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ON FRAMES DEFINED BY HORIZONTAL SPACES

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1. INTRODUCTION

Let M be a n-dimensional differentiable manifold and $F^2(M)$ be its second order holonomic frame bundle with canonical projection $F^2(M) \longrightarrow M$ and structure group $G^2(n)$ (see [2], Chapter 10, for a detailed description). It is shown in [1] that $G^2(n)$ can be expressed as a semidirect product $Gl(n,\mathbb{R}) \propto S^2(n)$ where $S^2(n)$ stands for the additive Lie group of symmetric bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. We express here the action of $S^2(n)$ on $F^2(M)$ in terms of the action of $\operatorname{Hom}(\mathbb{R}^n, \mathfrak{g})$ on the horizontal spaces in $F(M)$ where $\mathfrak{g}$ is the Lie algebra of $Gl(n,\mathbb{R})$. This latter action plays a basic role in the theory of prolongations of G-structures (see, for instance, [9, 15]). To make the geometric content of this action more transparent, we construct an auxiliary bundle $H^2(M) \longrightarrow M$ using arbitrary horizontal spaces in $F(M)$ and we identify $H^2(M) \longrightarrow M$ with $\hat{F}^2(M) \longrightarrow M$ (Theorem 3.1), where $\hat{F}^2M$ is the second order semi-holonomic frame bundle of M. Then, we express the universal connection on the 1-jets of sections $J^1(F(M))$ of $F(M)$ as studied in [10] (see also [2]) in terms of the $\mathfrak{g}$-component of the canonical form $\hat{\theta}^{(2)}$ of $\hat{F}^2(M)$. As an application, we prove that a semi-holonomic parallelism σ of second order is characterized by an ordinary parallelism plus a linear connection. We recover in this way the well-known result of P. Libermann ([13]) and P. C. Yuen [16]). The integrability of σ is obtained in terms of the two connections associated to σ. A semi-holonomic parallelism of second order may be viewed as a semi-holonomic second order $(1,0)$-structure. Semi-holonomic parallelisms of second order appears in a natural way in the characterization of the local homogeneity of Cosserat continua (see [6]).

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Let M be n-dimensional differentiable manifold and $F(M)$ the linear frame bundle of M. We denote by $\pi_0^1\colon F(M) \longrightarrow M$ the canonical projection. Let $j_0^1\varphi$ be the 1-jet of a differentiable mapping from a neighborhood of 0 in $\mathbb{R}^n$ into $F(M)$ such that $\pi_0^1 \circ \varphi$ is a diffeomorphism. Then $j_0^1\varphi$ is called a non-holonomic frame of second order at the point $\pi_0^1(\varphi(0)) \in M$. We know that the set $\tilde{F}^2(M)$ of all non-holonomic frames of second order at all points of M is a principal bundle over M with canonical projection $\tilde{\pi}_0^2\colon \tilde{F}^2(M) \longrightarrow M$ and structure group $\tilde{G}^2(n) = J_0^1(\mathbb{R}^n, J_0^1(\mathbb{R}^n,\mathbb{R}^n))$ of non-holonomic 2-jets with source and target $0 \in \mathbb{R}^n$. We also have a canonical projection $\tilde{\pi}_1^2\colon \tilde{F}^2(M) \longrightarrow F(M)$ given by $\tilde{\pi}_1^2(j_0^1\varphi) = \varphi(0)$. A non-holonomic 2-jet $j_0^1\varphi$ is said to be semi-holonomic if $\tilde{\pi}_1^2(j_0^1\varphi) = j_0^1(\pi_0^1 \circ \varphi)$. Let $\hat{F}^2(M) \subset \tilde{F}^2(M)$ be the set of semi-holonomic frames. We know that $\hat{\pi}_0^2\colon \hat{F}^2(M) \longrightarrow M$, where $\hat{\pi}_0^2$ is the restriction of $\tilde{\pi}_0^2$, is a frame bundle with structure group $\hat{G}^2(n) = Gl(n,\mathbb{R}) \times B^2(n)$ where $B^2(n)$ stands for the additive Lie group of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ (see [14, 16] for a detailed description). The group operation is given by

