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SOME GENERAL MEANS

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1. Introduction

Using the mean value theorem for integrals (with two functions), a method of construction of some means is given in [9]: if $0 < a < b$, p is a positive integrable function on $[a,b]$ and f is a continuous strictly monotone function on $[a,b]$, then we get a mean V_f^p by setting

$$V_f^p(a,b) = f^{-1}\left(\int_a^b f(t)p(t)\,\mathrm{d}t \Big/ \int_a^b p(t)\,\mathrm{d}t\right). \tag{1}$$

Some of them can be defined also for $a = 0$.

For p fixed, the means A^p, G^p, L^p and I^p are defined in [4] as V_f^p for $f(x) = x$, $f(x) = x^{-2}$, $f(x) = x^{-1}$ and $f(x) = \log x$, respectively. Also it is proved that

$$G^p < L^p < I^p < A^p. \tag{2}$$

For $p(x) = 1$, we get the classical means A, G, L and I, i.e. the arithmetic, geometric, logarithmic and identric means, defined by

$$A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{ab}, \quad L(a,b) = \frac{a-b}{\log a - \log b}$$

and

$$I(a,b) = \frac{1}{\mathrm{e}}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}},$$

respectively.

For $p(x) = 1$, let us denote $V_f^p = V_f$. We also denote by M_r the r-th power mean, defined by

$$M_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}}, \quad r \neq 0,$$

and $M_0 = G$.

Some other examples of means, relations with other methods of construction and many references can be found in [9]. Properties of means are also given in [4] and [5].

In what follows we consider generalized logarithmic means and extend (2) for them. Then we define and study some means by fixing the weight function p.

2. Generalized logarithmic means

We define the generalized-logarithmic mean L_r^p as V_f^p for $f(x) = x^r$. We have

$$L_1^p = A^p, \quad L_{-2}^p = G^p, \quad L_{-1}^p = L^P.$$

As

$$\lim_{r \to 0} L_r^p(a,b) = I^p(a,b)$$

we put also $L_0^p = I^p$. For $p(x) = 1$, $L_r^p = L_r$ are the usual r-th logarithmic means defined by

$$L_r(a,b) = \left[\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \right]^{\frac{1}{r}}$$

for $r \neq -1$ and $r \neq 0$, while $L_{-1} = L$, $L_0 = I$.

From Jensen's inequality (see [3]) we have the following general result:

Theorem 1. *If the function $g \circ f^{-1}$ is convex and g^{-1} is increasing, then*

$$V_f^p(a,b) \leqslant V_g^p(a,b). \tag{3}$$

Proof. Jensen's inequality for $g \circ f^{-1}$ gives

$$g \circ f^{-1} \left(\frac{\int_a^b p(x) f(x)\, \mathrm{d}x}{\int_a^b p(x)\, \mathrm{d}x} \right) \leqslant \frac{\int_a^b p(x) \left(g \circ f^{-1} \circ f\right)(x)\, \mathrm{d}x}{\int_a^b p(x)\, \mathrm{d}x}$$

and applying g^{-1} we get (3).

Remark 1. For other combinations of convexity/concavity of $g \circ f^{-1}$ and monotonicity of g, we also get (3) or its reverse.

Consequence 1. *If $q < r$ then*

$$L_q^p(a,b) < L_r^p(a,b). \tag{4}$$

This generalizes (2) where the values

$$-2 < -1 < 0 < 1$$

are used.

3. Exponential means

In [8] the exponential mean is defined by

$$E(a,b) = \frac{b\mathrm{e}^b - a\mathrm{e}^a}{\mathrm{e}^b - \mathrm{e}^a} - 1.$$

In [6] it is remarked that $E = A^p$ with $p(x) = \mathrm{e}^x$, and some relations with other means are proved.

Analogously for $p(x) = \mathrm{e}^x$ we consider the means from (2): the exponential-geometric mean G^e, the exponential-logarithmic mean L^e, the exponential-identric mean I^e and we use the term exponential-arithmetic mean for $A^e = E$. Of course, from (2) we have

$$G^e < L^e < I^e < A^e = E.$$

More generally, we have the exponential-logarithmic means L_r^e which is L_r^p for $p(x) = \exp(x)$.

