Czechoslovak Mathematical Journal

Radu Tunaru

An extension of invertibility of Hammerstein-type operators

Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 2, 341–346

Persistent URL: `http://dml.cz/dmlcz/127491`

Czechoslovak Mathematical Journal, 49 (124) (1999), 341–346

AN EXTENSION OF INVERTIBILITY OF HAMMERSTEIN-TYPE OPERATORS

Radu Tunaru, London

(Received August 7, 1996)

Abstract. My aim is to show that some properties, proved to be true for the square matrices, are true for some not necessarily linear operators on a linear space, in particular, for Hammerstein-type operators.

If A and B are operators satisfying angleboundedness assumption (for details see [1]), or the strong monotonicity condition following [2], there are sufficient conditions for invertibility of a Hammerstein-type operator $I + AB$ as described in [3].

In this paper we shall provide some generalizations of the results from [3]. We will mark the end of a proof with Halmos' square $\square$.

Proposition 1. *Let K_1 and K_2 be two linear spaces and $A\colon K_1 \to K_2$, $B\colon K_2 \to K_1$ two operators such that the operator ABA is linear.*

1. *If $I_2 - ABAB\colon K_2 \to K_2$ has a left inverse U_l then $I_1 - BABA\colon K_1 \to K_1$ has a left inverse $V_l = I_1 + BU_lABA$.*
2. *If $I_2 - ABAB\colon K_2 \to K_2$ has a right inverse U_d then $I_1 - BABA\colon K_1 \to K_1$ has a right inverse $V_d = I_1 + BU_dABA$.*

P r o o f. 1. We know that $U_l(I_2 - ABAB) = I_2$. Then we can write

$$\begin{aligned} V_l(I_1 - BABA) &= (I_1 + BU_lABA)(I_1 - BABA) \\ &= I_1 - BABA + BU_lABA(I_1 - BABA) \\ &= I_1 - BABA + BU_l(ABA - ABABABA) \\ &= I_1 - BABA + BU_l(I_2 - ABAB)ABA \\ &= I_1 - BABA + BI_2ABA \\ &= I_1 - BABA + BABA \\ &= I_1. \end{aligned}$$

So V_l is a left inverse for $I_1 - BABA$.

2. Because $(I_2 - ABAB)U_d = I_2$ we can write

$$\begin{aligned} (I_1 - BABA)(I_1 + BU_dABA) &= I_1 + BU_dABA - BABA(I_1 + BU_dABA) \\ &= I_1 + BU_dABA - B(ABA + ABABU_dABA) \\ &= I_1 + BU_dABA - B(I_2 + ABABU_d)ABA. \end{aligned}$$

But

$$\begin{aligned} (I_2 - ABAB)U_d &= I_2, \\ U_d - ABABU_d &= I_2, \\ I_2 + ABABU_d &= U_d. \end{aligned}$$

Therefore $(I_1 - BABA)V_d = I_1 + BU_dABA - BU_dABA = I_1$ so we can say that V_d is a right inverse of $I_1 - BABA$. □

Observation 1. For the above conditions we get that

$$ABAV_d = U_dABA.$$

P r o o f.

$$\begin{aligned} ABAV_d &= ABA(I_1 + BU_dABA) = ABA + ABABU_dABA \\ &= (I_2 + ABABU_d)ABA = U_dABA. \end{aligned}$$

□

Corollary 1. *Let ABA be linear and $I_2 - ABAB$ invertible. Then $I_1 - BABA$ is invertible, too.*

P r o o f. This is obvious because $U_l = U_d = U$ implies that

$$V_l = V_d = I_1 + BUABA.$$

$\square$

Remark 1. We cannot make a weaker assumption than ABA to be linear because there are operators A and B such that $I_2 - ABAB$ is invertible and $I_1 - BABA$ is not invertible.