$$(a,f) \,\square\, (b,g) = (ab, a \circ g + f(b,b)).$$

Let $\tau\colon \hat{G}^2(n) \longrightarrow \hat{G}^2(n)$ be the diffeomorphism defined by $\tau(a,f) = (a,0) \,\square\, (1,f) = (a, a \circ f)$. We define a second group structure $\bullet$ on $\hat{G}^2(n)$ by $u \bullet v = \tau^{-1}(\tau(u) \,\square\, \tau(v)), u, v \in \hat{G}^2(n)$. It follows that $(\hat{G}^2(n), \bullet)$ is given by the semidirect product

$$(a,f) \bullet (b,g) = (ab, b^{-1} \circ f(b,b) + g)$$

and $\tau\colon (\hat{G}^2(n), \bullet) \longrightarrow (\hat{G}^2(n), \square)$ is an isomorphism of Lie groups. For $w \in \hat{F}^2(M)$ and $c \in (\hat{G}^2(n), \bullet)$, using the same notation $\bullet$, we define $w \bullet c = w \cdot \tau(c)$. The maps $\mathrm{id}: \hat{F}^2(M) \longrightarrow \hat{F}^2(M)$ and τ define now an isomorphism of the principal bundles $(\hat{F}^2(M), M, (\hat{G}^2(n), \bullet))$ and $(\hat{F}^2(M), M, (\hat{G}^2(n), \square))$.

Remark 2.1. We can alternatively define $\tau(a,f) = (1,f) \,\square\, (a,0)$. This gives $(a,f) \bullet (b,g) = (ab, f + a \circ g(a^{-1}, a^{-1}))$, which is the convention adopted in [1].

Our aim is to study the actions of the subgroups $Gl(n,\mathbb{R})$ and $B^2(n)$ of $(\hat{G}^2(n), \square)$ on $\hat{F}^2(M)$. Although this can be done directly (see Section 3), to make the situation more transparent, we now introduce a bundle $H^2(M) \longrightarrow M$ with group $(\hat{G}^2(n), \bullet)$ which is interesting in its own right.

Let $\pi_0^1\colon F(M) \longrightarrow M$ be the canonical projection, $z \in F(M)$ and H_z a horizontal space at z over $\pi_0^1(z)$. The restriction of the canonical form $\theta^{(1)}$ of $F(M)$ to H_z gives an isomorphism $\theta^{(1)}\colon H_z \longrightarrow \mathbb{R}^n$. For $A \in \mathfrak{g} = gl(n,\mathbb{R})$, let A^* be the fundamental vector field corresponding to A. We obtain now an isomorphism $u\colon \mathbb{R}^n \oplus \mathfrak{g} \longrightarrow$

$T_z(F(M))$ defined by $u(\xi, A) = h + A_z^*$, where $h \in H_z$ with $\theta^{(1)}(h) = \xi$ (see [9, 15]). Choosing the canonical basis of $\mathbb{R}^n$ and $\mathfrak{g}$, u defines a frame at z and thus $u \in F(F(M))$ and $\pi_1^2(u) = z$ where $\pi_1^2 \colon F(F(M)) \longrightarrow F(M)$ is the canonical projection. From now on, we will identify u with the pair (z, H_z).

Definition 2.1. $H^2(M)$ is the subset of $F(F(M))$ consisting of all frames in $F(M)$ defined by some horizontal space as described above.

The additive group $\mathfrak{g}^{(1)} = \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g})$ acts on $H^2(M)$ as follows ([9, 15]): Let $u = (z, H_z)$ and $f \in \mathfrak{g}^{(1)}$. Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$ and choose $h_i \in H_z$ with $\theta^{(1)}(h_i) = e_i$. Let L_z be the horizontal space spanned by the vectors $h_i + f(e_i)_z^*$. We define $u \cdot f = (z, L_z)$. Clearly, this is a free group action. Now let (z, H_z) and (z, L_z) be given. Let $\xi \in \mathbb{R}^n$ and choose $h \in H_z, l \in L_z$ with $\theta^{(1)}(h) = \theta^{(1)}(l) = \xi$. Then $(\pi_1^2)_*(l - h) = 0$ and therefore $l - h = A_z^*$ for some $A \in \mathfrak{g}$. Define the map $f \colon \mathbb{R}^n \longrightarrow \mathfrak{g}$ by $f(\xi) = A$. It follows that $f \in \mathfrak{g}^{(1)}$ and $(z, H_z) \cdot f = (z, L_z)$. Thus the action is also transitive. Consequently, $\mathrm{g}^{(1)}$ acts simply transitively on the fibers of the projection $\bar{\pi}_1^2 \colon H^2(M) \longrightarrow F(M)$, where $\bar{\pi}_1^2$ denotes the restriction of $\pi_1^2 \colon F(F(M)) \longrightarrow F(M)$ to $H^2(M)$.

