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Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 2, 265–278

Czechoslovak Mathematical Journal, 50 (125) (2000), 265–278

REMARK TO TRANSFORMATIONS OF LINEAR DIFFERENTIAL AND FUNCTIONAL-DIFFERENTIAL EQUATIONS

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(Received April 29, 1997)

Abstract. For linear differential and functional-differential equations of the n-th order criteria of equivalence with respect to the pointwise transformation are derived.

Keywords: ordinary differential equations, functional-differential equations, transformations

MSC 2000: 34A30, 34A34, 34K05, 34K15

1. Introduction

The theory of global pointwise transformations of homogeneous linear differential equations was developed in the monograph of F. Neuman [6] (see historical remarks, definitions, results and applications). The criterion of global equivalence of the second order equations was published by O. Borůvka [1], of the third and higher order equations by F. Neuman [6]. Transformations of functional-differential equations were studied in [2, 3, 4, 5, 7, 8]. In this paper we derive criteria of equivalence for ordinary differential equations and functional-differential equations of the first and higher orders, exploiting some results from [6, 9].

This research has been conducted at the Department of Mathematics as part of the research project "Qualitative Behaviour of Solutions of Functional Differential Equations Describing Mathematical Models of Technical Phenomena" and has been supported by CTU grant No. 460078.

2. Notation

Let $\mathbf{V}_{n+1}$ denote an $(n+1)$-dimensional vector space, $\vec{c} = [c_0, c_1, \dots, c_n]^T = [c_i]_{i=0}^n \in \mathbf{V}_{n+1}$ being a point, a vector of the space written in the column form; ^{T} means the transposition. Let $\mathbf{V}_{n+1}$ be equipped with the scalar product $(\vec{p}, \vec{q}) = \sum_{i=0}^{n} p_i q_i$ for any pair $\vec{p}, \vec{q}$ of its vectors. Let $\vec{p}_0, \vec{p}_1, \dots, \vec{p}_m$ be $m+1$ vectors from $\mathbf{V}_{n+1}$. Notation $P = [\vec{p}_0, \vec{p}_1, \dots, \vec{p}_m] = [p_{ij}]_{j=0,\dots,m}^{i=0,\dots,n}$ denotes a matrix and $(P, Q) = \sum_{j}^{i} p_{ij} q_{ij}$ the scalar product of two matrices of the same type. Similarly $P_{(j,\dots,k)} = [\vec{p}_j, \dots, \vec{p}_k]$ means a submatrix, $PQ = P_{(0,\dots,n)} Q_{(0,\dots,n)}$ is the matrix multiplication. Consider real functions $y \in C^{n+1}(I)$, $I \subseteq \mathbb{R}$ being an interval, $\xi_1, \xi_2, \dots, \xi_m \in C^n(I)$, $\xi_j \colon I \to I$, $\xi_0 = \mathrm{id}_I$, $\xi_j \neq \xi_k$ for $j \neq k$; $j, k \in \{0, \dots, m\}$; $m, n \in \mathbb{N} = \{1, 2, \dots\}$. We denote $(y(\xi_j(x)))^{(i)} = d^i y(\xi_j(x))/dx^i$, $y^{(i)}(\xi_j(x)) = d^i y(\xi_j(x))/d\xi_j(x)^i$, $x \in I$ and $y_i(x) = y^{(i)}(x), y_{ij}(x) = y^{(i)}(\xi_j(x))$. Then $\vec{y}(x) = [y_0(x), y_1(x), \dots, y_n(x)]^T = [y(x), y'(x), \dots, y^{(n)}(x)]^T \in \mathbf{V}_{n+1}$ for each $x \in I$ and $Y_{(j,\dots,k)}(x) = [\vec{y}(\xi_j(x)), \dots, \vec{y}(\xi_k(x))]$, $x \in I$.

3. Preliminary results

Lemma 1 (A modification of Lemma 8.5.1, [6]). *Let $n \in \mathbb{N}$ and let the relation*

$$z(t) = L(t) y(\varphi(t))$$

be satisfied where the real functions $y \colon I \to \mathbb{R}$, $z \colon J \to \mathbb{R}$ belong to the classes $C^{n+1}(I)$, $C^{n+1}(J)$ respectively, and $L \colon J \to \mathbb{R}$, $L \in C^r(J)$, $L(t) \neq 0$ on J, and φ is a C^r diffeomorphism of J onto I for some integer $r \geqslant n+1$. Then

$$\begin{aligned} z^{(i)}(t) &= \sum_{j=0}^{i} a_{ij}(t) y^{(j)}(\varphi(t)) \\ &= a_{i0}(t) y(\varphi(t)) + a_{i1}(t) y'(\varphi(t)) + \dots + a_{ii}(t) y^{(i)}(\varphi(t)), \\ &\quad i \in \{0, 1, \dots, n+1\}, \end{aligned}$$

where

$$\begin{aligned} a_{00}(t) &= L(t), \dots, a_{i0}(t) = a'_{i-10}(t), \quad i \geqslant 1; \\ a_{ij}(t) &= a'_{i-1j}(t) + a_{i-1j-1}(t)\varphi'(t), \quad i > j > 1; \\ a_{ii}(t) &= a_{i-1i-1}(t)\varphi'(t), \quad i \in \{0, 1, \dots, n+1\} \end{aligned}$$

