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Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 2, 401–414

Persistent URL: `http://dml.cz/dmlcz/127496`

Czechoslovak Mathematical Journal, 49 (124) (1999), 401–414

GEOMETRIC PROPERTIES OF A SEQUENCE OF STANDARD MINIMAL IMMERSIONS BETWEEN SPHERES

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(Received September 30, 1996)

Introduction

In the theory of minimal isometric immersions of Riemannian manifolds, minimal immersions into spheres have particular interest.

Takahashi in [T]proved that if φ is an isometric immersion of a compact Riemannian manifold (M, g) of dimension n into $\mathbb{R}^{m+1}$, such that all the components of φ are eigenfunctions of the Laplace-Beltrami operator Δ on (M, g) corresponding to the *same eigenvalue* λ, then $\varphi(M)$ is contained in a sphere $S^m(r) \subset \mathbb{R}^{m+1}$ with radius $r = \sqrt{n/\lambda}$ and is minimal in $S^m(r)$. The manifolds M which admit such immersions are the irreducible Riemannian homogeneous spaces, namely those in which the isotropy group of a point acts irreducibly on the tangent space.

One then has, for these manifolds, an explicit method for building up minimal immersions into spheres: any orthonormal basis of the eigenspace V_{λ_k} of dimension m_k associated to the eigenvalue λ_k for each integer $k \geqslant 1$, such that the multiplicity of λ_k is sufficiently high in order to provide the coordinate functions of the immersion, gives rise to a minimal isometric immersion $\varphi_{n,k}$ of M into a sphere $S_1^{m_k-1} \subset V_{\lambda_k}$ and these immersions are called *standard minimal immersions*: *s.m.i.*.

In this paper we shall consider the particular homogeneous spaces $M = SO(n+1)/SO(n)$.

Our point of view is to study the geometric properties of a single *s.m.i.* $\varphi_{n,k}$ by means of the sequence of the previous maps $\varphi_{n,r}$ where n is fixed and $r < k$, basing our study on the intrinsic properties of these maps, namely that they are full, equivariant, and that [d-W]$_2$ there is a universal isomorphism between the normal bundle of degree $r-1$ of a *s.m.i.* $\varphi_{n,r}$ and the spherical harmonic of degree r on the $n-1$ unitary sphere.

This universal pattern allows us to state that the sequence of *s.m.i.* $\varphi_{n,r}$ defines a sequence of osculating immersions of order $r-1$ which we call *generalized*, and that the normal bundles to the osculating spaces to the image of the immersions are parallel (Theorem 3.3).

From [Sa], [Ts], we know that the standard, minimal immersions are helical geodesic immersions, namely immersions such that for each geodesic γ of the domain of $\varphi_{n,k}$ the curve $\varphi_{n,k}\gamma$ has constant curvatures which do not depend on γ.

In Theorem 3.7 substituting the maps $\varphi_{n,2}, \varphi_{n,3}, \dots, \varphi_{n,k}$, with equivalent maps, namely equal modulo an isometry of the ambient space, we prove that given any geodesic γ in the sphere $S^n_{c_k}$, sphere of constant sectional curvature c_k, the curve $\varphi_{n,k}\gamma \subset S_1^{m_k-1}$ has exactly $(k-1)$ curvatures which are expressed by the eigenvalues of the Laplacian respectively on the spheres $S^n_{c_2}, S^n_{c_3}, \dots, S^n_{c_k}$. This geometric point of view of considering, for the study of a *s.m.i.*, all the previous immersions like a sort of approximation has not been considered before and can be applied in the study of those spaces which admit *s.m.i.*, for instance in the case of generalized symmetric spaces.

Naturally some modifications are necessary as, in these cases, the spectrum of the Laplacian is generally unknown.

We think moreover that a possible application could be in the field of isoparametric maps. As a matter of fact for $r = 2$ a *s.m.i.* is an isoparametric immersion since the osculating space of order 1 of $S^n_{c_2}$ is the tangent space to $S_1^{m_2-1}$. This is the case of the Veronese surface. For $r > 2$ the map $\varphi_{n,r}$ can be considered at the same time as a particular and also as a generalized isoparametric map since it is the generalized osculating space of order $r-1$ of S_{c_r} which is isomorphic, not equal, to the tangent space to $S_1^{m_r-1}$.

It will be interesting to see which machinery used in the study of isoparametric maps can be used in the case of a *s.m.i.* and which results remain valid.

