ideals of G, and thus $H = \{0\}$, a contradiction. Therefore $H' = \{0\}$, which means G' is representable. □

Let us return to the identities (1) and (2) which characterize the variety of representable l-groups in **L**. We have proved that there are representable *wal*-groups not satisfying these conditions. Therefore, there is a natural question whether, in the class of representable *wal*-groups, l-groups are the only ones that satisfy both (1) and (2). However, the answer to this question is negative.

For instance, consider the *wal*-group $G = (\mathbb{Z}, +, \leqslant)$, where $G^+ = \{2^k;\, k \geqslant 0\} \cup -(\mathbb{Z}^+ \setminus \{2^k;\, k \geqslant 0\}) = \{0, 1, 2, -3, 4, -5, -6, -7, 8, -9, -10, \dots, -15, 16, -17, \dots\}$. Evidently G is a *to*-group, hence it is representable. Moreover, G is not an *o*-group.

Let $0 \leqslant k \in \mathbb{Z}$. Then

$$2^k - (-2^k) = 2^{k+1} \in G^+,$$

hence $-2^k \leqslant 2^k$ and we have $-2^k \leqslant 0$.

Let $k \geqslant 1$. Then

$$-(2k+1) - (2k+1) = -2(2k+1) \in G^+,$$

hence $2k + 1 < -(2k+1)$, and $2k + 1 < 0$.

Let $k \geqslant 3$, $k \neq 2^l$, $\forall l \geqslant 0$. Then

$$-2k - 2k = -4k \neq -2^m, \ \forall m \geqslant 0,$$

hence $2k < -2k$, and $2k < 0$.

Therefore

$$\forall x \in G;\ (x \wedge -x) \vee 0 = 0,$$

and so the identity (1) is satisfied.

Now, let $x, y \in G$. Since x, y are comparable, we can suppose e.g. $x \leqslant y$, hence $2(x \wedge y) = 2x$.

Let $y - x = 2^k$, $k \geqslant 0$. Then

$$2y - 2x = 2(x + 2^k) - 2x = 2^{k+1} \in G^+,$$

hence $2x \leqslant 2y$, i.e. $2x \wedge 2y = 2x$.

Let $y - x = -(2k+1)$, $k \geqslant 1$. Then $2y - 2x = -2(2k+1) \in G^+$, so $2x \wedge 2y = 2x$.

Finally, let $y - x = -2k$, $k \geqslant 3$, $k \neq 2^l$, $\forall l \geqslant 0$. Then

$$2y - 2x = -2(2k) \neq 2^m,\ \forall m \geqslant 0,$$

hence $2x \wedge 2y = 2x$.

Therefore G satisfies also the condition (2), and thus the variety of representable *wal*-groups satisfying both (1) and (2) is larger than the variety $\mathcal{R}_l$.

Now, let us consider the identity

$$\text{(3)} \qquad (x \vee 0) \wedge ((-y - x + y) \vee 0) = 0,$$

which is in the case of l-groups equivalent to the identity (1).

Let G be a *to*-group, $x \in G$. If $x \geqslant 0$, then

$$(x \vee 0) \wedge ((-y - x + y) \vee 0) = x \wedge 0 = 0.$$

If $x < 0$, then

$$(x \vee 0) \wedge ((-y - x + y) \vee 0) = 0 \wedge (-y - x + y) = 0.$$

Hence G satisfies (3), and therefore, in contrast to the condition (1), every representable *wal*-group also satisfies (3). But not even the condition (3) is sufficient to the characterization of the variety $\mathcal{R}_{wal}$, because any abelian *wal*-group also satisfies (3).

Therefore, let us consider the identity

$$\text{(4)} \quad (2x \wedge ((y + x) \wedge (2y \wedge (x + y)))) \wedge (2y \wedge ((x + y) \wedge (2x \wedge (y + x)))) = 2x \wedge 2y,$$

which is in the case of l-groups equivalent to (2).

Let G be a *to*-group, $x, y \in G$. Let $x \geqslant y$. Then

$$2x \wedge ((y+x) \wedge (2y \wedge (x+y))) = 2x \wedge ((y+x) \wedge 2y) = 2x \wedge 2y,$$

$$2y \wedge ((x+y) \wedge (2x \wedge (y+x))) = 2y \wedge ((x+y) \wedge (y+x)),$$

and if $x + y \leqslant y + x$, then

$$2y \wedge ((x+y) \wedge (y+x)) = 2y \wedge (x+y) = 2y.$$

Similarly for $y + x \leqslant x + y$.