are real functions, $a_{ij}(t) \in C^{r-(i-j)-1}(J)$ *for* $j > 0$, *and* $a_{i0}(t) \in C^{r-i}(J)$. *Moreover*

$$\begin{aligned}
a_{i0}(t) &= L^{(i)}(t), \quad i \geqslant 0;\\
a_{i1}(t) &= (L(t)\varphi(t))^{(i)} - L^{(i)}(t)\varphi(t) = \sum_{j=0}^{i-1} \binom{i}{j} L^{(j)}(t)\varphi^{(i-j)}(t), \quad i \geqslant 1;\\
&\dots\\
a_{ij}(t) &= \binom{i}{j} L^{(i-j)}(t)\varphi'(t)^j + \binom{i}{j-1} L(t)\varphi'(t)^{j-1}\varphi^{(i-j+1)}(t)\\
&\quad + r_{ij}(L, \dots, L^{(i-j-1)}, \varphi', \dots, \varphi^{(i-j)})(t), \quad i > j > 1;\\
&\dots\\
a_{ii-2}(t) &= \binom{i}{2} L''(t)\varphi'(t)^{i-2} + \binom{i}{3}(L(t)\varphi'''(t) + 3L'(t)\varphi''(t))\varphi'(t)^{i-3}\\
&\quad + 3\binom{i}{4} L(t)\varphi'(t)^{i-4}\varphi''(t)^2, \quad i \geqslant 2;\\
a_{ii-1}(t) &= \binom{i}{1} L'(t)\varphi'(t)^{i-1} + \binom{i}{2} L(t)\varphi'(t)^{i-2}\varphi''(t), \quad i \geqslant 2;\\
a_{ii}(t) &= L(t)\varphi'(t)^i, \quad i \geqslant 0
\end{aligned}$$

and

$$\begin{aligned}
&a_{i0}(t) = a_{i0}(L^{(i)})(t), \qquad i \geqslant 0;\\
&a_{ij}(t) = a_{ij}(L, \dots, L^{(i-j)}, \varphi', \dots, \varphi^{(i-j+1)})(t), \qquad i \geqslant j > 0;\ i \in \{0, 1, \dots, n+1\}.
\end{aligned}$$

P r o o f. From the relation

$$z(t) = a_{00}(t)y(\varphi(t)), \quad a_{00}(t) = L(t),$$

we have

$$z'(t) = a'_{00}(t)y(\varphi(t)) + a_{00}(t)\varphi'(t)y'(\varphi(t)) = a_{10}(t)y(\varphi(t)) + a_{11}(t)y'(\varphi(t)).$$

Suppose by induction that

$$\begin{aligned}
z^{(i-1)}(t) &= a_{i-10}(t)y(\varphi(t)) + \dots + a_{i-1j-1}(t)y^{(j-1)}(\varphi(t))\\
&\quad + a_{i-1j}(t)y^{(j)}(\varphi(t)) + \dots + a_{i-1i-1}(t)y^{(i-1)}(\varphi(t)).
\end{aligned}$$

Then

$$\begin{aligned}
z^{(i)}(t) &= a'_{i-10}(t)y(\varphi(t)) + \dots + (a_{i-1j-1}(t)\varphi'(t)\\
&\quad + a'_{i-1j}(t))y^{(j)}(\varphi(t)) + \dots + a_{i-1i-1}(t)\varphi'(t)y^{(i)}(\varphi(t))\\
&= a_{i0}(t)y(\varphi(t)) + \dots + a_{ij}(t)y^{(j)}(\varphi(t)) + \dots + a_{ii}(t)y^{(i)}(\varphi(t)).
\end{aligned}$$

Hence

$$z^{(i)}(t) = \sum_{j=0}^{i} a_{ij}(t) y^{(j)}(\varphi(t))$$

where

$$a_{00}(t) = L(t), \dots, a_{i0}(t) = a'_{i-10}(t), \qquad i \geqslant 1;$$
$$a_{ij}(t) = a'_{i-1j}(t) + a_{i-1j-1}(t)\varphi'(t), \qquad i > j > 1;$$
$$a_{ii}(t) = a_{i-1i-1}(t)\varphi'(t); \qquad i \in \{0, 1, \dots, r\}.$$

By induction we get

$$a_{00}(t) = L(t), \quad a_{10}(t) = a'_{00}(t) = L'(t), \quad \dots$$

and

$$a_{i0}(t) = a'_{i-10}(t) = L^{(i)}(t), \qquad i \geqslant 0.$$

Similarly

$$a_{00}(t) = L(t), \quad a_{11}(t) = a_{00}(t)\varphi'(t) = L(t)\varphi'(t), \quad \dots$$

and

$$a_{ii}(t) = a_{i-1i-1}(t)\varphi'(t) = L(t)\varphi'(t)^i, \qquad i \geqslant 0.$$

We have

$$a_{21}(t) = a'_{11}(t) + a_{10}(t)\varphi'(t) = 2L'(t)\varphi'(t) + L(t)\varphi''(t) = (L(t)\varphi(t))'' - L''(t)\varphi(t)$$

and by induction we get

$$a_{i1}(t) = (L(t)\varphi(t))^{(i)} - L^{(i)}(t)\varphi(t); \qquad i \geqslant 2$$

because

$$\begin{aligned} a_{i+11}(t) &= a'_{i1}(t) + a_{i0}(t)\varphi'(t) = ((L(t)\varphi(t))^{(i)} - L^{(i)}(t)\varphi(t))' + L^{(i)}(t)\varphi'(t) \\ &= (L(t)\varphi(t))^{(i+1)} - L^{(i+1)}(t)\varphi(t). \end{aligned}$$

