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Nan-jing Huang

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COMPLETELY GENERALIZED NONLINEAR VARIATIONAL INCLUSIONS FOR FUZZY MAPPINGS

Nan-jing Huang, Chengdu

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Abstract. In this paper, we introduce and study a new class of completely generalized nonlinear variational inclusions for fuzzy mappings and construct some new iterative algorithms. We prove the existence of solutions for this kind of completely generalized nonlinear variational inclusions and the convergence of iterative sequences generated by the algorithms.

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1. Introduction

Variational inequalities, introduced and studied by Hartman and Stampacchia [12] in the early sixties, are very powerful tools of the current mathematical technology. These inequalities have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, operations research, nonlinear programming, economics and transportation equilibrium and engineering sciences etc. Quasivariational inequalities are generalized forms of variational inequalities in which the constraint set depends on the solution. They were introduced and studied by Bensoussan, Goursat and Lions [3] in 1973. For further details we refer to [1, 2, 4, 5, 21].

In 1991, Chang and Huang [7, 8] introduced and studied a new class of complementarity problems and variational inequalities for set-valued mappings with compact values in Hilbert spaces. In 1994, Hassouni and Moudafi [13] studied a new class of variational inclusions, which included many variational and quasivariational inequalities considered by Noor [23–25], Isac [18], Siddiqi and Ansari [28, 29] as special cases. In 1996, the author [14] introduced and studied a new class of set-valued nonlinear generalized variational inclusions, which have improved and extended many results in recent years.

On the other hand, Chang and Zhu [11] were the first to introduce and study a class of variational inequalities for fuzzy mappings in 1989. Recently, several kinds of variational inequalities and complementarity problems for fuzzy mappings were considered and studied by Chang [6], Chang and Huang [9, 10], Huang [15, 16], Noor [26] and Lee et al. [19, 20]. These papers may lead to new and significant results in these areas [27].

Inspired and motivated by the recent research papers going on in this filed, in this paper we introduce and study a new class of completely generalized nonlinear variational inclusions for fuzzy mappings which include many known classes as special cases. We also construct some new iterative algorithms, and discuss the existence of solutions for this class of completely generalized nonlinear variational inclusions and the convergence of iterative sequences generated by these algorithms. Our results extend and improve some known results in this field.

2. Preliminaries

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathscr{F}(H)$ be a collection of all fuzzy sets on H. A mapping F from H into $\mathscr{F}(H)$ is called a fuzzy mapping on H. If F is a fuzzy mapping on H, then $F(x)$(we denote it by F_x in the sequel) is a fuzzy set on H and $F_x(y)$ is the membership function of y in F_x.

Let $M \in \mathscr{F}(H)$, $q \in [0,1]$. Then the set

$$(M)_q = \{x \in H\colon\ M(x) \geqslant q\}$$

is called a $q-$cut set of M.

Let $T, A\colon\ H \to \mathscr{F}(H)$ be two fuzzy mappings satisfying the following condition (I):

(I) There exist two mappings $a, b\colon\ H \to [0,1]$ such that for all $x \in H$, we have $(T_x)_{a(x)} \in CB(H)$ and $(A_x)_{b(x)} \in CB(H)$, where $CB(H)$ denotes the family of all nonempty bounded closed subsets of H.

By using the fuzzy mappings T and A, we can define two set-valued mappings $\widetilde{T}$ and $\widetilde{A}$ as follows:

$$\begin{aligned}
\widetilde{T}\colon\ H \to CB(H),\ x \longmapsto (T_x)_{a(x)},\\
\widetilde{A}\colon\ H \to CB(H),\ x \longmapsto (A_x)_{b(x)}.
\end{aligned}$$

In the sequel, $\widetilde{T}$ and $\widetilde{A}$ are called the set-valued mappings induced by the fuzzy mappings T and A, respectively.

Given mappings $a, b\colon H \to [0,1]$, fuzzy mappings $T, A\colon H \to \mathscr{F}(H)$, single-valued mappings $f, p\colon H \to H$, and a set-valued mapping $g\colon H \to 2^H$ with $\operatorname{Im} g \bigcap \operatorname{dom}(\partial\varphi) \neq \emptyset$ (where 2^H denotes the family of all nonempty subsets of H), we consider the following problem:

Find $u, w, y, z \in H$ such that

$$\begin{cases} T_u(w) \geqslant a(u),\ A_u(y) \geqslant b(u),\ z \in g(u) \bigcap \operatorname{dom}(\partial\varphi), \\ \langle f(w) - p(y), v - z\rangle \geqslant \varphi(z) - \varphi(v),\ \forall v \in H, \end{cases} \tag{2.1}$$

where $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semicontinuous function $\varphi\colon H \to R \cup \{+\infty\}$. This problem is called a completely generalized nonlinear variational inclusion for a fuzzy mapping.