In n. 1 we will summarize the basic notions on higher fundamental forms, in n. 2 the notion of standard, minimal immersions and spherical harmonics, and in n. 3 we give the theorems on osculating immersions and helical immersions.

In this paper the differentiability of all geometric objects will be C^∞.

Let $(\overline{M}, g)$ be a Riemannian manifold, n and m respectively the dimensions of M and $\overline{M}$, $f\colon\ M \longrightarrow \overline{M}$ an isometric immersion and $\langle , \rangle$ the inner product.

The pull back bundle $f^{-1}(T(\overline{M}))$ of the tangent bundle $T(\overline{M})$ on M splits into the orthogonal direct sum

$$f^{-1}(T(\overline{M})) = T(M) \oplus N(M) \tag{1.1}$$

of Riemannian vector bundles, where $N(M)$ is the normal bundle to the bundle $T(M)$ tangent to M.

We denote by $(\ )^T$ and by $(\ )^N$ the tangential and orthogonal projection associated to the splitting (1.1).

Remark 1.1. In a small neighborhood U of any point $p \in M$, f is a topological embedding, thus locally we can identify a vector field Y on M with its image $f \cdot Y$ defined on $f(M) \subset \overline{M}$.

Denote by $\overline{\nabla}$ the Levi Civita connection on $\overline{M}$ and by ∇ the induced connection on M via the projection on $T(M)$.

The second fundamental form $\overset{o}{\mathrm{s}}$ of f at p is defined by

$$\overset{o}{\mathrm{s}}(X_p, Y_p) = (\overline{\nabla}_{X_p} Y)^N \tag{1.2}$$

where $X_p, Y_p \in T_p(M)$ and Y is a generic extension of Y_p.

Definition 1.2. The *first normal space* N_p^1 is the linear space spanned by the second fundamental form at p and *the second osculating space* $(O_2)_p$is given by

$$(O_2)_p = T_pM \oplus N_p^1. \tag{1.3}$$

To simplify the notations, we omit sometimes indicating the points in which the fundamental forms, the normal and osculating spaces are calculated.

The higher fundamental forms are then defined inductively.

Definition 1.3. The third fundamental form $\overset{1}{\mathrm{s}}$ at p is

$$\overset{1}{\mathrm{s}}(X, Y, Z) = (\overline{\nabla}_X \overset{o}{\mathrm{s}}(Y, Z))^{{o_2}^{\perp}}. \tag{1.4}$$

The *second normal space* N_p^2 is the linear span of $\overset{1}{\mathrm{s}}$ and the *third osculating space* $(O_3)_p$ is

$$(O_3)_p = T_pM \oplus N_p^1 \oplus N_p^2. \tag{1.5}$$

If k is any positive integer, proceeding inductively, one can define the fundamental forms $\overset{r-2}{\mathrm{s}}$ at p, the normal spaces of order $r-1$ and the osculating space O_r of order r:

$$(O_r)_p = T_p(M) \oplus N_p^1 \ldots \oplus N_p^{r-1}. \tag{1.6}$$

We define $(O_1)_p = T_p(M) = N_p^o$.

We can see that that the k^{th}-osculating space of f at p is the subspace of $f^{-1}(T(\overline{M}))$ spanned by those vectors obtained by taking covariant derivative up to the $(k-1)^{\text{th}}$ order.

(For further informations on higher fundamental forms see [Sp]).

As $\dim(T_p(M) \oplus N_p^1 \ldots \oplus N_p^k) \leqslant \dim T_p(\overline{M})$ this process must end.

Remark 1.4. If the manifold M has constant curvature, the fundamental forms are symmetric. The generic fundamental form of order k at p define then a map

$$\overset{k-2}{\mathrm{s}}\colon\ S^k(T_p(M)) \longrightarrow N_p^{k-1} \tag{1.7}$$

where $S^k(T_p(M))$ is the symmetric product of k copies of $T_p(M)$.

Definition 1.5. Let q be the first integer $\geqslant 1$ such that $\dim N_p^q \neq 0$, but $\dim N_p^{q+1} = 0$. We call q the *normal degree* of the immersion in p.

In general the dimension of the normal and of the associated osculating spaces is not constant.

We call *normally regular domain* an open set $M' \subset M$ such that in any point $p \in M'$ the dimension of all the normal spaces N^r $(1 \leqslant r \leqslant q)$ is maximal.