Then

$$\begin{aligned} a_{21}(t) &= 2L'(t)\varphi'(t) + L(t)\varphi''(t), \\ a_{32}(t) &= a'_{22}(t) + a_{21}(t)\varphi'(t) \\ &= \binom{3}{1} L'(t)\varphi'(t)^{3-1} + \binom{3}{2} L(t)\varphi'(t)^{3-2}\varphi''(t) \end{aligned}$$

and

$$a_{ii-1}(t) = \binom{i}{1} L'(t)\varphi'(t)^{i-1} + \binom{i}{2} L(t)\varphi'(t)^{i-2}\varphi''(t), \quad i \geqslant 2$$

since

$$\begin{aligned}a_{i+1i}(t) &= a'_{ii}(t) + a_{ii-1}(t)\varphi'(t)\\ &= (L(t)\varphi'(t)^i)' + \binom{i}{1}L'(t)\varphi'(t)^i + \binom{i}{2}L(t)\varphi'(t)^{i-1}\varphi''(t)\\ &= \binom{i+1}{1}L'(t)\varphi'(t)^i + \binom{i+1}{2}L(t)\varphi'(t)^{i-1}\varphi''(t).\end{aligned}$$

In the same way,

$$\begin{aligned}a_{20}(t) &= L''(t),\\ a_{31}(t) &= a'_{21}(t) + a_{20}(t)\varphi'(t) = L'''(t)\varphi(t) + 3L'(t)\varphi''(t) + 3L''(t)\varphi'(t)\\ &= \binom{3}{2}L''(t)\varphi'(t)^{3-2} + \binom{3}{3}(L(t)\varphi'''(t) + 3L'(t)\varphi''(t))\varphi'(t)^{3-3}\end{aligned}$$

and

$$\begin{aligned}a_{ii-2}(t) = \binom{i}{2}L''(t)\varphi'(t)^{i-2} &+ \binom{i}{3}(L(t)\varphi'''(t) + 3L'(t)\varphi''(t))\varphi'(t)^{i-3}\\ &+ 3\binom{i}{4}L(t)\varphi'(t)^{i-4}\varphi''(t)^2; \quad i \geqslant 2\end{aligned}$$

because then

$$\begin{aligned}a_{i+1i-1}(t) &= a'_{ii-1}(t) + a_{ii-2}(t)\varphi'(t)\\ &= \left(i + \binom{i}{2}\right) L''(t)\varphi'(t)^{i-1} + \left(\binom{i}{2} + \binom{i}{3}\right) L(t)\varphi'(t)^{i-3}\varphi'''(t)\\ &\quad + \left(i(i-1) + \binom{i}{2} + 3\binom{i}{3}\right) L'(t)\varphi'(t)^{i-2}\varphi''(t)\\ &\quad + \left((i-2)\binom{i}{2} + 3\binom{i}{4}\right) L(t)\varphi'(t)^{i-3}\varphi''(t)^2\\ &= \binom{i+1}{2}L''(t)\varphi'(t)^{i-1} + \binom{i+1}{3}(L(t)\varphi'''(t) + 3L'(t)\varphi''(t))\varphi'(t)^{i-2}\\ &\quad + 3\binom{i+1}{4}L(t)\varphi'(t)^{i-3}\varphi''(t)^2.\end{aligned}$$

Using the above results we can suppose that

$$\begin{aligned}a_{ij}(t) = \binom{i}{j}L^{(i-j)}(t)\varphi'(t)^j &+ \binom{i}{j-1}L(t)\varphi'(t)^{j-1}\varphi^{(i-j+1)}(t)\\ &+ r_{ij}(L, \ldots, L^{(i-j-1)}, \varphi', \ldots, \varphi^{(i-j)})(t).\end{aligned}$$