If $g\colon H \to H$ is a single-valued mapping, then the problem (2.1) is equivalent to finding $u, w, y \in H$ such that

$$\begin{cases} T_u(w) \geqslant a(u),\ A_u(y) \geqslant b(u),\ g(u) \bigcap \operatorname{dom}(\partial\varphi) \neq \emptyset, \\ \langle f(w) - p(y), v - g(u)\rangle \geqslant \varphi(g(u)) - \varphi(v),\ \forall v \in H, \end{cases} \tag{2.2}$$

which is called a generalized nonlinear variational inclusion for a fuzzy mapping.

As examples, we now consider some particular variational inclusions for fuzzy mappings.

Example 1. If $F, G, g\colon H \to 2^H$ are classical set-valued mappings, by using F and G, we can define two fuzzy mappings:

$$T\colon H \to \mathscr{F}(H),\ x \longmapsto \chi_{F(x)},$$
$$A\colon H \to \mathscr{F}(H),\ x \longmapsto \chi_{G(x)},$$

where $\chi_{F(x)}$ and $\chi_{G(x)}$ are the characteristic functions of the set $F(x)$ and $G(x)$, respectively. Taking $a(x) = 1$, $b(x) = 1$ for all $x \in H$, the problem (2.1) is equivalent to finding $u, w, y, z \in H$ such that

$$\begin{cases} w \in Fu,\ y \in Gu,\ z \in g(u) \bigcap \operatorname{dom}(\partial\varphi), \\ \langle f(w) - p(y), v - z\rangle \geqslant \varphi(z) - \varphi(v),\ \forall v \in H, \end{cases} \tag{2.3}$$

which is called a completely generalized nonlinear variational inclusion for set-valued mappings.

Example 2. If $F, G\colon H \to 2^H$ are classical set-valued mappings and $g\colon H \to H$ is a single-valued mapping, by using F and G we define two fuzzy mappings T and A as in Example 1. Taking $a(x) = 1$, $b(x) = 1$ for all $x \in H$, the problem (2.1) is equivalent to finding $u, w, y \in H$, such that

$$\tag{2.4} \begin{cases} w \in Fu, \ y \in Gu, \ g(u) \bigcap \operatorname{dom}(\partial\varphi) \neq \emptyset, \\ \langle f(w) - p(y), v - g(u)\rangle \geqslant \varphi(g(u)) - \varphi(v), \ \forall v \in H, \end{cases}$$

which is called a set-valued nonlinear generalized variational inclusion which was introduced and studied by Huang [14].

Remark 2.1. For appropriate and suitable choices of the mappings f, p, g, T, A and the functions a, b, φ, the variational inclusion (2.1) includes a number of known classes of variational inequalities and quasi-variational inequalities studied previously by many authors in [5, 7, 8, 10, 13, 14, 17, 18, 23–25, 28–30] as special cases.

3. Iterative Algorithms

First, let us prove the following lemma.

Lemma 3.1. *u, w, y and z are a solution of problem* (2.1) *if and only if $w \in \widetilde{T}u, y \in \widetilde{A}u$ and $z \in g(u)$ such that*

$$u = u - z + J_\alpha^\varphi(z - \alpha(f(w) - p(y))),$$

where $\alpha > 0$ is a constant and $J_\alpha^\varphi = (I + \alpha\partial\varphi)^{-1}$ is the so-called proximal mapping on H.

P r o o f. From the definition of J_α^φ one has

$$z - \alpha(f(w) - p(y)) \in z + \alpha\partial\varphi(z),$$

hence

$$p(y) - f(w) \in \partial\varphi(z).$$

From the definition of $\partial\varphi$ we have

$$\varphi(v) \geqslant \varphi(z) + \langle p(y) - f(w), v - z\rangle, \forall v \in H.$$

Thus u, w, y and z are a solution of problem (2.1).