There is also an action of $Gl(n, \mathbb{R})$ on $H^2(M)$ by right translation: $(z, H_z) \cdot a = (za, (R_a)_*(H_z)), a \in Gl(n, \mathbb{R})$. We identify $\mathfrak{g}^{(1)}$ with the additive group $B^2(n)$ of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $f(e_i)(e_j) = f(e_j, e_i) = f_{ji}^k e_k$ An element (a, f) of the set $Gl(n, \mathbb{R}) \times B^2(n)$ defines a map $\overline{(a, f)} \colon H^2(M) \longrightarrow H^2(M)$ by $(u)\overline{(a, f)} = (u \cdot a) \cdot f$. Clearly $\overline{(a, f)}$ is a bijection on the fibers of $\bar{\pi}_0^2 \colon H^2(M) \longrightarrow M$, $\bar{\pi}_0^2$ being the canonical projection.

Lemma 2.1. $\overline{(a, f)} \circ \overline{(b, g)} = \overline{(ab, b^{-1} \circ f(b, b) + g)}$, *where we compose from left to right on the left hand side of the equality.*

Proof. First note that for a given $a \in Gl(n, \mathbb{R})$ and $f \in B^2(n)$, there exists a unique $f^{(a)} \in B^2(n)$ satisfying $((u \cdot a) \cdot f) \cdot a^{-1} = u \cdot f^{(a)}$ for all $u \in H^2(M)$. Omitting dots and paranthesis, we have $(u)\overline{(a, f)} \circ \overline{(b, g)} = (u)afbg = (u)abb^{-1}fbg = (u)abf^{(b^{-1})}g = (u)\overline{(ab, f^{(b^{-1})} + g)}$. It suffices therefore to prove that $f^{(b^{-1})} = b^{-1} \circ f(b, b)$. This can be checked by a straightforward computation using $(R_b)^*\theta^{(1)} = b^{-1}\theta^{(1)}$ and $(R_b)_* A^* = ((Ad(b^{-1})A)^*$ and we will omit the details. □

Thus, in this way, we recover the group $(\hat{G}^2(n), \bullet)$ and identifying (a, f) with $\overline{(a, f)}$, the group acts on $H^2(M)$.

It is easy to see that $H^2(M)$ is a submanifold of $F(F(M))$ and the action of $(\hat{G}^2(n), \bullet)$ on $H^2(M)$ is differentiable. Also, let $\Theta^{(1)}$ be the restriction of the canonical form of $F(F(M)) \longrightarrow F(M)$ to $H^2(M)$. Then $\Theta^{(1)}$ is a $\mathbb{R}^n \oplus \mathfrak{g}$ valued 1-form: If X is a tangent vector at $u = (z, H_z)$ and $(\bar{\pi}_1^2)_*(X) = h_z + A_z^*, h_z \in H_z$, then the

$\mathbb{R}^n$-component of $\Theta^{(1)}(X)$ is $\theta^{(1)}(h_z)$, and its $\mathfrak{g}$-component of $\Theta^{(1)}(X)$ is A. In fact, we have

Theorem 2.2. (i) $\hat{\pi}_1^2\colon H^2(M) \longrightarrow F(M)$ *is a trivial principal bundle with group* $B^2(n)$ *and* $\hat{\pi}_0^2\colon H^2(M) \longrightarrow M$ *is a principal bundle with group* $(\hat{G}^2(n), \bullet)$.

(ii) $(R_\alpha)^*\Theta^{(1)} = \alpha^{-1}\Theta^{(1)}$, *where* $\alpha = (a, f) \in (\hat{G}^2(n), \bullet)$ *acts on* $\mathbb{R}^n \oplus \mathfrak{g}$ *by* $(a, f)(\xi, \beta) = (a\xi, f(a\xi) + Ad(a)\beta)$.

Rather than working out the details of Theorem 2.1, we shall identify $H^2(M)$ with $\hat{F}^2(M)$ (Theorem 3.1) and these properties will be immediate. However, it is worth noting that the opposite route can be taken: All properties of $\hat{F}^2(M)$ and its canonical form $\hat{\theta}^{(2)}$ can be derived simply from $H^2(M)$ and $\theta^{(1)}$.

We also have the following theorem whose proof is immediate from definitions.