Then

$$\begin{aligned}
a_{i+1k}(t) &= a'_{ik}(t) + a_{ik-1}(t)\varphi'(t) \\
&= \Bigg(\binom{i}{k} L^{(i-k)}(t)\varphi'(t)^k + \binom{i}{k-1} L(t)\varphi'(t)^{k-1}\varphi^{(i-k+1)}(t) \\
&\quad + r_{ik}(L,\dots,L^{(i-k-1)},\varphi',\dots,\varphi^{(i-k)})(t)\Bigg)' \\
&\quad + \Bigg(\binom{i}{k-1} L^{(i-k+1)}(t)\varphi'(t)^{k-1} + \binom{i}{k-2} L(t)\varphi'(t)^{k-2}\varphi^{(i-k+2)}(t) \\
&\quad + r_{ik-1}(L,\dots,L^{(i-k)},\varphi',\dots,\varphi^{(i-k+1)})(t)\Bigg)\varphi'(t) \\
&= \left(\binom{i}{k} + \binom{i}{k-1}\right) L^{(i-k+1)}(t)\varphi'(t)^k \\
&\quad + \left(\binom{i}{k-1} + \binom{i}{k-2}\right) L(t)\varphi'(t)^{k-1}\varphi^{(i-k+2)}(t) \\
&\quad + [r'_{ik}(L,\dots,L^{(i-k-1)},\varphi',\dots,\varphi^{(i-k)})(t) \\
&\quad + r_{ik-1}(L,\dots,L^{(i-k)},\varphi',\dots,\varphi^{(i-k+1)})(t)\varphi'(t) \\
&\quad + k\binom{i}{k} L^{(i-k)}(t)\varphi'(t)^{k-1}\varphi''(t) + \binom{i}{k-1} L'(t)\varphi'(t)^{k-1}\varphi^{(i-k+1)}(t) \\
&\quad + (k-1)\binom{i}{k-1} L(t)\varphi'(t)^{k-2}\varphi''(t)\varphi^{(i-k+1)}(t)] \\
&= \binom{i+1}{k} L^{((i+1)-k)}(t)\varphi'(t)^k + \binom{i+1}{k-1} L(t)\varphi'(t)^{k-1}\varphi^{((i+1)-k+1)}(t) \\
&\quad + r_{i+1k}(L,\dots,L^{((i+1)-k+1)},\varphi',\dots,\varphi^{((i+1)-k)})(t)
\end{aligned}$$

for every k, $0 < k \leqslant i+1$. Moreover,

$$a_{i0}(t) = a_{i0}(L^{(i)}(t)) = L^{(i)}(t).$$

We can suppose by induction that

$$a_{ij}(t) = a_{ij}(L(t),\dots,L^{(i-j)}(t),\varphi'(t),\dots,\varphi^{(i-j+1)}(t))$$

for $i \geqslant j > 0$ because $a_{11}(t) = L(t)\varphi'(t) = a_{11}(L(t),\varphi'(t))$ and

$$\begin{aligned}
a_{i+1k}(t) &= a'_{ik}(L(t),\dots,L^{(i-k)}(t),\varphi'(t),\dots,\varphi^{(i-k+1)}(t)) \\
&\quad + a_{ik-1}(L(t),\dots,L^{(i-k+1)}(t),\varphi'(t),\dots,\varphi^{(i-k+2)}(t))\varphi'(t) \\
&= a_{i+1k}(L(t),\dots,L^{((i+1)-k)}(t),\varphi'(t),\dots,\varphi^{((i+1)-k+1)}(t))
\end{aligned}$$

for every k, $0 < k \leqslant i+1$.

Finally,

$$a_{i0}(t) = L^i(t) \in C^{r-i}(J),$$
$$a_{ij}(t) = a_{ij}(L(t), \dots, L^{(i-j)}(t), \varphi'(t), \dots, \varphi^{(i-j+1)}(t)) \in C^{r-(i-j)-1}(J)$$

for $j > 0$, with respect to the assumptions $L, \varphi \in C^r(J)$. □

4. Linear differential equations

Consider two ordinary homogeneous linear differential equations

$$(1) \qquad y_{n+1}(x) = (\vec{p}(x), \vec{y}(x)) = p_0(x)y_0(x) + p_1(x)y_1(x) + \dots + p_n(x)y_n(x),$$
$$y_i(x) = y^{(i)}(x), \quad x \in I;$$
$$(2) \qquad z_{n+1}(t) = (\vec{q}(t), \vec{z}(t)) = q_0(t)z_0(t) + q_1(t)z_1(t) + \dots + q_n(t)z_n(t),$$
$$z_i(t) = z^{(i)}(t), \quad t \in J$$

with real coefficients, defined on an interval $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}$, respectively.

Definition 1 ([6], p. 15). We say that (1) is globally transformable into (2) if there exist two functions L, φ such that

– the function L is of the class $C^{n+1}(J)$ and is nonvanishing on J,

– the function φ is a C^{n+1} diffeomorphism of the interval J onto I,

and the function

$$(3) \qquad z(t) = L(t)y(\varphi(t))$$

is a solution of (2) whenever y is a solution of (1).

If (1) is globally transformable into (2) then we say that (1), (2) are *equivalent equations.* We say that (3) is a *stationary transformation* if it globally transforms an equation (1) into itself on I, i.e. if $L \in C^{n+1}(I)$, $L(x) \neq 0$ on I, φ is a C^{n+1} diffeomorphism of I onto $I = \varphi(I)$ and the function $z(x) = L(x)y(\varphi(x))$ is a solution of $z_{n+1}(x) = (\vec{p}(x), \vec{z}(x))$ whenever y is a solution of $y_{n+1}(x) = (\vec{p}(x), \vec{y}(x))$, $x \in I$.

Theorem 1. *Let $n, r \in \mathbb{N}$ and $r \geqslant n+1$. Let L, φ satisfy the assumptions of Lemma 1. Then (1) is globally transformable into (2) by means of a transformation $z(t) = L(t)y(\varphi(t))$ if and only if*

$$(4) \qquad \vec{a}_{n+1}(t) = A^T(t)\vec{q}(t) - a_{n+1n+1}(t)\vec{p}(\varphi(t)), \quad t \in J$$

is satisfied for the vectors of coefficients of the equations (1), (2) *and*

$$A(t) = [a_{ij}(t)]_{j=0,\dots,n}^{i=0,\dots,n}, \vec{a}_{n+1}(t) = [a_{n+10}(t), a_{n+11}(t), \dots, a_{n+1n}(t)]^T$$

where the functions $a_{ij}(t)$ *are defined by Lemma* 1.