Conversely, if u, w, y and z are a solution of problem (2.1), we know that $w \in \widetilde{T}u$, $y \in \widetilde{A}u$, $z \in g(u) \bigcap \mathrm{dom}(\partial\varphi)$ and

$$\varphi(v) \geqslant \varphi(z) + \langle p(y) - f(w), v - z \rangle, \forall v \in H.$$

This implies

$$p(y) - f(w) \in \partial\varphi(z)$$

and

$$z - \alpha(f(w) - p(y)) \in z + \alpha\partial\varphi(z).$$

Therefore

$$u = u - z + J_\alpha^\varphi(z - \alpha(f(w) - p(y))).$$

This completes the proof. □

Based on Lemma 3.1, we construct our algorithms.

Suppose that $T, A\colon H \to \mathscr{F}(H)$ satisfy the conditon (I) and $g\colon H \to CB(H)$. Let $\widetilde{T}, \widetilde{A}\colon H \to CB(H)$ be set-valued mappings induced by T, A, respectively. For a given $u_0 \in H$, let $w_0 \in \widetilde{T}u_0$, $y_0 \in \widetilde{A}u_0$, $z_0 \in g(u_0)$ and

$$u_1 = u_0 - z_0 + J_\alpha^\varphi(z_0 - \alpha(f(w_0) - p(y_0))).$$

By [22] there exist $w_1 \in \widetilde{T}u_1$, $y_1 \in \widetilde{A}u_1$ and $z_1 \in g(u_1)$ such that

$$\begin{aligned} \|w_1 - w_0\| &\leqslant (1+1)\widehat{\mathbf{H}}(\widetilde{T}u_1, \widetilde{T}u_0), \\ \|y_1 - y_0\| &\leqslant (1+1)\widehat{\mathbf{H}}(\widetilde{A}u_1, \widetilde{A}u_0), \\ \|z_1 - z_0\| &\leqslant (1+1)\widehat{\mathbf{H}}(g(u_1), g(u_0)), \end{aligned}$$

where $\widehat{\mathbf{H}}$ is the Hausdorff metric on $CB(H)$. By induction we can obtain our algorithm as follows.

Algorithm 3.1. *Suppose that $T, A\colon H \to \mathscr{F}(H)$ satisfy the conditon* (I) *and $g\colon H \to CB(H)$. Let $\widetilde{T}, \widetilde{A}\colon H \to CB(H)$ be set-valued mappings induced by T, A, respectively, and $f, p\colon H \to H$. For a given $u_0 \in H$, we can get an algorithm for* (2.1) *as follows:*

$$\text{(3.1)} \qquad \begin{cases} u_{n+1} = u_n - z_n + J_\alpha^\varphi(z_n - \alpha(f(w_n) - p(y_n))), \\ w_n \in \widetilde{T}u_n, \ \|w_{n+1} - w_n\| \leqslant (1 + (1+n)^{-1})\,\widehat{\mathbf{H}}(\widetilde{T}u_{n+1}, \widetilde{T}u_n), \\ y_n \in \widetilde{A}u_n, \ \|y_{n+1} - y_n\| \leqslant (1 + (1+n)^{-1})\,\widehat{\mathbf{H}}(\widetilde{A}u_{n+1}, \widetilde{A}u_n), \\ z_n \in g(u_n), \ \|z_{n+1} - z_n\| \leqslant (1 + (1+n)^{-1})\,\widehat{\mathbf{H}}(g(u_{n+1}), g(u_n)), \end{cases}$$

for $n = 0, 1, 2, \ldots$.

Similarly, we have

Algorithm 3.2. *Suppose that* $T, A\colon H \to \mathscr{F}(H)$ *satisfy the conditon* (I). *Let* $\widetilde{T}, \widetilde{A}\colon H \to CB(H)$ *be set-valued mappings induced by* T, A*, respectively, and* $f, p, g\colon H \to H$*. For a given* $u_0 \in H$*, we can get an algorithm for* (2.2) *as follows:*

$$\begin{aligned} u_{n+1} &= u_n - g(u_n) + J_\alpha^\varphi(g(u_n) - \alpha(f(w_n) - p(y_n))), \\ w_n &\in \widetilde{T}u_n,\ \|w_{n+1} - w_n\| \leqslant (1 + (1+n)^{-1})\,\widehat{\mathbf{H}}(\widetilde{T}u_{n+1}, \widetilde{T}u_n), \\ y_n &\in \widetilde{A}u_n\ \|y_{n+1} - y_n\| \leqslant (1 + (1+n)^{-1})\,\widehat{\mathbf{H}}(\widetilde{A}u_{n+1}, \widetilde{A}u_n), \end{aligned}$$

for $n = 0, 1, 2, \ldots$.