Definition 1.6. If $f\colon\ M \longrightarrow \overline{M}$ is an isometric immersion of a homogeneous Riemannian manifold $M = G/K$ in a manifold $\overline{M}$, we say that f is *equivariant*, if there exists a continuous homomorphism ϱ from G into the group $I(\overline{M})$ of the isometries of $\overline{M}$ such that

$$f(g \cdot p) = \varrho(g) f(p) \quad \forall p \in \overline{M}, \quad g \in G \tag{1.8}$$

If the map is *equivariant* we have

$$\varrho(g) N_p^h = N_{g\cdot p}^h \quad \forall p \in M, \quad h \leqslant q \tag{1.9}$$

In this case the dimensions of the normal spaces are constants and we obtain a decomposition of the tangent bundle $f^{-1}(T(\overline{M}))$ in the Whitney sum

$$f^{-1}(T(\overline{M})) = T(M) \oplus N^1 \ldots \oplus N^q \oplus N \tag{1.10}$$

Definition 1.7. The *mean curvature* H of an isometric immersion $f\colon\ M \longrightarrow \overline{M}$ is the trace of the second fundamental form.

If $(e_1, e_2, \dots e_n)$ is a local orthonormal frame field, then

$$H = 1/n \sum_{i=1}^{n} (\overline{\nabla}_{e_i} e_i)^{N^1} = 1/n \sum_{i=1}^{n} \overset{o}{\mathrm{s}}(e_i, e_i). \tag{1.11}$$

Definition 1.8. The Weingarten operator A^k of order k is defined by

$$(\overline{\nabla}_X \xi^k)^{N^{k-1}} = -A^k(X, \xi^k)$$

for $\xi^k \in \Gamma N^k$, section of N^k.

In the sequel the following generalized Frenet formula [Sp] will be used:

$$\overline{\nabla}_X \xi^k = -A^k(X, \xi^k) + (\overline{\nabla}_X \xi^k)^{N^k} + \overset{k}{\mathrm{s}}(X, \xi^k). \tag{1.12}$$

2. Standard immersions and spherical harmonics

Let $M = G/K$ be a compact homogeneous Riemannian manifold with metric g, and assume that the linear isotropy group acts irreducibly on the tangent space.

If $\lambda \neq 0$ is a real number, we shall denote by V_λ the set of functions solution of the Laplace-Beltrami equations:

$$\Delta f + \lambda f = 0. \tag{2.1}$$

Since M is compact, each V_λ is a finite dimensional vector space.

Considering that G is a transitive group of isometries of M, to each element $g \in G$ we can associate an operator L_g on V_λ which transforms the functions $f(p) \in V_\lambda$ where $p \in M$ under the rule

$$L_g f(p) = (g \cdot f)(p) = f(g^{-1} \cdot p). \tag{2.2}$$

L_g defines a *representation of* G *in* V_λ.

Moreover V_λ can be endowed with the inner product

$$(f, g) = \int_M f \cdot g \, \mathrm{d}v. \tag{2.3}$$

For convenience we shall normalize it in such a way that the integral over M of the canonical measure $\mathrm{d}v$ is the dimension m_λ of V_λ.

An orthonormal basis $(f_1, f_2, \ldots, f_{m_\lambda})$ of V_λ, defines a map: $\varphi\colon\ M \longrightarrow \mathbb{R}^{m_\lambda}$ by $\varphi(p) = (f_1(p), f_2(p), \ldots, f_{m_\lambda}(p))$ with $p \in M$.

As $\Delta\varphi = (\Delta f_1, \Delta f_2, \ldots, \Delta f_{m_\lambda})$, we see that φ is a solution of (2.1).

Due to the normalization, $\sum\limits_i (f_i)^2 = 1.$ for all $p \in M$. It turns out that φ defines a map of M into $\varphi(M) \subset S_1^{m_\lambda - 1} \in \mathbb{R}^{m_\lambda}$. The identification of V_λ with $\mathbb{R}^{m_\lambda}$ is possible after the choice of the orthonormal basis.

Remark 2.1. It can be proved that the action of G on M defined by (2.2) leaves the eigenspace V_λ invariant and is an isometry for the inner product (2.3).

If now we change the metric g on M with the metric $\widetilde{g} = \sum\limits_{i=1}^{m_\lambda} (\,\mathrm{d}f_i)^2$ induced on M by the Euclidean metric on $\mathbb{R}^{m_\lambda}$, the immersion $\varphi\colon\ (M, \widetilde{g})$ into $S^{m_\lambda - 1}$ becomes an isometry. Since both metrics g and $\widetilde{g}$ are G-invariant,and hence in p they are invariant under the irreducible action of the isotropy group K, the metric g is a multiple of the metric $\widetilde{g}$,namely

$$\widetilde{g} = cg \tag{2.4}$$

for $c > 0$ as the functions f_i are not constants.