Example. $A\colon \mathbb{R} \to \mathbb{R}$, $Ax = \log\sqrt{x}$ if $x > 0$ and $Ax = 0$ otherwise, and $B : \mathbb{R} \to \mathbb{R}$, $Bx = \exp\frac{2x}{3}$. Then $(I - ABAB)x = \frac{8}{9}x$ which is invertible. At the same time $(I - BABA)x = x - \sqrt[9]{x}$ for $x > 0$ and $(I - BABA)x = x - 1$ for $x \leqslant 0$. As can be easily seen this is not invertible.

Proposition 2. *Let K be a linear complex space and let operators $A, B\colon K \to K$ be given, A being linear. If there is any nonzero complex number λ such that*

$$\lambda AB + A + B = 0$$

then $\lambda A + I$ has a right inverse and $\lambda B + I$ has a left inverse.

P r o o f.

$$\lambda AB + A + B = 0, \quad \lambda \neq 0,$$
$$\lambda^2 AB + \lambda A + \lambda B = 0.$$

Adding the identity operator to both sides we get

$$\lambda^2 AB + \lambda A + \lambda B + I = I,$$
$$\lambda(\lambda AB + B) + \lambda A + I = I,$$
$$\lambda(\lambda A + I)B + \lambda A + I = I,$$
$$(\lambda A + I)(\lambda B + I) = I.$$

Hence $(\lambda A + I)$ is a left inverse for $(\lambda B + I)$ and $(\lambda B + I)$ is a right inverse for $(\lambda A + I)$. $\square$

Corollary 2. *Let $A, B\colon K \to K$ be operators and let A be linear. Let λ be a nonzero complex number such that $\lambda AB + A + B = 0$. Then*

1. *If $\lambda A + I$ is left invertible then $\lambda A + I$ is invertible and its inverse is $\lambda B + I$.*
2. *If $\lambda B + I$ is right invertible then $\lambda B + I$ is invertible and its inverse is $\lambda A + I$.*

Observation 2. There are examples when A, B are not linear and there is a nonzero complex number λ for which $(\lambda A + I)(\lambda B + I) = I$.

Let $Ax = \frac{\tan x - x}{\lambda}$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $Ax = 0$ otherwise.

Let $Bx = \frac{\arctan x - x}{\lambda}$ for every $x \in \mathbb{R}$.

Then $(\lambda A + I)(\lambda B + I)x = x$ and it is obvious that A, B are not linear.

Proposition 3. *Let K_1, K_2 be two complex linear spaces and let $A\colon K_1 \to K_2$ be linear. Consider $B\colon K_2 \to K_1$ and m a positive integer.*

1. *If $I_2 + (AB)^m$ has a left inverse U_l then $I_1 + (BA)^m$ has a left inverse V_l, where $V_l = I_1 - B(AB)^{m-1}U_lA$.*
2. *If $I_2 + (AB)^m$ has a right inverse U_d then $I_1 + (BA)^m$ has a right inverse V_d, where $V_d = I_1 - B(AB)^{m-1}U_dA$.*

Moreover, $AV_d = U_dA$.

P r o o f. For $m = 1$ this result was proved in [3].

1. For $m > 1$ we observe that $I_2 + (AB)^m = I_2 + AC$, where $C = B(AB)^{m-1}$. Therefore

$$\begin{aligned} &I_1 + (BA)^m = I_1 + CA, \\ &U_l[I_2 + (AB)^m] = I_2, \\ &U_l[I_2 + AC] = I_2. \end{aligned}$$

We know that the result is true for $m = 1$, hence there is $V_l = I_1 - CU_lA$ which is a left inverse for $I_1 + CA$. This means that $V_l = I_1 - B(AB)^{m-1}U_lA$ is a left inverse for $I_1 + (BA)^m$.