Algorithm 3.3. *Let* $F, G, g\colon H \to CB(H)$ *be set-valued mappings and* $f, p\colon H \to H$*. For a given* $u_0 \in H$*, we can get an algorithm for* (2.3) *as follows:*

$$\begin{aligned} u_{n+1} &= u_n - z_n + J_\alpha^\varphi(z_n - \alpha(f(w_n) - p(y_n))), \\ w_n &\in Fu_n,\ \|w_{n+1} - w_n\| \leqslant (1 + (1+n)^{-1})\,\widehat{\mathbf{H}}(Fu_{n+1}, Fu_n), \\ y_n &\in Gu_n,\ \|y_{n+1} - y_n\| \leqslant (1 + (1+n)^{-1})\,\widehat{\mathbf{H}}(Gu_{n+1}, Gu_n), \\ z_n &\in g(u_n),\ \|z_{n+1} - z_n\| \leqslant (1 + (1+n)^{-1})\,\widehat{\mathbf{H}}(g(u_{n+1}), g(u_n)), \end{aligned}$$

for $n = 0, 1, 2, \ldots$.

Remark 3.1. The algorithms 3.1–3.3 include several known algorithms of [5, 7, 8, 10, 13, 14, 18, 23–25, 28–30] as special cases.

4. Existence and Convergence

Definition 4.1. A mapping $g\colon H \to H$ is said to be
(i) strongly monotone if there exists $\delta > 0$ such that

$$\langle g(u_1) - g(u_2), u_1 - u_2\rangle \geqslant \delta\|u_1 - u_2\|^2,\ \forall u_i \in H, i = 1, 2;$$

(ii) Lipschitz continuous if there exists $\sigma > 0$ such that

$$\|g(u_1) - g(u_2)\| \leqslant \sigma\|u_1 - u_2\|,\ \forall u_i \in H,\ i = 1, 2.$$

Definition 4.2. A set-valued mapping $T\colon H \to 2^H$ is said to be
(i) strongly monotone if there exists $\delta > 0$ such that

$$\langle w_1 - w_2, u_1 - u_2\rangle \geqslant \delta\|u_1 - u_2\|^2,\ \forall u_i \in H,\ w_i \in Tu_i, i = 1, 2;$$

(ii) strongly monotone with respect to a mapping $f\colon H\to H$ if there exists $\beta>0$ such that

$$\langle f(w_1)-f(w_2),u_1-u_2\rangle\geqslant\beta\|u_1-u_2\|^2,\ \forall u_i\in H,\ w_i\in Tu_i, i=1,2;$$

(iii) $\widehat{\mathbf{H}}$-Lipschitz continuous if there exists $\gamma>0$ such that

$$\widehat{\mathbf{H}}(Tu_1,Tu_2)\leqslant\gamma\|u_1-u_2\|,\ \forall u_i\in H, i=1,2.$$

Theorem 4.1. *Let $g\colon H\to CB(H)$ be strongly monotone and $\widehat{\mathbf{H}}$-Lipschitz continuous, $f,p\colon H\to H$ Lipschitz continuous, let $T,A\colon H\to\mathscr{F}(H)$ be fuzzy mappings satisfying the condition* (I). *Let $\widetilde{T},\widetilde{A}\colon H\to CB(H)$ be set-valued mappings induced by T,A, respectively, and let $\widetilde{T},\widetilde{A}$ be $\widehat{\mathbf{H}}$-Lipschitz continuous and $\widetilde{T}$ strongly monotone with respect to f. If the following conditions hold:*

$$\left|\alpha-\frac{\beta+\varepsilon\mu(k-1)}{\eta^2\gamma^2-\varepsilon^2\mu^2}\right|<\frac{\sqrt{(\beta+(k-1)\varepsilon\mu)^2-(\gamma^2-\varepsilon^2\mu^2)k(2-k)}}{\eta^2\gamma^2-\varepsilon^2\mu^2},\tag{4.1}$$