This relation is true in any point of M since G acts transitively and isometrically on M

Denoting by $\widetilde{M}$ the manifold M with the metric $\widetilde{g}$, the Laplacian of $\widetilde{M}$ is given by $\widetilde{\Delta} = 1/c\Delta$. Thus $\varphi\colon\ \widetilde{M} \longrightarrow S_1^{m_\lambda - 1}$ becomes an isometric immersion satisfying $\widetilde{\Delta}\varphi + \widetilde{\lambda}\varphi = 0$ where $\widetilde{\lambda} = \lambda/c$.

From Takahashi's theorem it follows that φ is a minimal immersion into a sphere of radius $r = \sqrt{n/\widetilde{\lambda}}$. As here r=1 we obtain $c = \lambda/n$ and this determines $\widetilde{g}$.

This immersion is called *standard, minimal immersion* (s.m.i) of degree k if k is the k-th non zero eigenvalue associated to the immersion.

Definition 2.2. Two minimal standard immersions are called *equivalent or congruent* if they differ by an isometry of the ambient space. Note that a different choice of the orthonormal basis of V_λ gives rise to an equivalent immersion.

Let us now consider a particular homogeneous space, namely $M = SO(n+1)/SO(n)$. M can be realized as a sphere S^n of the Euclidean space $\mathbb{R}^{n+1}$ with a metric g of constant curvature 1. In that case the spectrum and the eigenfunction of M are known.

The eigenspace V_{λ_k} of the Laplacian on (S_1^n, g) associated to the eigenvalue λ_k, for each $k \in \mathbb{Z}_+$, are the restrictions to S_1^n of the homogeneous polynomials P_n^k of degree k defined on $\mathbb{R}^{n+1}$ which satisfy on S_1^n the equation $\Delta P = 0$. Such restrictions are called *spherical harmonics of* S_1^n.

It is proved that all the harmonic homogeneous polynomials of degree k restrict to S_1^n are eigenfunctions of Δ with the same eigenvalue.

The value of the eigenvalues are the following:

$$\lambda_k = k(k+n-1)$$

and the dimension of the associated eigenspaces V_{λ_k} are

$$m_k = (n+2k-1)\frac{(n+k-2)!}{k!(n-1)!}. \tag{2.5}$$

From the general considerations that we have seen above, it follows that an orthonormal basis of the vector space V_{λ_k} consisting of the spherical harmonics of S^n of order k gives a *standard minimal isometric immersion of order k*. We shall denote it $\varphi_{n,k}$:

$$\varphi_{n,k}\colon\ S^n_{c_k} \longrightarrow S_1^{m_k-1} \in \mathbb{R}^{m_k} \tag{2.6}$$

where $S^n_{c_k}$ is the n-sphere with constant sectional curvature c_k. The curvature c_k is defined by the fact that the metric $\tilde{g}$ in $S^n_{c_k}$ is $(\lambda_k/n)g$.

We obtain then:

$$c_k = \frac{n}{k(k+n-1)}. \tag{2.7}$$

For odd k the standard minimal isometric immersion is a minimal isometric embedding of $S^n_{c_k}$ into $S_1^{m_k-1}$.

For even k all the components of the immersion are invariant under the antipodal map, and we get a minimal isometric embedding of RP^n into S^{m_k-1}

Remark 2.3. The standard minimal isometric immersions between spheres

$$\varphi_{n,k}\colon\ S^n_{c_k} \longrightarrow S_1^{m_k-1} \in \mathbb{R}^{m_k}$$

have some nice properties:

1) they are *full*; namely $\varphi_{n,k}(S^n_{c_k})$ is not contained in a proper vector subspace of $\mathbb{R}^{m_k}$ or equivalently, in a totally geodesic submanifold of $S_1^{m_k-1}$

2) they are *equivariant* (see definition 1.6). we have then a decomposition (1.10) for the tangent bundle in normal spaces

Let $\varphi_{n,k}\colon S^n_{c_k} \longrightarrow S^{m_k-1}_1$ be a standard minimal immersion. Fixing an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space to the sphere $S^n_{c_k}$ in a point p, we have an isomorphism between $T_p(S^n_{c_k})$ and $\mathbb{R}^n$. This isomorphism extends to an isomorphism from the symmetric product $S^h(T_p(S^n_{c_h}))$ of h-copies of the tangent space of $S^n_{c_h}$ to the vector space P^h_n of the homogeneous polynomials of degree h on $\mathbb{R}^n$.