P r o o f. Using Lemma 1, we obtain

$$z_i(t) = z^{(i)}(t) = \sum_{j=0}^{i} a_{ij}(t) y^{(j)}(\varphi(t)) = \sum_{j=0}^{i} a_{ij}(t) y_j(\varphi(t)), i \in \{0, 1, \dots, n+1\}, t \in J.$$

Thus

$$\begin{aligned}\vec{z}(t) &= [z_i(t)]_{i=0}^{n} = A(t)\vec{y}(\varphi(t)),\\ z_{n+1}(t) &= a_{n+10}(t)y_0(\varphi(t)) + \dots + a_{n+1n}(t)y_n(\varphi(t)) + a_{n+1n+1}(t)y_{n+1}(\varphi(t))\\ &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t)(\vec{p}(\varphi(t)), \vec{y}(\varphi(t)))\\ &= (\vec{q}(t), \vec{z}(t)) = (\vec{q}(t), A(t)\vec{y}(\varphi(t))), \quad t \in J.\end{aligned}$$

Hence

$$(\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) = (A^T(t)\vec{q}(t) - a_{n+1n+1}(t)\vec{p}(\varphi(t)), \vec{y}(\varphi(t)))$$

and

$$\vec{a}_{n+1}(t) = A^T(t)\vec{q}(t) - a_{n+1n+1}(t)\vec{p}(\varphi(t)), \tag{5}$$

i.e.

$$\begin{aligned}a_{n+10}(t) &= a_{00}(t)q_0(t) + a_{10}(t)q_1(t) + \dots + a_{n0}(t)q_n(t) - a_{n+1n+1}(t)p_0(\varphi(t)),\\ a_{n+11}(t) &= a_{11}(t)q_1(t) + \dots + a_{n1}(t)q_n(t) - a_{n+1n+1}(t)p_1(\varphi(t)),\\ &\dots\\ a_{n+1n}(t) &= a_{nn}(t)q_n(t) - a_{n+1n+1}(t)p_n(\varphi(t)),\\ &t \in J.\end{aligned}$$

□

Corollary 1. *If* (1) *is globally transformable into* (2) *by means of the transformation* (3), *then the* $(n+1)$*-st derivatives* $(n \geqslant 1)$ *of the functions* L *and* φ *satisfy* *differential equations*

$$\begin{aligned}
L^{(n+1)}(t) &= h(t, \varphi(t), \varphi'(t), L(t), \ldots, L^{(n)}(t))\\
&= q_0(t)L(t) + \ldots + q_n(t)L^{(n)}(t) - L(t)\varphi'(t)^{n+1}p_0(\varphi(t));\\
\varphi^{(n+1)}(t) &= g(t, \varphi(t), \ldots, \varphi^{(n)}(t), L(t), \ldots, L^{(n)}(t))\\
&= \frac{1}{L(t)}\sum_{k=1}^{n}(a_{k1}(t)q_k(t) - \binom{n+1}{k}L^{(k)}(t)\varphi^{(n+1-k)}(t)) - p_1(\varphi(t))\varphi'(t)^{n+1},\\
a_{k1}(t) &= a_{k1}(L, \ldots, L^{(k-1)}, \varphi', \ldots, \varphi^{(k)})(t);\ t \in J.
\end{aligned}$$

We obtain the assertion using the first two equations in (5) *together with the definitions of the functions* $a_{ij}(t)$ *in Lemma* 1*.*

Corollary 2. *The transformation* (3) *is a stationary transformation of* (1) *if and only if* $\varphi(I) = I$ *and the functions* $L, \varphi \in C^{n+1}(I)$ *satisfy the conditions*

$$\vec{a}_{n+1}(x) = A^T(x)\vec{p}(x) - a_{n+1n+1}(x)\vec{p}(\varphi(x)), \quad x \in I,$$

where the functions a_{ij} *are defined by Lemma* 1*;* $i, j \in \{0, 1, \ldots, n+1\}$*.*

Remark 1. Using Theorem 1, if we suppose that $q_n(t) \equiv 0$ on J, $p_n(x) \equiv 0$ on I, then $a_{n+1n}(t) = (n+1)\varphi'(t)^{n-1}(L''(t)\varphi(t) + \frac{n}{2}L(t)\varphi(t)) \equiv 0$ on J and $a_{n+1n}(t) = -\{\varphi, t\}\binom{n+2}{3}L(t)\varphi'(t)^{n-1}$ for $\varphi \in C^r(J)$, $r > n+1$ ($r \geqslant n+1$ if $n > 1$), $\varphi'(t) \neq 0$, where the symbol $\{\varphi, t\} := \frac{1}{2}\frac{\varphi'''(t)}{\varphi'(t)} - \frac{3}{4}(\frac{\varphi''(t)}{\varphi'(t)})^2$ is used for the Schwarzian derivative of φ; $t \in J$ (See [6], p. 7). In the case $n = 1$ we obtain for $\varphi \in C^3(J)$ the Kummer equation $q_0(t) = p_0(\varphi(t))\varphi'(t)^2 - \{\varphi, t\}$, $t \in J$ (see [6], p. 89). We can obtain more useful formulas (as in [6], pp. 49–52) using (5).