$$\beta>(1-k)\varepsilon\mu+\sqrt{(\eta^2\gamma^2-\varepsilon^2\mu^2)k(2-k)},\ \eta\gamma>\varepsilon\mu,\tag{4.2}$$

$$\alpha\mu\varepsilon<1-k,\ k=2\sqrt{1-2\delta+\sigma^2},\ k<1,\tag{4.3}$$

where β and δ are constants of strong monotonicity of $\widetilde{T}$ and g, respectively, σ, γ and μ are $\widehat{\mathbf{H}}$-Lipschitz constants of g, $\widetilde{T}$ and $\widetilde{A}$, respectively, η and ε are the Lipschitz constants of f and p, respectively, then there exist $u,w,y,z\in H$ such that (2.1) *is satisfied. Moreover, $u_n\to u$, $w_n\to w$, $y_n\to y$, $z_n\to z$, $n\to\infty$, where $\{u_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined in Algorithm* 3.1.

Proof. From (3.1) we have

$$\|u_{n+1}-u_n\|=\|u_n-u_{n-1}-(z_n-z_{n-1})+J_\alpha^\varphi(h(u_n))-J_\alpha^\varphi(h(u_{n-1}))\|,$$

where $h(u_n)=z_n-\alpha(f(w_n)-p(y_n))$. Also we have

$$\begin{aligned}\|J_\alpha^\varphi(h(u_n))-J_\alpha^\varphi(h(u_{n-1}))\|&\leqslant\|h(u_n)-h(u_{n-1})\|\\&\leqslant\|u_n-u_{n-1}-\alpha(f(w_n)-f(w_{n-1}))\|\\&\quad+\|u_n-u_{n-1}-(z_n-z_{n-1})\|+\alpha\|p(y_n)-p(y_{n-1})\|.\end{aligned}$$

That is,

$$\begin{aligned}\|u_{n+1}-u_n\|\leqslant{}&2\|u_n-u_{n-1}-(z_n-z_{n-1})\|\\&+\|u_n-u_{n-1}-\alpha(f(w_n)-f(w_{n-1}))\|\\&+\alpha\|p(y_n)-p(y_{n-1})\|.\end{aligned}\tag{4.4}$$

By $\widehat{\mathbf{H}}$-Lipschitz continuity and strong monotonicity of g we obtain

$$\|u_n - u_{n-1} - (z_n - z_{n-1})\|^2 \leqslant (1 - 2\delta + (1 + n^{-1})^2\sigma^2)\|u_n - u_{n-1}\|^2. \tag{4.5}$$

Also from $\widehat{\mathbf{H}}$-Lipschitz continuity and strong monotonicity of $\widetilde{T}$ with respect to f, and Lipschitz continuity of f, we have

$$\|u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1}))\|^2 \leqslant (1 - 2\beta\alpha + \alpha^2\eta^2(1 + n^{-1})^2\gamma^2)\|u_n - u_{n-1}\|^2. \tag{4.6}$$

By $\widehat{\mathbf{H}}$-Lipschitz continuity of $\widetilde{A}$, Lipschitz continuity of p and (3.1), we know

$$\alpha\|p(y_n) - p(y_{n-1})\| \leqslant \alpha\varepsilon(1 + n^{-1})\mu\|u_n - u_{n-1}\|. \tag{4.7}$$

So by combining (4.4)–(4.7), we have

$$\|u_{n+1} - u_n\| \leqslant \theta_n\|u_n - u_{n-1}\|,$$

where

$$\theta_n := 2\sqrt{1 - 2\delta + (1 + n^{-1})^2\sigma^2} + \sqrt{1 - 2\beta\alpha + \alpha^2\eta^2(1 + n^{-1})^2\gamma^2} + \alpha\varepsilon(1 + n^{-1})\mu.$$

Letting $\theta := 2\sqrt{1 - 2\delta + \sigma^2} + \sqrt{1 - 2\beta\alpha + \alpha^2\eta^2\gamma^2} + \alpha\varepsilon\mu$, we know that $\theta_n \searrow \theta$. It follows from (4.1)–(4.3) that $\theta < 1$. Hence $\theta_n < 1$, for n sufficiently large. Therefore $\{u_n\}$ is a Cauchy sequence and we can suppose that $u_n \to u \in H$.