For natural values of r we can give an explicit formula for L_r^e as that for E. Using the Green-Lagrange formula

$$\int_a^b g^{(r)}(x)\, f(x)\, \mathrm{d}x = \Big[g^{(r-1)}(x)f(x) - g^{(r-2)}(x)f'(x) + \dots$$
$$+(-1)^{r-1}g(x)f^{(r-1)}(x)\Big]\Big|_a^b + (-1)^r \int_a^b g(x)\, f^{(r)}(x)\, \mathrm{d}x$$

for $g(x) = \mathrm{e}^x$ and $f(x) = x^r$, we get

$$\int_a^b (\mathrm{e}^x)^{(r)}\, x^r\, \mathrm{d}x = \left[\mathrm{e}^x P_r(x)\right]\big|_a^b + (-1)^r\, r! \int_a^b \mathrm{e}^x\, \mathrm{d}x$$

where

$$P_r(x) = \sum_{k=0}^{r-1} (-1)^k \, k! \binom{r}{k} x^{r-k}.$$

Consequently,

$$(L_r^e(a,b))^r = \frac{\mathrm{e}^b P_r(b) - \mathrm{e}^a P_r(a)}{\mathrm{e}^b - \mathrm{e}^a} + (-1)^r r!.$$

4. Applications of Hermite-Hadamard's inequalities

If the function $f\colon [a,b] \to \mathbb{R}$ is convex, then following the classical Hermite-Hadamard's inequalities we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}. \tag{5}$$

The inequalities are reversed if f is a concave function. We remark that these inequalities are also named the Hadamard or Jensen-Hadamard inequalities.

To deduce inequalities for the means L_r^e using (5), we consider functions f_r defined by

$$f_r(x) = x^r \mathrm{e}^x \quad \text{for} \quad x > 0, \quad r \neq 0$$

and

$$f_0(x) = \mathrm{e}^x \log x \quad \text{for} \quad x > 0.$$

Obviously,

$$f_r''(x) = \mathrm{e}^x x^{r-2} \left(x^2 + 2rx + r^2 - r\right), \quad r \neq 0$$

and

$$f_0''(x) = \mathrm{e}^x \left(x^2 \log x + 2x - 1\right)/x^2.$$

So, f_r is convex if $r \geqslant 1$ or $r < 0$. If $0 < r < 1$, f_r is convex for $x \geqslant \sqrt{r} - r$ and concave for $0 < x \leqslant \sqrt{r} - r$. Finally, there is a unique $x_0 \in (1/2, 1)$ such that the function f_0 is convex for $x \geqslant x_0$ and concave on $(0, x_0)$.

Thus we have the following results.

Theorem 2. *The exponential-logarithmic means L_r^e satisfy the estimates*

i) *if $r \geqslant 1$, or $0 < r < 1$ and $a, b \geqslant \sqrt{r} - r$ then*

$$A(a,b) \left(\frac{2\mathrm{e}^{(a+b)/2}}{\mathrm{e}^a + \mathrm{e}^b}\right)^{\frac{1}{r}} < L_r^e(a,b) < M_r(a,b)\, \mathrm{e}^{\frac{|b-a|}{2r}}; \tag{6}$$

ii) *if* $r < 0$ *the inequalities in* (6) *are reversed;*

iii) *if* $0 < r < 1$ *and* $a, b \leqslant \sqrt{r} - r$ *then*

$$M_r(a,b) \leqslant L_r^e(a,b) \leqslant A(a,b)\,; \tag{7}$$

iv) *if* $r = 0$ *and* $a, b \geqslant 1$, *then*

$$[A(a,b)]^{\exp(-|b-a|/2)} \leqslant I^e(a,b) \leqslant [G(a,b)]^{\exp(|b-a|/2)}\,; \tag{8}$$

v) *if* $r = 0$ *and* $0 < a,\ b < x_0$ *(where* $x_0^2 \log x_0 + 2x_0 = 1$*), then*

$$G(a,b) \leqslant I^e(a,b) \leqslant A(a,b). \tag{9}$$

P r o o f. If f_r is convex, then (5) implies

$$\frac{b-a}{\mathrm{e}^b - \mathrm{e}^a}\left(\frac{a+b}{2}\right)^r \mathrm{e}^{\frac{a+b}{2}} \leqslant \frac{\int_a^b x^r \mathrm{e}^x \,\mathrm{d}x}{\int_a^b \mathrm{e}^x \,\mathrm{d}x} \leqslant \frac{b-a}{\mathrm{e}^b - \mathrm{e}^a}\,\frac{a^r \mathrm{e}^a + b^r \mathrm{e}^b}{2}. \tag{6'}$$

Using (2) we have $G(x,y) < L(x,y) < A(x,y)$. Taking $x = \mathrm{e}^a$ and $y = \mathrm{e}^b$ we get

$$\frac{2}{\mathrm{e}^a + \mathrm{e}^b} < \frac{b-a}{\mathrm{e}^b - \mathrm{e}^a} < \mathrm{e}^{-\left(\frac{a+b}{2}\right)} \tag{10}$$

and from (6′) we conclude

$$\frac{2\mathrm{e}^{\frac{a+b}{2}}}{\mathrm{e}^a + \mathrm{e}^b}\left(\frac{a+b}{2}\right)^r \leqslant [L_r^e(a,b)]^r \leqslant \frac{\mathrm{e}^{\frac{a-b}{2}} a^r + \mathrm{e}^{\frac{b-a}{2}} b^r}{2} < \frac{a^r + b^r}{2}\mathrm{e}^{\frac{|b-a|}{2}}.$$