We recall two propositions of do-Carmo-Wallach that will be useful in the following.

Proposition 3.1. *Let $\varphi_{n,k}$ be a standard, minimal immersion then* $\ker \overset{k}{\mathrm{s}} \supseteq r^2 \cdot P^k_n$ *where $r^2 = \sum\limits_{i=1}^{n} e_i^2 \in P^2_n$ is the distance function from the origin of $\mathbb{R}^n$ and $\overset{k}{\mathrm{s}}$ is the fundamental form of order $k+2$. [d-W]$_2$.*

P r o o f. The proposition is proved by induction on k. For $k=0$ the minimality condition gives $\overset{o}{\mathrm{s}}(r^2)=0$.

Suppose now $\overset{k-1}{\mathrm{s}}(r^2.S^{k-1}(T_p(S^n_{c_k}))=0$. As

$$\overset{k}{\mathrm{s}}(Y, X_1, \ldots, X_{n+1}) = (\nabla_Y \overset{k-1}{\mathrm{s}}(X_1, X_2, \ldots, X_{k+1}))^{N^{k+1}}$$

from the inductive hypothesis we conclude that for every $t \in S^k(T_p(S^n_{c_k}))$

$$\overset{k}{\mathrm{s}}(r^2.t) = 0, \quad \text{i.e.} \quad \overset{k}{\mathrm{s}}(r^2.P^k_n) = 0 \tag{3.1}$$

namely $\ker \overset{k}{\mathrm{s}} \supseteq r^2 \cdot P^k_n$. □

Proposition 3.2. *Let $\varphi_{n,k}$ be a standard minimal immersion. Then the vector normal space N^{h-1} of order $h-1$ is isomorphic to the vector space H^h_n of the spherical harmonics of order h on S^{n-1}_1, for every $h \leqslant k$* [d-W]$_2$.

P r o o f. We recall [B] that the space P^h_n of the homogeneous polynomials of degree h of $\mathbb{R}^n$ admits the following decomposition in the direct sum of the subspace H^h_n of the harmonic polynomials homogeneous of degree h in n variables and of the polynomials of the form $r^2P^{h-2}_n$:

$$P^h_n = H^h_n \oplus r^2 \cdot P^{h-2}_n. \tag{3.2}$$

Using the isomorphism $S^h(T_p(S^n_{c_h})) \cong P^h_n$, from the epimorphism (cf. 1.7)

$$\overset{h-2}{\mathrm{s}}\colon\ S^h(T_p(S^n_{c_h})) \longrightarrow N^{h-1}_p \tag{3.3}$$

and from Proposition 3.1 and the decomposition (3.2) of P^h_n, we obtain the injective map:

$$\overset{h-2}{\mathrm{s}}\colon\ S^h(T_p(S^n_{c_h})\longrightarrow H^h_n$$

and thus the bijective map

$$\overset{h-2}{\mathrm{s}}\colon\ H^h_n\longrightarrow N^{h-1} \tag{3.4}$$

of vector spaces. Then the isomorphism between the set of the spherical harmonics of order h on S^{n-1}_1 and N^{h-1} for every $h\leqslant k$ follows. □

Theorem 3.3. *The sequence $\varphi_{n,2},\varphi_{n,3},\ldots,\varphi_{n,k}$ of minimal standard immersions between spheres, with $k>1$, $n>1$, defines a sequence of generalized osculating immersions of order $2,3,\ldots,k$. Moreover the normal bundles $N^1,N^2,\ldots,N^{k-1}$ to the osculating spaces $O_1,O_2,\ldots,O_{k-1}$ of the immersions $\varphi_{n,2},\varphi_{n,3},\ldots,\varphi_{n,k}$ are parallel.*

Proof. We exclude the case $k=1$ as for $k=1$ the map $\varphi_{n,1}$ gives the standard immersion $S^n_1\longrightarrow S^n_1\subset\mathbb{R}^{n+1}$.

We exclude also the case $n=1$ as for any k and $n=1$ we have a map from $S^1_{c_k}$ to S^1_1.

For $k=2$, the map $\varphi_{n,2}\colon\ S^n_{c_2}\longrightarrow S^{m_2-1}_1$, considering that the immersion is *full*, gives

$$T(S^{m_2-1}_1)=N^o\oplus N^1=T(S^n_{c_2})\oplus N^1=O_1\oplus N^1 \tag{3.5}$$

with $\dim N^1=m_2-n-1$, equal to the dimension of the space of the harmonic polynomials of degree two on $\mathbb{R}^n$ restricted to S^{n-1}_1. The decomposition (3.5) is the Whitney sum of the tangent bundle $\varphi^{-1}(T(S^{m_2-1}_1))$, since the standard maps are equivariant (cf. 1.4). We semplify the notations omitting φ^{-1}.