Theorem 2.3. *There is a one to one correspondence between*

(i) *Connections on* $F(M) \longrightarrow M$.

(ii) $Gl(n, \mathbb{R})$*-reductions of* $H^2(M) \longrightarrow M$.

(iii) $Gl(n, \mathbb{R})$*-invariant sections of* $H^2(M) \longrightarrow F(M)$.

Remark 2.2. The equivalence (i)–(iii) was proved by P. Libermann [13] (see also [16]).

3. Identifications with $\hat{F}^2(M)$ and $J^1(F(M))$

Recall that an element $w \in \hat{F}^2(M)$ is a 1-jet $j_0^1\varphi$, $\varphi\colon \mathbb{R}^n \longrightarrow F(M)$ such that $\pi_0^1 \circ \varphi$ is a diffeomorphism and $\hat{\pi}_1^2(j_0^1\varphi) = \varphi(0) = j_0^1(\pi_0^1 \circ \varphi)$. Given w, we define a horizontal space at $z = \varphi(0)$ by the image of $T(\mathbb{R}^n)_0$ under the map $(\varphi)_*(0)$, which we will denote by $H_z(w)$. By a straightforward computation in local coordinates (see the proof of Lemma 3.1), it follows that any horizontal space at z is obtained in this way and if $\hat{\pi}_1^2(w) = \hat{\pi}_1^2(v) = z$ and $H_z(w) = H_z(v)$, then $w = v$. We thus obtain a bijective map $\nu\colon \hat{F}^2(M) \longrightarrow H^2(M)$ defined by $\nu(v) = (z, H_z(w))$.

Lemma 3.1. (i) $H_z(w \cdot (1, f)) = H_z(w) \cdot f$;

(ii) $H_{za}(w \cdot (a, 0)) = (R_a)_* H_z(w)$, $z = \hat{\pi}_1^2(w)$.

Proof. (i) Let $p \in M$, $(x^1, \ldots, x^n; U)$ be a coordinate neighborhood containing p, (x^i, x^i_j, x^i_{jk}) the induced coordinates on $W = (\hat{\pi}_0^2)^{-1}(U)$ and $w \in W$. Notice that in general the coordinates x^i_{jk} are not symmetric with respect to the indices jk. Then $H_z(w)$ is spanned by the vectors

$$\alpha_i = x_i^k \frac{\partial}{\partial x^k}\Big|_z + x^j_{ki} \frac{\partial}{\partial x^j_k}\Big|_z.$$

In fact, if $w = j_0^1\varphi$, where $\varphi(t^k) = (\varphi^i(t^k), \varphi^i_j(t^k))$, $t = (t^1, \cdots, t^n) \in \mathbb{R}^n$, then we have

$$(\varphi)_*(0)(\frac{\partial}{\partial t^i}) = \frac{\partial \varphi^k}{\partial x^i}(0)\frac{\partial}{\partial x^k} + \frac{\partial \varphi^j_k}{\partial x^i}(0)\frac{\partial}{\partial x^j_k} = x^k_i\frac{\partial}{\partial x^k}\Big|_z + x^j_{ki}\frac{\partial}{\partial x^j_k}\Big|_z.$$

By definition, $H_z(w)\cdot f$ is then spanned by the vectors $\alpha_i + f(e_i)^*_z$. Since for $A \in \mathfrak{g}, A = (A)^i_j$, we have

$$A^*_z = A^s_j x^i_s \frac{\partial}{\partial x^i_j}\Big|_z,$$

it follows that

$$\alpha_i + (f(e_i))^*_z = x^k_i\frac{\partial}{\partial x^k}\Big|_z + (x^k_{ji} + x^k_m f^m_{ji})\frac{\partial}{\partial x^k_j}\Big|_z.$$

On the other hand, we also have $w\cdot(1,f) = (x^i, x^i_j, x^i_{jk})\cdot(\delta^i_j, f^i_{jk}) = (x^i, x^i_j, x^i_{jk} + x^i_m f^m_{jk})$, which proves (i).