Hence we have

$$\begin{aligned}(2x \wedge ((y+x) \wedge (2y \wedge (x+y)))) \wedge (2y \wedge ((x+y) \wedge (2x \wedge (y+x))))\\ = (2x \wedge 2y) \wedge 2y = 2x \wedge 2y.\end{aligned}$$

Since x and y on both sides of (4) appear symmetrically, the same result is valid also for the case $x < y$.

Thus G satisfies (4), and therefore every representable *wal*-group satisfies (4), too.

At the same time, there are abelian *wal*-groups not satisfying the property (4). For instance, let $G = (\mathbb{Z}, +)$, where $G^+ = \{0, 1, 2, 4, 6, \ldots, 2n, \ldots\}$. Let us consider $x = 3$, $y = 8$. Then

$$\begin{aligned}(2 \cdot 3 \wedge ((8+3) \wedge (2 \cdot 8 \wedge (3+8)))) \wedge (2 \cdot 8 \wedge ((3+8) \wedge (2 \cdot 3 \wedge (8+3))))\\ = (6 \wedge (11 \wedge (16 \wedge 11))) \wedge (16 \wedge (11 \wedge (6 \wedge 11)))\\ = (6 \wedge (11 \wedge 10)) \wedge (16 \wedge (11 \wedge 5))\\ = (6 \wedge 10) \wedge (16 \wedge 5) = 6 \wedge 4 = 4,\end{aligned}$$

but

$$2 \cdot 3 \wedge 2 \cdot 8 = 6 \wedge 16 = 6.$$

Hence the identity (4) separates the varieties $\mathcal{A}\mathrm{b}_{wal}$ and $\mathcal{R}_{wal}$. The following question remains open.

Question. *Does the identity (4) characterize the variety R_{wal}?*

Now, we can draw a fragment of the lattice **WAL**. It is clear that the variety $\mathcal{A}\mathrm{b}_l$ is still an atom of **WAL**. Indeed, let $\mathcal{V}$ be a variety of *wal*-groups such that $\mathcal{A}\mathrm{b}_l \not\subseteq V$ and let V contain a non-trivial l-group G. Then

$$\mathcal{A}\mathrm{b}_l \subseteq \mathcal{V}\mathrm{ar}_l(G) \subseteq \mathcal{V}\mathrm{ar}_{wal}(G) \subseteq \mathcal{V},$$

a contradiction. Therefore $\mathcal{V}$ contains no non-trivial l-group.

But $\mathcal{A}b_l$ is not the least non-trivial variety of **WAL** because it is non-comparable, for example, with the variety $\mathcal{T}_3$ of *wal*-groups satisfying the identity

$$(T_3) \qquad\qquad 3x = 0.$$

Let $\mathcal{X}_i$ denote the variety of *wal*-groups satisfying the identity (i), $i = 1, 2, 3, 4$, and $\mathcal{T}_n$ the variety of *wal*-groups satisfying the identity

$$(T_n) \qquad\qquad nx = 0,$$

where $n > 1$, n odd, $\mathcal{G}_{wal}$ the variety of all *wal*-groups, $\mathcal{G}_l$ the variety of all l-

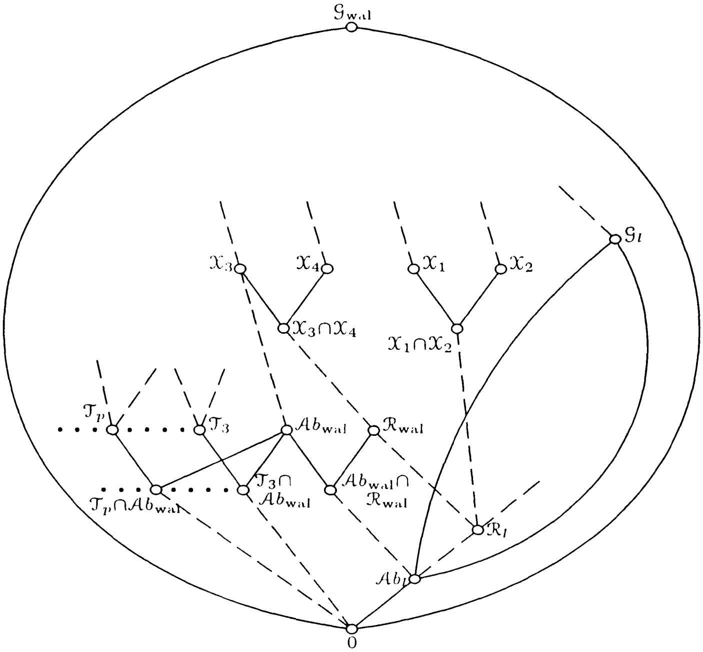

Fig. 1

groups, and $\mathcal{O}$ the trivial variety. Then the connections among these varieties are demonstrated in Figure 1.

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