For $k=3$ we obtain, considering (3.5),

$$T(S^{m_3-1}_1)\cong T(S^{m_2-1}_1)\oplus N^2\cong N^o\oplus N^1\oplus N^2\cong O_2\oplus N^2 \tag{3.6}$$

where we have called N^2 the supplementary of $S^{m_2-1}_1$ in $S^{m_3-1}_1$ and where the N^1, which appears in (3.5), is isomorphic to the space N^1 which appears in (3.6) as both for Prop. 3.2 are isomorphic to H^2_n, spherical harmonic of order 2 on the sphere S^{n-1}_1. With O_3 is denoted the generalized osculating bundle of order 3 on $S^n_{c_3}$.

By recurrence, for the pull-back bundle of $T(S^{m_k-1}_1)$, we obtain the following decomposition in “generalized normal bundles”

$$T(S^{m_k-1}_1)=N^0\oplus N^1\oplus\ldots\oplus N^{k-1}. \tag{3.7}$$

We call $O_s = N^o \oplus N^1 \oplus \dots N^{s-1}$ *generalized osculating bundle of order* s. Here the adjective generalized is used to point out that the normal spaces are associated to *different maps* between spheres having the radius of the sphere domain varying homothetically with r ($r \leqslant k-1$) and the dimension of the sphere image varying as well at every step.

The isomorphism between these different normal spaces relative to different maps is possible because the normal spaces $N^1, N^2, \dots, N^{k-1}$ are isomorphic to the spherical harmonics respectively of order $2, 3, \dots, k$ on the sphere of dimension $n-1$.

We can therefore state that the sequence of the maps $\varphi_{n,2}, \varphi_{n,3}, \dots, \varphi_{n,k}$ defines a sequence of *generalized osculating immersions*, namely the map $\varphi_{n,k}$ defines a *generalized osculating immersion of order* k of $S^n_{c_k}$ in a sphere of radius one contained in the eigenspace V_{λ_k}.

Moreover the immersions $j_r\colon\ S_1^{m_r-1} \longrightarrow S_1^{m_{r+1}-1}$ with $r = 2, 3, \dots, k-1$, are totally geodesic since they are inclusions of the unitary euclidean sphere S^{m_r-1} into the unitary euclidean sphere $S^{m_{r+1}-1}$ induced by the inclusions $i_r\colon\ \mathbb{R}^{m_r} \longrightarrow \mathbb{R}^{m_{r+1}}$.

The normal bundles of the immersions j_r consist of those tangent vectors to $S^{m_{r+1}-1}$ supplementary and orthogonal to the tangent vectors to S^{m_r-1}.

We deduce that

$$\overline{\nabla}_X \xi^r = 0 \tag{3.8}$$

for any $X \in \varphi_{n,r}(S^n_{c_r})$, $\xi^r \in \Gamma N^r$, section of the bundle N^r normal to the osculating space O_r, and $\overline{\nabla}$ covariant derivative in $S_1^{m_r-1}$.

The normal bundles are then parallel. □

Corollary 3.4. *For any $k \in \mathbb{Z}_+$, the dimension of the spherical harmonic of order k on S_1^{n-1} is given by the difference between the dimensions of the eigenspaces V_{λ_k} and $V_{\lambda_{k-1}}$ of the Laplacian on the unitary n-sphere.*

Definition 3.5. We recall [Sa], that an isometric immersion $f\colon\ M \longrightarrow \overline{M}$ of a connected complete Riemannian manifold M into a Riemannian manifold $\overline{M}$ is called *helical geodesic* immersion of order s if, for each geodesic γ of M, the curve $f \cdot \gamma$ has constant curvatures $k_1, k_2, \dots, k_s$ which do not depend on γ.

It is known that strongly harmonic manifolds admit a helical geodesic minimal immersion into a sphere [Be], and standard immersions between spheres admit such immersions [Ts].

On the subject of helical geodesic immersions of Riemannian manifolds there are interesting papers of [Sa] and of [Ts]. Our approach of considering a standard map as approximated by the previous standard maps seems to show some new aspects as it gives, in the case of *s.m.i.* between spheres, the values of the various curvatures expressing them by means of eigenvalues of the Laplacian and the dimension n.