(ii) is proved similarly and we will omit the details. □

We now have

$$\begin{aligned}
\nu(w\bullet(a,f)) &= \nu(w\cdot\tau(a,f))\\
&= \nu((w\cdot(a,0))\cdot(1,f))\\
&= (za, H_{za}(w\cdot(a,0))\cdot f)\ \text{(Lemma 3.1, (i))}\\
&= (za, \{(R_a)_*(H_z(w))\}\cdot f)\ \text{(Lemma 3.1, (ii))}\\
&= \{(z, H_z(w))\cdot a\}\cdot f\\
&= \nu(w)\cdot(a,f).
\end{aligned}$$

Introducing a diferentiable structure on $H^2(M)$ by ν (which is easily checked to be none other than the one in Theorem 2.1), we obtain

Theorem 3.2. *We have the principal bundle isomorphisms*

$$\begin{aligned}
&(\nu, \mathrm{id})\colon (\hat{F}^2(M), M, (\hat{G}^2(n), \bullet)) \longrightarrow (H^2(M), M, (\hat{G}^2(n), \bullet))\ \textit{and}\\
&(\nu, \mathrm{id})\colon (\hat{F}^2(M), F(M), B^2(n)) \longrightarrow (H^2(M), F(M), B^2(n)).
\end{aligned}$$

Note that, in view of description of $\Theta^{(1)}$ above and the definitions of $\hat{\theta}^{(2)}$ and τ, it follows that $\nu^*(\Theta^{(1)}) = \hat{\theta}^{(2)}$. In fact, ν is the identity when one considers the induced coordinates in $\hat{F}^2(M)$ and $H^2(M)$, the latter considered as a submanifold of $F(F(M))$.

Now let $J^1(F(M))$ be the manifold of 1-jets of sections of $\pi_0^1\colon F(M) \longrightarrow M$, i.e., $J^1(F(M)) = \{j_x^1(s) \mid s \text{ is a section of } \pi_0^1\}$. Given w, we define a horizontal space at $s(x)$ by the image of $T(M)_x$ under the map $(s)_*(x)$. In this way, we obtain a bijective map $\mu\colon J^1(F(M)) \longrightarrow H^2(M)$ by which we identify $J^1(F(M))$ and $H^2(M)$. Notice that μ is locally given by $\nu(\bar{x}^i, \bar{x}^i_j, \bar{x}^i_{jk}) = (x^i = \bar{x}^i, x^i_j = \bar{x}^i_j, x^i_{jk} = \bar{x}^r_k \bar{x}^i_{jr})$, where $(\bar{x}^i, \bar{x}^i_j, \bar{x}^i_{jk})$ are the induced coordinates in $J^1(F(M))$ (see [10]). In [10], it is proved that there is a $\mathfrak{g}$ valued 1-form Ψ defined on $J^1(F(M))$ such that for any $Gl(n,\mathbb{R})$-invariant section γ of $J^1(F(M)) \longrightarrow F(M)$, $\gamma^*(\Psi)$ is a connection 1-form on $F(M)$ and this sets up a one to one correspondence between connection 1-forms ω and invariant sections γ (see [10] for a more general result). Ψ is also called the universal connection on $J^1(F(M))$ and plays an important role in the study of systems of connections (see [2], Chapter 9, and the references therein). Let us remark that $Gl(n,\mathbb{R})$ acts on $J^1(F(M))$ in such a way that $J^1(F(M))$ is a $Gl(n,\mathbb{R})$-principal bundle over $J^1(F(M))/Gl(n,\mathbb{R})$.

Theorem 3.3. *The universal connection Ψ on $\hat{F}^2(M)$ coincides with the $\mathfrak{g}$-component $(\hat{\theta}^{(2)})^{\mathfrak{g}}$ of $\hat{\theta}^{(2)}$, i.e., we have*

$$\Psi = \mu^*(\nu^{-1})^*((\hat{\theta}^{(2)})^{\mathfrak{g}}).$$

This can be showed by using the local expressions of Ψ (see [10]) and $\theta^{(2)}$ (see [14]). Indeed, we have

$$\hat{\theta}^{(2)} = \hat{\theta}^i e_i + \hat{\theta}^i_j E^j_i,$$

where

$$\hat{\theta}^i = y^i_k dx^k,\ \hat{\theta}^i_j = y^i_k dx^k_j - y^i_s y^t_r x^s_{tj} dx^r,$$

and

$$\Psi = \Psi^i_j,\ \Psi^i_j = \bar{y}^i_k d\bar{x}^k_j - \bar{y}^i_r \bar{x}^r_{jk} d\bar{x}^k,$$