We get (6) if $r > 0$ but the reverse inequalities for $r < 0$. We also have the reverse inequalities in (6′) if f_r is concave and using (10) we deduce

$$\frac{a^r \mathrm{e}^a + b^r \mathrm{e}^b}{\mathrm{e}^a + \mathrm{e}^b} \leqslant [L_r^e(a,b)]^r \leqslant [A(a,b)]^r\,,$$

which gives (7).

For $x_0 < 1 \leqslant a < b$, f_0 is convex on $[a,b]$, thus

$$\begin{aligned} \frac{b-a}{\mathrm{e}^b - \mathrm{e}^a}\mathrm{e}^{\frac{a+b}{2}} \log\left(\frac{a+b}{2}\right) &\leqslant \int_a^b \mathrm{e}^x \log x \,\mathrm{d}x \Big/ \int_a^b \mathrm{e}^x \,\mathrm{d}x \\ &\leqslant \frac{b-a}{2}\,\frac{\mathrm{e}^a \log a + \mathrm{e}^b \log b}{\mathrm{e}^b - \mathrm{e}^a} \end{aligned} \tag{8'}$$

and using (10) we get

$$\mathrm{e}^{\frac{a-b}{2}}\log\left(\frac{a+b}{2}\right)\leqslant\log I^{e}\left(a,b\right)\leqslant\mathrm{e}^{\frac{b-a}{2}}\frac{\log a+\log b}{2}.$$

Consequently,

$$\left(\frac{a+b}{2}\right)^{\exp((a-b)/2)}\leqslant I^{e}\left(a,b\right)\leqslant\sqrt{(ab)^{\exp((b-a)/2)}},$$

which gives (8). If $0<a<b<x_0$, the inequalities ($8'$) are reversed and using (10) we get (9). □

Consequence 2. *We have the estimates*

$$A(a,b)<E(a,b)<A(a,b)\mathrm{e}^{\frac{|b-a|}{2}}.$$

Indeed, the first inequality was proved in [8] while the second is deduced from (6) for $r=1$. We remark that the first inequality is better than the first inequality of (6) for $r=1$.

5. Applications of Cauchy's mean value Theorem

Integrating by parts we have

$$\int\limits_a^b \mathrm{e}^x\log x\,\mathrm{d}x=\mathrm{e}^b\log b-\mathrm{e}^a\log a-\int\limits_a^b\frac{\mathrm{e}^x}{x}\,\mathrm{d}x$$

and

$$\int\limits_a^b x^r\mathrm{e}^x\,\mathrm{d}x=b^r\mathrm{e}^b-a^r\mathrm{e}^a-r\int\limits_a^b x^{r-1}\mathrm{e}^x\,\mathrm{d}x,\ r\neq0,$$

thus

$$\log I^e=\frac{\mathrm{e}^b\log b-\mathrm{e}^a\log a}{\mathrm{e}^b-\mathrm{e}^a}-\frac{1}{L^e}$$

and

$$\left[L_r^e\right]^r=\frac{b^r\mathrm{e}^b-a^r\mathrm{e}^a}{\mathrm{e}^b-\mathrm{e}^a}-r\left[L_{r-1}^e\right]^{r-1}.$$

Cauchy's theorem gives $c,d\in(a,b)$ such that

$$\frac{\mathrm{e}^b\log b-\mathrm{e}^a\log a}{\mathrm{e}^b-\mathrm{e}^a}=k\left(c\right)$$

and

$$\frac{b^r e^b - a^r e^a}{e^b - e^a} = h\,(d)$$

where

$$k(c) = \log c + 1/c$$

and

$$h(d) = d^{r-1}(r + d).$$

Since

$$k'(c) = (c-1)/c^2, \quad h'(d) = d^{r-2} r(r + d - 1)$$

we get the following results:

Theorem 3. *If $0 < a < b$ we have*

$$1 \leqslant \log a + 1/a \leqslant \log I^e + 1/L^e \leqslant \log b + 1/b, \quad \text{if } a \geqslant 1 \tag{11}$$

and

$$1 \leqslant \log b + 1/b \leqslant \log I^e + 1/L^e \leqslant \log a + 1/a, \quad \text{if } b \leqslant 1. \tag{12}$$

Theorem 4. *If $a < b$, then*

$$a^{r-1}(r + a) \leqslant [L_r^e]^r - r\left[L_{r-1}^e\right]^{r-1} \leqslant b^{r-1}\,(r + b)\,, \text{ if } r > 1$$

and

$$b^{r-1}(b + r) \leqslant [L_r^e]^r - r\left[L_{r-1}^e\right]^{r-1} \leqslant a^{r-1}(a + r), \text{ if } r < 0 \text{ and } a \geqslant 1 - r.$$

Remark 2. Generally, we have

$$\log a + 1/b \leqslant \log I^e + 1/L^e \leqslant \log b + 1/a,$$

which is improved by (11) and (12) for certain special values of a and b.