Remark 3.6. It will be convenient to indicate some notations that will be used in the following theorem.

We will indicate $h_{r,s}$ the homothety

$$h_{r,s}\colon\ S^n_{c_r} \longrightarrow S^n_{c_s}, \tag{3.9}$$

by $j_{r,s}$ the totally geodesic immersion

$$j_{r,s}\colon\ S_1^{m_r-1} \longrightarrow S_1^{m_s-1}, m_r < m_s, \tag{3.10}$$

by i_r the canonical immersion

$$i_r\colon\ S^n_{c_r} \longrightarrow \mathbb{R}^{n+1}, \tag{3.11}$$

and by $\overline{\nabla}$ the covariant derivative in the ambient space.

Theorem 3.7. *Let $\varphi_{n,k}$ be a s.m.i. of order k, if γ is any geodesic of $S^n_{c_k}$ then, in the equivalence class of the minimal, standard immersions $\varphi_{n,r}$ with $r \leqslant k$, the principal curvatures of the curve $\varphi_{n,k}\gamma$ are $\sqrt{n/\lambda_2}, \sqrt{n/\lambda_3}, \dots, \sqrt{n/\lambda_k}$ with $\lambda_2, \lambda_3, \dots, \lambda_k$ eigenvalues of the Laplacian respectively on the spheres $S^n_{c_2}, S^n_{c_3}, \dots, S^n_{c_k}$.*

P r o o f. We start by considering the geodesic γ' obtained from the geodesic γ by the homothety $h_{k,2}\colon\ S^n_{c_k} \longrightarrow S^n_{c_2}$. Let $\gamma'\colon\ I \longrightarrow S^n_{c_2} \subset \mathbb{R}^{n+1}$ be parametrized by the arc length t with $\gamma(o) = p$ point of $S^n_{c_2}$ and let X be the unitary tangent vector to γ' in p.

The value of the second fundamental form of the immersion i_2 on the couple of vectors (X,X) is

$$\overset{o}{\mathrm{s}}_{i_2}(X,X) = \overline{\nabla}_X X - \nabla_X X = \sqrt{c_2}\cdot \xi^1 \tag{3.12}$$

as the radius of the osculating circle to γ' is the radius of the sphere in which the curve lies and where ξ^1 is the first unitary principal normal to γ'.

As $\varphi_{n,2}$ is *full*, $\varphi_{n,2}(S^n_{c_2})$ cannot be contained in a subspace of $\mathbb{R}^{m_2}$. It turns out then that ξ^1 must belong to N^1.

The value of the second fundamental form of the immersion $\varphi_{n,2}$ on the same couple of vectors is:

$$\overset{o}{\mathrm{s}}_{\varphi_{n,2}}(X,X) = (\overline{\nabla}_X X)^{N^1} = \sqrt{c_2}\cdot \eta^1 \tag{3.13}$$

with $\eta^1 \in \Gamma N^1$. Moreover as $N^1 \in T_p(S_1^{m_2-1})$, with an isometry in S^{m_2-1} we obtain a map $\widetilde{\varphi}_{n,2}$ equivalent to $\varphi_{n,2}$ satisfying the equality:

$$\overset{o}{\mathrm{s}}_{\widetilde{\varphi}_{n,2}}(X,X) = \sqrt{c_2}\cdot\xi^1. \tag{3.14}$$

From the factorization of $\varphi_{n,k}$ in the product of an homothety and a totally geodesic immersion,namely $\varphi_{n,k} = j_{2,k}\cdot\varphi_{n,2}\cdot h_{k,2}$ and considering that the fundamental forms of an homothety and of a totally geodesic immersions are zero, from the formula of the fundamental forms of a product of two maps (cf. [E.S], [E.R.] we obtain

$$\overset{o}{\mathrm{s}}_{\widetilde{\varphi}_{n,k}}(X,X) = \sqrt{c_2}\cdot\xi^1. \tag{3.15}$$

To get information on the second curvature of $\varphi_{n,k}\gamma$, we need to consider $\varphi_{n,3}$, osculating map of third order.

By the homothety $h_{2,3}\colon S^n_{c_2} \longrightarrow S^n_{c_3}$ we obtain the geodesic $h_{2,3}\gamma' \subset S^n_{c_3} \subset \mathbb{R}^{n+1}$.