being $(y^i_j) = (x^i_j)^{-1}$. Let Γ now be the section of $\pi^2_1\colon \hat{F}^2(M) \longrightarrow F(M)$ given in Theorem 2.2, and ω the connection form of the connection defined by Γ. It is then easy to see that $\omega = \Gamma^*((\theta^{(2)})^{\mathfrak{g}})$. Indeed, if $\Gamma\colon (x^i, x^i_j) \longrightarrow (x^i, x^i_j, \Gamma^l_{rs}(x^i, x^i_j))$ since Γ is $Gl(n,\mathbb{R})$-invariant, then $\Gamma^l_{rs}(x^i, x^i_j) = x^u_r x^v_s \Gamma^k_{uv}(x^i, \delta^i_j) y^l_k$. Hence a straightforward computation shows that $\Gamma^*(\theta^i_j) = y^i_t(dx^t_j) - x^u_j \Gamma^i_{ru} dx^r$ and the functions $-\Gamma^t_{sm}(x^i, \delta^i_j)$ are the Christoffel symbols of the connection 1-form $\omega = \Gamma^*(\theta^{\mathfrak{g}_2})$ (see [11]).

R e m a r k 3.1. (1) A $Gl(n,\mathbb{R})$-invariant section of $H^2(M) \longrightarrow M$ is called an E-connection of order 1 in [16].

(2) If $\omega(H_z) = 0$, ω and Γ as above, it is easy to see that $\Gamma(F(M)) \subset F^2(M)$ iff ω is torsionfree. Therefore, we recover Proposition 7.1. (p. 147) in [12].

4. Semi-holonomic parallelisms

A global section $\sigma\colon M \longrightarrow \hat{F}^2(M)$ will be called a *semi-holonomic parallelism of second order*. In this section we shall give a geometric interpretation of a semi-holonomic parallelism.

First, notice that σ induces by projection a global section of $p\colon M \longrightarrow F(M)$, i.e., an ordinary parallelism on M and a $Gl(n,\mathbb{R})$-invariant section $q\colon F(M) \longrightarrow \hat{F}^2(M)$, or, equivalently, a linear connection Λ on M. Conversely, an ordinary parallelism and a linear connection on M defines a semi-holonomic parallelism of second order on M.

Next, we shall study the integrability of a semi-holonomic parallelism σ. We say that σ is *integrable* if there exist local coordinates (x^i) around each point of M such that $\sigma(x^i) = (x^i, 1, 0)$. Let us recall that an ordinary parallelism $p\colon M \longrightarrow F(M)$ induces a flat linear connection Γ defined by $\nabla_{p_i} p_j = 0$, where $p = (p_1, \cdots, p_n)$, and $\{p_1, \cdots, p_n\}$ are n linearly independent vector fields on M. In general, Γ is not symmetric.

Suppose that σ is integrable. Hence p is integrable, i.e., $p(x^i) = (x^i, 1)$, or, in other words, $p_i = \frac{\partial}{\partial x^i}$. Then the Christoffel symbols of Γ vanish in the coordinates (x^i). Furthermore, we have $q(x^i, x^i_j) = (x^i, x^i_j, q^i_{jk}(x^u, x^u_v))$. Since $q(x^i, 1) = \sigma(x^i) = (x^i, 1, 0)$ we deduce that the Christoffel symbols of Λ are zero.

Conversely, let T be the torsion tensor of Γ and $D = \Gamma - \Lambda$ the difference tensor of the two connections. Suppose that T and D simultaneously vanish. Hence there exist local coordinates (x^i) around each point of M such that $p(x^i) = (x^i, 1)$, or, in other words, the Christoffel symbols Γ^i_{jk} vanish. Since $\Gamma = \Lambda$ the same is true for Λ and consequently we obtain $\sigma(x^i) = (x^i, 1, 0)$. Summing up, we have proved the following.

Theorem 4.1. *A semi-holonomic parallelism of second order σ is integrable if and only if T and D simultaneously vanish.*

These results generalize the previous ones obtained in [7, 3, 4, 5, 8] for holonomic parallelisms of second order.

5. Final remark

Finally, we would like to point out the following interesting point: The group operation $\bullet$ on $\hat{G}^2(n)$ is simply the second order jet composition, whereas, the bundle $H^2(M) \longrightarrow M$ is constructed on purely geometric grounds using right translation and "Hom-action". Still, the two groups and the corresponding bundles turn out to be isomorphic. This brings up the following somewhat vague question: Is it possible to construct the analogous bundles $H^k(M)$ for $k \geqslant 3$ using translations and Hom-actions and recover semi-holonomic jet groups and the frame bundles?

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