2. As above $I_1 + CA$ has a right inverse $V_d = I_1 - CU_dA$. So $I_1 + (BA)^m$ has a right inverse $V_d = I_1 - B(AB)^{m-1}U_dA$.

$$\begin{aligned} AV_d &= A\left[I_1 - B(AB)^{m-1}U_dA\right] \\ &= A - (AB)^m U_dA \\ &= \left[I_2 - (AB)^m U_d\right] A. \end{aligned}$$

However, $I_2 - (AB)^m U_d = U_d$ from where we conclude that $AV_d = U_dA$. □

Corollary 3. *Let m be a positive integer. If $I_2 + (AB)^m$ is invertible then $I_1 + (BA)^m$ is invertible.*

P r o o f. Let U be the inverse of $I_2 + (AB)^m$. Then for C as above

$$[I_1 + (BA)^m]^{-1} = I_1 - CUA = I_1 - B(AB)^{m-1}UA.$$

□

Proposition 4. *Let K_1, K_2 be two complex linear spaces and m a nonnegative integer. Let $A\colon K_1 \to K_2$, $B\colon K_2 \to K_1$ be two operators such that the operator $B(AB)^{m-1}$ is invertible and linear. Then*

1. *If $I_2 + (AB)^m$ has U_d as a right inverse then $I_1 + (BA)^m$ has V_d as a right inverse, where $V_d = B(AB)^{m-1}U_d \left[B(AB)^{m-1}\right]^{-1}$;*
2. *If $I_2 + (AB)^m$ has U_l as a left inverse then $I_1 + (BA)^m$ has V_l as a left inverse, where $V_l = B(AB)^{m-1}U_l \left[B(AB)^{m-1}\right]^{-1}$.*

Proof. Let $C = B(AB)^{m-1}$, $C\colon K_2 \to K_1$, which is linear and invertible.

1. We know that $[I_2 + (AB)^m]\, U_d = I_2$. Hence we can write

$$\begin{aligned}[I_1 + (BA)^m]\, V_d &= [I_1 + (BA)^m]\left(CU_dC^{-1}\right)\\ &= CU_dC^{-1} + (BA)^m CU_dC^{-1}\\ &= [C + (BA)^m C]\, U_dC^{-1}\\ &= C\,[I_2 + (AB)^m]\, U_dC^{-1}\\ &= CI_2C^{-1} = CC^{-1} = I_1,\end{aligned}$$

2.

$$\begin{aligned}V_l\,[I_1 + (BA)^m] &= B(AB)^{m-1}U_l\left[B(AB)^{m-1}\right]^{-1}[I_1 + (BA)^m]\\ &= CU_lC^{-1}\,[I_1 + (BA)^m]\\ &= CU_l\left[C^{-1} + C^{-1}(BA)^m\right]\\ &= CU_l\,[I_2 + (AB)^m]\, C^{-1}.\end{aligned}$$

However, we know that $U_l\,[I_2 + (AB)^m] = I_2$. Therefore we obtain that

$$V_l\,[I_1 + (BA)^m] = CC^{-1} = I_1.$$

□

Corollary 4. *Let A, B be two operators as in Proposition 4. If $I_2 + (AB)^m$ is invertible then $I_1 + (BA)^m$ is invertible and*

$$[I_1 + (BA)^m]^{-1} = B(AB)^{m-1}\,[I_2 + (AB)^m]^{-1}\left[B(AB)^{m-1}\right]^{-1}.$$

Observation 3. For $m = 1$ we get the principal results from [3]. Let K_1 and K_2 be Banach spaces such that the hypotheses of the last corollary are satisfied and $B(AB)^{m-1}$ is bounded. Then using the Open Map Theorem we can say that $[I_1 + (BA)^m]^{-1}$ is (Lipschitz) continuous if $[I_2 + (AB)^m]^{-1}$ is (Lipschitz) continuous.

References

[1] *Brezis, H. and Browder, F.F.*: Nonlinear integral equations and systems of Hammerstein-type. Advances in Math. *18* (1975), 115–147.

[2] *Doležal, V.*: Monotone Operators and Applications in Control and Network Theory. Elsevier, 1979.

[3] *Doležal, V.*: A remark on inversability of Hammerstein type operators. Applied Mathematical Letters *5* (1992), 29–31.

Author's address: Middlesex University, School of Mathematics and Statistics, The Burroughs, London, NW4 4BT, U.K., email: `r.tunaru@mdx.ac.uk`.