5. Linear functional-differential equations

Let $y \in C^{n+1}(I)$; $\xi_1, \xi_2, \ldots, \xi_m \in C^n(I)$, $I \subseteq \mathbb{R}$ being an interval, $\xi_j\colon I \to I$, $\xi_0 = \mathrm{id}_I$, $\xi_j \neq \xi_k$ for $j \neq k$; $j, k \in \{0, \ldots, m\}$; $m, n \in \mathbb{N}$. We consider a homogeneous linear functional-differential equation with real coefficients p_i, p_{ij} and m delays $\xi_1, \ldots, \xi_m$ of the form

$$\begin{aligned}
y_{n+1}(x) &= (\vec{p}(x), \vec{y}(x)) + (P(x), Y_{(1,\ldots,m)}(x))\\
&= ([p_i(x)]_{i=0}^n, [y_i(x)]_{i=0}^n) + ([p_{ij}(x)]_{j=1,\ldots,m}^{i=0,\ldots,n}, [y_i(\xi_j(x))]_{j=1,\ldots,m}^{i=0,\ldots,n})\\
&= \sum_{i=0}^{n}(p_i(x)y_i(x) + \sum_{j=1}^{m}p_{ij}(x)y_i(\xi_j(x))), y_i(x) = y^{(i)}(x), \quad x \in I.
\end{aligned}$$

Consider two such equations

$$(6)\qquad y_{n+1}(x)=(\vec{p}(x),\vec{y}(x))+(P(x),Y(x)),\quad Y(x)=[y_i(\xi_j(x))]_j^i,\quad x\in I,$$

$$(7)\qquad z_{n+1}(t)=(\vec{q}(t),\vec{z}(t))+(Q(t),Z(t)),\quad Z(t)=[z_i(\eta_j(t))]_j^i,\quad t\in J.$$

In accordance with Definition 1 (see [2], [5], [7], [8]), an equation (6) is globally transformable into an equation (7) with respect to the transformation $z(t)=L(t)y(\varphi(t))$, if $L\in C^{n+1}(J)$, $L(t)\neq 0$ on J, φ is a C^{n+1} diffeomorphism of J onto I and the function z is a solution of (7) whenever y is a solution of (6). Moreover,

$$(8)\qquad \xi_j(\varphi(t))=\varphi(\eta_j(t)),\quad j=1,2,\dots,m;\ t\in J$$

is true for the deviations $\xi_j(x)$, $\eta_j(t)$ and the function φ.

Theorem 2. *Let $m,n,r\in\mathbb{N}$ and $r\geqslant n+1$. Let L, φ satisfy the assumptions of Lemma* 1. *Then* (6) *is globally transformable into* (7) *by means of a transformation $z(t)=L(t)y(\varphi(t))$ if and only if the function φ is a solution of*

$$\xi_j(\varphi(t))=\varphi(\eta_j(t)),\quad j=1,2,\dots,m;\ t\in J,\ \varphi(J)=I,$$

and

$$(9)\qquad \vec{a}_{n+1}(t)=A^T(t)\vec{q}(t)-a_{n+1n+1}(t)\vec{p}(\varphi(t));$$

$$(10)\qquad A^T(\eta_j(t))\vec{q}_j(t)=a_{n+1n+1}(t)\vec{p}_j(\varphi(t)),\ j=1,2,\dots,m$$

are satisfied on J, where $a_{ij}(t)$ are the functions defined by Lemma 1, *$\vec{a}_{n+1}(t)=[a_{n+10}(t),a_{n+11}(t),\dots,a_{n+1n}(t)]^T$, $\vec{p}_j(\varphi(t))$ and $\vec{q}_j(t)$ denotes the j-th column of the matrix $P(\varphi(t))$ and $Q(t)$ respectively.*

Proof. By using Lemma 1 we have

$$z_i(t)=z^{(i)}(t)=\sum_{j=0}^{i}a_{ij}(t)y^{(j)}(\varphi(t))=\sum_{j=0}^{i}a_{ij}(t)y_j(\varphi(t)),i\in\{0,1,\dots,n+1\},\ t\in J.$$

Thus

$$\vec{z}(t)=[z_i(t)]_{i=0}^n=A(t)\vec{y}(\varphi(t))$$

and for $\xi_j(\varphi(t))=\varphi(\eta_j(t))$ we get

$$\vec{z}(\eta_j(t))=A(\eta_j(t))\vec{y}(\varphi(\eta_j(t)))=A(\eta_j(t))\vec{y}(\xi_j(\varphi(t)));\quad j=1,2,\dots,m;\ t\in J.$$