6. Applications of Chebyshev's inequality

The classical Chebyshev's inequality asserts that if f and g have the same monotonicity then

$$(b-a)\int_a^b f(x)\,g(x)\,\mathrm{d}x \geqslant \int_a^b f(x)\,\mathrm{d}x \int_a^b g(x)\,\mathrm{d}x. \tag{13}$$

The inequality is reversed if f and g have different monotonicities.

Theorem 5. *If f and p are monotone, then the inequality*

$$V_f^p(a,b) \geqslant V_f(a,b) \tag{14}$$

holds if p is increasing while its reverse holds if p is decreasing.

Proof. If p is increasing then we have from (13)

$$f(V_f^p(a,b)) \geqslant f(V_f(a,b))$$

if f is increasing or the reverse inequality if f is decreasing. But in both cases we get (14). The proof proceeds analogously for p decreasing.

For example, we obtain

$$A^e = E \geqslant A,$$

which was proved in a different way in [8], but also

$$I^e \geqslant I, \quad L^e \geqslant L, \quad G^e \geqslant G$$

and generally, for logarithmic means

$$L_r^e \geqslant L_r. \tag{15}$$

□

Consequence 3. *If $r > 1$, then*

$$L_r^e(a,b) > A(a,b). \tag{16}$$

Indeed, in this case (4) gives $L_r > A$, so that (15) gives (16).

We remark that (16) improves the left hand side of (6).

7. Applications of Seiffert type results

In [2] a general statement of the Hermite-Hadamard inequality for functionals is given. For the special case when the functional T is given by

$$T(f) = \int_a^b f(x)p(x)\,\mathrm{d}x \Big/ \int_a^b p(x)\,\mathrm{d}x,$$

with p a positive continuous function on $[a,b]$, and for $f\colon\ [a,b] \to [c,d]$ continuous, the result of [2] is: if h is a convex function on $[c,d]$, then

$$h\left(T\left(f\right)\right) \leqslant T\left(h\left(f\right)\right) \leqslant \left[\left(d - T\left(f\right)\right)h\left(c\right) + \left(T\left(f\right) - c\right)h\left(d\right)\right]/(d-c). \tag{17}$$

Starting from a result of H.-J. Seiffert from [7], developed also by H. Alzer [1], a general result, based on (17) is given in [10]. Taking $h = g \circ f^{-1}$ and noting that

$$T\left(f\right) = f\left(V_f^p\left(a,b\right)\right)$$

we get the following

Lemma 1. *If $f\colon\ [a,b] \to [c,d]$ and $g\colon\ [a,b] \to \mathbb{R}$ are strictly increasing continuous functions, $g \circ f^{-1}$ is convex and $p\colon\ [a,b] \to \mathbb{R}$ is positive, then*

$$\begin{aligned} V_f^p\left(a,b\right) &\leqslant V_g^p\left(a,b\right) \\ &\leqslant g^{-1}\left(\frac{g\left(a\right)\left[f\left(b\right) - f\left(V_f^p\left(a,b\right)\right)\right] + g\left(b\right)\left[f\left(V_f^p\left(a,b\right)\right) - f\left(a\right)\right]}{f\left(b\right) - f\left(a\right)}\right) \end{aligned} \tag{18}$$

Again for different combinations of monotony and/or convexity, we have (18) *or their reverses.*

We also remark that the first inequality of (18) is in fact (3).

For example, we get

Theorem 6. *The following inequality holds:*

$$\frac{L\left(a,b\right)L^e\left(a,b\right)}{G^2\left(a,b\right)} \geqslant 1 \Big/ \log\frac{eI\left(a,b\right)}{I^e\left(a,b\right)}. \tag{19}$$

P r o o f. Taking $p(x) = \mathrm{e}^x$, $f(x) = \log x$ and $g(x) = 1/x$, we have in (18) the reverse inequalities

$$I^e\left(a,b\right) \geqslant L^e\left(a,b\right) \geqslant \frac{ab\left(\log b - \log a\right)}{\log\left(b^b/a^a\right) - \left(b-a\right)\log I^e\left(a,b\right)},$$

which give (19). $\square$

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