Now we prove that $w_n \to w \in \widetilde{T}u$, $y_n \to y \in \widetilde{A}u$, $z_n \to z \in g(u)$. In fact, it follows from Algorithm 3.1 that

$$\begin{aligned}
\|w_n - w_{n-1}\| &\leqslant (1 + n^{-1})\gamma\|u_n - u_{n-1}\|,\\
\|y_n - y_{n-1}\| &\leqslant (1 + n^{-1})\mu\|u_n - u_{n-1}\|,\\
\|z_n - z_{n-1}\| &\leqslant (1 + n^{-1})\sigma\|u_n - u_{n-1}\|,
\end{aligned}$$

i.e. $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences. Let $w_n \to w$, $y_n \to y$ and $z_n \to z$. Further we have

$$\begin{aligned}
\varrho(w, \widetilde{T}u) &= \inf\{\|w - z\| \colon\ z \in \widetilde{T}u\}\\
&\leqslant \|w - w_n\| + \varrho(w_n, \widetilde{T}u)\\
&\leqslant \|w - w_n\| + \widehat{\mathbf{H}}(\widetilde{T}u_n, \widetilde{T}u)\\
&\leqslant \|w - w_n\| + \gamma\|u_n - u\| \to 0.
\end{aligned}$$

Hence, $w \in \widetilde{T}u$. Similarly, $y \in \widetilde{A}u$ and $z \in g(u)$. This completes the proof. $\square$

Remark 4.1. (i) If we replace the conditions (4.1)–(4.3) by

$$\theta = 2\sqrt{1 - 2\delta + \sigma^2} + \sqrt{1 - 2\beta\alpha + \alpha^2\eta^2\gamma^2} + \alpha\varepsilon\mu < 1,$$

the conclusions of Theorem 4.1 are still true.

(ii) For an appropriate and suitable choice of the constants α, β, γ, η, μ, σ, δ, ε, the conditions (4.1)–(4.3) in Theorem 4.1 can be satisfied.

From Theorem 4.1 we can get the following results.

Theorem 4.2. *Let $g\colon H \to H$ be strongly monotone and Lipschitz continuous, $f, p\colon H \to H$ Lipschitz continuous, let $T, A\colon H \to \mathscr{F}(H)$ be fuzzy mappings satisfying the condition* (I). *Let $\widetilde{T}, \widetilde{A}\colon H \to CB(H)$ be set-valued mappings induced by T, A, respectively, and let $\widetilde{T}$, $\widetilde{A}$ be $\widehat{\mathbf{H}}$-Lipschitz continuous and $\widetilde{T}$ strongly monotone with respect to f. If the conditions* (4.1)–(4.3) *of Theorem* 4.1 *hold, where β and δ are constants of strong monotonicity of $\widetilde{T}$ and g, respectively, γ and μ are $\widehat{\mathbf{H}}$-Lipschitz constants of $\widetilde{T}$ and $\widetilde{A}$, respectively, σ, η and ε are the Lipschitz constants of g, f and p, respectively, then there exist $u, w, y \in H$ such that* (2.2) *is satisfied. Moreover, $u_n \to u$, $w_n \to w$, $y_n \to y$, $n \to \infty$, where $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are defined in Algorithm* 3.2.

Theorem 4.3. *Let $g\colon H \to CB(H)$ be strongly monotone and $\widehat{\mathbf{H}}$-Lipschitz continuous, $f, p\colon H \to H$ Lipschitz continuous, let $F, G\colon H \to CB(H)$ be $\widehat{\mathbf{H}}$-Lipschitz continuous and F strongly monotone with respect to f. If the conditions* (4.1)–(4.3) *of Theorem* 4.1 *hold, where β and δ are constants of strong monotonicity of F and g, respectively, σ, γ and μ are $\widehat{\mathbf{H}}$-Lipschitz constants of g, F and G, respectively, η and ε are the Lipschitz constants of f and p, respectively, then there exist $u, w, y, z \in H$ such that* (2.3) *is satisfied. Moreover, $u_n \to u$, $w_n \to w$, $y_n \to y$, $z_n \to z$, $n \to \infty$, where $\{u_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined in Algorithm* 3.3.

Remark 4.2. Theorems 4.1–4.3 include some known results of [5, 7, 8, 10, 14, 18, 23–25, 29–30] as special cases.

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Author's address: Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P. R. China.