Then

$$\begin{aligned} z_{n+1}(t) &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) y_{n+1}(\varphi(t)) \\ &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t)(\vec{p}(\varphi(t)), \vec{y}(\varphi(t))) \\ &\quad + a_{n+1n+1}(t)(P(\varphi(t)), Y(\varphi(t))) \\ &= (\vec{a}_{n+1}(t) + a_{n+1n+1}(t)\vec{p}(\varphi(t)), \vec{y}(\varphi(t))) \\ &\quad + \sum_{j=1}^{n} (a_{n+1n+1}(t)\vec{p}_j(\varphi(t)), \vec{y}(\xi_j(\varphi(t)))), \quad t \in J \end{aligned}$$

where $\vec{p}_j$ denotes the j-th column of the matrix P. Hence

$$\begin{aligned} z_{n+1}(t) &= (\vec{q}(t), \vec{z}(t)) + (Q(t), Z(t)) = (\vec{q}(t), \vec{z}(t)) + \sum_{j=1}^{m} (\vec{q}_j(t), \vec{z}(\eta_j(t))) \\ &= (\vec{q}(t), A(t)\vec{y}(\varphi(t))) + \sum_{j=1}^{m} (\vec{q}_j(t), A(\eta_j(t))\vec{y}(\xi_j(\varphi(t)))) \\ &= (A^T(t)\vec{q}(t), \vec{y}(\varphi(t))) + \sum_{j=1}^{m} (A^T(\eta_j(t))\vec{q}_j(t), \vec{y}(\xi_j(\varphi(t)))). \end{aligned}$$

Comparison of the last two expressions yields

$$\begin{aligned} &(\vec{a}_{n+1}(t) + a_{n+1n+1}(t)\vec{p}(\varphi(t)) - A^T(t)\vec{q}(t), \vec{y}(\varphi(t))) \\ &\qquad + \sum_{j=1}^{m} (a_{n+1n+1}(t)\vec{p}_j(\varphi(t)) - A^T(\eta_j(t))\vec{q}_j(t), \vec{y}(\xi_j(\varphi(t)))) = 0, \quad t \in J. \end{aligned}$$

Thus

$$\begin{aligned} \vec{a}_{n+1}(t) &= A^T(t)\vec{q}(t) - a_{n+1n+1}(t)\vec{p}(\varphi(t)), \\ A^T(\eta_j(t))\vec{q}_j(t) &= a_{n+1n+1}(t)\vec{p}_j(\varphi(t)) \end{aligned}$$

on J and the assertion of Theorem 2 is proved. □

Corollary 3. *If* (6) *is globally transformable into* (7) *by means of the transformation* (3), *then the* $(n+1)$*-st derivatives* ($n \in \mathbb{N}$) *of the functions* L, φ *satisfy the* *differential equations*

$$
\begin{aligned}
L^{(n+1)}(t) &= h(t,\varphi(t),\varphi'(t),L(t),\dots,L^{(n)}(t)) \\
&= q_0(t)L(t)+\ldots+q_n(t)L^{(n)}(t)-L(t)\varphi'(t)^{n+1}p_0(\varphi(t)); \\
\varphi^{(n+1)}(t) &= g(t,\varphi(t),\dots,\varphi^{(n)}(t),L(t),\dots,L^{(n)}(t)) \\
&= \frac{1}{L(t)}\sum_{k=1}^{n}(a_{k1}(t)q_k(t)-\binom{n+1}{k}L^{(k)}(t)\varphi^{(n+1-k)}(t))-p_1(\varphi(t))\varphi'(t)^{n+1}, \\
a_{k1}(t) &= a_{k1}(L,\dots,L^{(k-1)},\varphi',\dots,\varphi^{(k)})(t); \quad t\in J.
\end{aligned}
$$

Moreover,

$$\varphi'(t)=G(t,\varphi(t),L(t),[L^{(i)}(\eta_j(t))]_{j=1,\dots,m}^{i=0,\dots,n}), \quad t\in J.$$

We obtain the assertion using the first two equations in (9) *and the first equation in* (10) *together with the definitions of the functions* $a_{ij}(t)$ *in Lemma* 1.

Corollary 4. *The transformation* (3) *is a stationary transformation of* (6) *if and only if the functions* L, $\varphi\in C^{n+1}(I)$ *satisfy the conditions*

$$
\begin{aligned}
\xi_j(\varphi(x)) &= \varphi(\xi_j(x)),\varphi(I)=I, \\
\vec{a}_{n+1}(x) &= A^T(x)\vec{p}(x)-a_{n+1n+1}(x)\vec{p}(\varphi(x)), \\
A^T(\xi_j(x))\vec{p}_j(x) &= a_{n+1n+1}(x)\vec{p}_j(\varphi(x)), \quad x\in I,
\end{aligned}
$$

where the functions a_{ij} *are defined by Lemma* 1 *and* $\vec{p}_j$ *denotes the* j*-th column of the matrix* P*;* $i,j\in\{0,1,\dots,n+1\}$, $x\in I$.

Remark 2 (see [7]; pp. 355, 357.). In a situation when the deviating arguments in equation (6) are constant deviations

$$\xi_j(x)=x-c_j, c_j\in\mathbb{R}-\{0\}; \quad j=1,2,\dots,m,$$

the condition $\xi_j(\varphi(t))=\varphi(\eta_j(t))$ become a system of Abel equations

$$\varphi(\eta_j(t))=\varphi(t)-c_j; \quad j=1,2,\dots,m.$$

When the deviating arguments in (7) are

$$\eta_j(t)=t-d_j, \quad d_j\in\mathbb{R}-\{0\},$$

then we get

$$\varphi(t-d_j)=\varphi(t)-c_j; \quad j=1,2,\dots,m.$$

If we require that the delayed arguments be converted into delayed ones (or the advanced into advanced), then we need $\varphi'(t) > 0$, $t \in J$. Let d_j/d_k be irrational for a pair $j, k \in \{1, 2, \dots, m\}$. Then for each fixed $j \in \{1, 2, \dots, m\}$, the Abel equation $\varphi(t - d_j) = \varphi(t) - c_j$ has a general solution $\varphi \in C^{n+1}(J)$, $\varphi'(t) > 0$, $\varphi(J) = I$, of the form

$$\varphi(t) = \frac{c_j}{d_j} t + k, \quad k \in \mathbb{R}.$$

For the existence of a simultaneous solution φ it is then sufficient and necessary to have $\varrho = c_j/d_j$ (a constant not depending on φ) for all $j \in \{1, 2, \dots, m\}$.