The value of the third fundamental form for the immersion i_3, gives

$$\overset{1}{\mathrm{s}}_{i_2}(X,\xi^1) = \sqrt{c_3}\cdot\xi^2 \tag{3.16}$$

with $\|\xi^2\| = 1$, and, since $\varphi_{n,3}$ is full, ξ^2 belongs to N^2.

By the generalized Frenet formula (cf. 1.12) for the immersion $\varphi_{n,3}$ we obtain

$$\overline{\nabla}_X\xi^1 = -A^1(X,\xi^1) \oplus (\overline{\nabla}_X\xi^1)^{N^1} \oplus \overset{1}{\mathrm{s}}_{\varphi_{n,3}}(X,\xi^1) \tag{3.17}$$

where from the (3.8) of theorem (3.3) is $(\overline{\nabla}_X\xi^1)^{N^1} = 0$.

Moreover $\left\langle A^1(X,\xi^1),X\right\rangle = \left\langle \overset{0}{\mathrm{s}}_{\varphi_{n,2}}(X,X),\xi^1\right\rangle = \sqrt{c_2}$.

Considering that $\varphi_{n,3}$ is full, for the same reasons valid in the case $k=2$, with an isometry of $S_1^{m_3-1}$ we can find an equivalent standard map such that

$$\overset{1}{\mathrm{s}}_{\widetilde{\varphi}_{n,3}}(X,\xi^1) = \sqrt{c_3}\cdot\xi^2. \tag{3.18}$$

As $\varphi_{n,k} = j_{3,k}\cdot\varphi_{n,3}\cdot h_{k,3}$, we see that

$$\overset{1}{\mathrm{s}}_{\widetilde{\varphi}_{n,k}}(X,\xi^1) = \sqrt{c_3}\cdot\xi^2.$$

Moreover (3.17) gives

$$\overline{\nabla}_X\xi^1 = -\sqrt{c_2}.X + \sqrt{c_3}\cdot\xi^2 \tag{3.19}$$

where $c_i = n/\lambda_i$ $(i=2,3)$ and λ_i is an eigenvalue of Δ on S_{c_i}.

By recurrence, we obtain, for the image of γ in $S^n_{c_k}$, the $k-1$ equations:

$$\overline{\nabla}_X \xi^{k-2} = -\sqrt{c_{k-1}} \cdot \xi^{k-3} + \sqrt{c_k} \cdot \xi^{k-1}. \tag{3.21}$$

Taking into account that the immersions j_r are totally geodesic, the covariant derivative which appears in the above $k-1$ equations (3.19), (3.20) can all be evaluated in $S_1^{m_k-1}$.

We conclude that the map $\varphi_{n,k}$ is an helical geodesic minimal immersion and the values of the principal curvatures of the curve image of any geodesic γ of $S^n_{c_k}$ by $\varphi_{n,k}$ are $\sqrt{n/\lambda_2}$, $\sqrt{n/\lambda_3}, \dots, \sqrt{n/\lambda_k}$ with $\lambda_2, \lambda_3, \dots, \lambda_k$ eigenvalues of the Laplacian respectively on the spheres $S^n_{c_2}, S^n_{c_3} \dots S^n_{c_k}$.

The equations (3.21) for $k = 2, 3, \dots, k$ give the Frenet equations for any geodesic of $S^n_{c_k}$ with respect to the Frenet frames $R(t) = (X(t), \xi^1(t), \dots, \xi^k(t))$ $\square$

Corollary 3.8. *Given any geodesic of $S^n_{c_k}$ we can always find in the equivalence class of the minimal standard immersions $\varphi_{n,r}$, $(r = 2, 3, \dots, k)$ a s.m.i. $\widetilde{\varphi}_{n,r}$ such that the following equalities are verified*

$$\overset{o}{\mathrm{s}}_{i_2}(X,X) = \overset{o}{\mathrm{s}}_{\widetilde{\varphi}_{n,2}}(X,X) = \overset{o}{\mathrm{s}}_{\widetilde{\varphi}_{n,k}}(X,X) = \sqrt{c_2} \cdot \xi^1,$$
$$\overset{s-2}{\mathrm{s}}_{i_s}(X,\xi^{s-2}) = \overset{s-2}{\mathrm{s}}_{\widetilde{\varphi}_{n,s}}(X,\xi^{s-2}) = \overset{s-2}{\mathrm{s}}_{\widetilde{\varphi}_{n,k}}(X,\xi^{s-2}) = \sqrt{c_s} \cdot \xi^{s-1}$$

with $s = 2, 3, \dots, k+1$.

We thank G. Romani for some useful remarks in the proof of Theorem 3.7.

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