Example. Consider two equations

$$y''(x) = \left(\begin{bmatrix} p_0 \\ p_1 \end{bmatrix}, \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} \right) + \left(\begin{bmatrix} p_{01} & p_{02} \\ p_{11} & p_{12} \end{bmatrix}, \begin{bmatrix} y(x - c_1) & y(x - c_2) \\ y'(x - c_1) & y'(x - c_2) \end{bmatrix} \right),$$
$$x \in I = [a, \infty)$$

and

$$z''(t) = \left(\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}, \begin{bmatrix} z(t) \\ z'(t) \end{bmatrix} \right) + \left(\begin{bmatrix} q_{01} & q_{02} \\ q_{11} & q_{12} \end{bmatrix}, \begin{bmatrix} z(t - d_1) & z(t - d_2) \\ z'(t - d_2) & z'(t - d_2) \end{bmatrix} \right),$$
$$t \in J = [b, \infty)$$

with constant coefficients and deviations; c_1, c_2, d_1, $d_2 \in \mathbb{R} - \{0\}$, $\frac{c_1}{d_1} > 0$, $\frac{d_1}{d_2}$ being irrational. Due to Theorem 2 they are equivalent with respect to the transformation (3) if and only if

$$\varphi(t - d_j) = \varphi(t) - c_j,$$
$$\begin{bmatrix} a_{20}(t) \\ a_{21}(t) \end{bmatrix} = \begin{bmatrix} a_{00}(t) & a_{10}(t) \\ 0 & a_{11}(t) \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} - a_{22}(t) \begin{bmatrix} p_0 \\ p_1 \end{bmatrix},$$
$$\begin{bmatrix} a_{00}(t - d_j) & a_{10}(t - d_j) \\ 0 & a_{11}(t - d_j) \end{bmatrix} \cdot \begin{bmatrix} q_{0j} \\ q_{1j} \end{bmatrix} = a_{22}(t) \begin{bmatrix} p_{0j} \\ p_{1j} \end{bmatrix},$$

$j = 1, 2$, where $a_{00}(t) = L(t)$, $a_{10}(t) = L'(t)$, $a_{11}(t) = L(t)\varphi'(t)$, $a_{20}(t) = L''(t)$, $a_{21}(t) = 2L'(t)\varphi'(t) + L(t)\varphi''(t)$, $a_{22}(t) = L(t)\varphi'(t)^2$, $t \in J$. In accordance with Remark 2 we have $\varphi(t) = \varrho t + k$, $\varrho = \frac{c_1}{d_1} = \frac{c_2}{d_2} > 0$, $k \in \mathbb{R} - \{0\}$ and $\varphi(t) = \varrho \cdot (t - b) + a$ for $\varphi(J) = I$.

Then

$$L''(t) = q_0 L(t) + q_1 L'(t) - p_0 \varrho^2 L(t),$$
$$2L'(t) = (q_1 - p_1 \varrho) L(t),$$
$$q_{0j} L(t - d_j) + q_{1j} L'(t - d_j) = p_{0j} \varrho^2 L(t),$$
$$q_{1j} L(t - d_j) = p_{1j} \varrho L(t), \qquad j = 1, 2; t \in J.$$

Using the second equation we obtain $L(t) = c\exp\{kt\}$, $k = \frac{q_1 - p_1\varrho}{2}$, $c \in \mathbb{R} - \{0\}$, and

$$\begin{aligned} k^2 &= q_0 + q_1 k - p_0\varrho^2, \\ q_{1j} &= p_{1j}\varrho\exp\{kd_j\}, \\ q_{0j} + q_{1j}k &= p_{0j}\varrho^2\exp\{kd_j\},\ j = 1, 2. \end{aligned}$$

Thus the equations are equivalent with respect to the transformation $z(t) = L(t)y(\varphi(t))$ if and only if there exists $\varrho \in \mathbb{R}$, $\varrho > 0$, such that $c_j = \varrho d_j$,

$$\begin{aligned} q_1^2 + 4q_0 &= \left(p_1^2 + 4p_0\right)\varrho^2, \\ q_{0j} &= \varrho\Big(p_{0j}\varrho - \frac{1}{2}p_{1j}(q_1 - p_1\varrho)\Big)\exp\Big\{\frac{q_1 - p_1\varrho}{2}d_j\Big\}, \\ q_{1j} &= p_{1j}\varrho\exp\Big\{\frac{q_1 - p_1\varrho}{2}d_j\Big\},\ j = 1, 2. \end{aligned}$$

For the functions L, φ we have

$$L(t) = c\exp\Big\{\frac{q_1 - p_1\varrho}{2}t\Big\}, \quad \varphi(t) = \varrho(t - b) + a; \quad c \in \mathbb{R} - \{0\}.$$

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