Czechoslovak Mathematical Journal

Radomír Halaš

Relative polars in ordered sets

Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 2, 415–429

Persistent URL: http://dml.cz/dmlcz/127580

Czechoslovak Mathematical Journal, 50 (125) (2000), 415–429

RELATIVE POLARS IN ORDERED SETS

Radomír Halaš, Olomouc

(Received December 29, 1997)

Abstract. In the paper, the notion of relative polarity in ordered sets is introduced and the lattices of R-polars are studied. Connections between R-polars and prime ideals, especially in distributive sets, are found.

Keywords: Ordered set, distributive set, ideal, prime ideal, R-polar, annihilator

MSC 2000: 06A99

Introduction

Polarity is a useful tool for studying the properties of many mathematical structures. For example (see [1]), in the theory of lattice ordered groups (that means groups endowed with a lattice order relation compatible with the binary group operation), the normality of polars yields that a given lattice ordered group belongs to the variety generated by linearly ordered groups, etc. Polars (and their generalizations) have been studied also for lattices in [16], semilattices in [17], and for the general case in [24]. This paper is a generalization of results obtained in [15].

The notion of a distributive ordered set was introduced in [6] and [18] and the theory of such ordered sets has been recently intensively developed.

In the paper, R-polars of ordered sets are defined and some structural properties of them are found. Especially, R-polars in distributive ordered sets in connection with prime and minimal prime ideals are studied.

In the theory of ordered sets, the problem of their classifications is very important. The study of order varieties due to D. Duffus and I. Rival [5] represents one of the possibilities. A generalization of the classification used in the lattice theory, where classes of lattices are determined by conditions concerning lattice terms (varieties, quasivarieties), is another possibility.

In the theory of lattices, formulations of such conditions are based on using the binary lattice operations join and meet that are determined by the order relation, and this makes it possible to study lattices as special cases of algebras. However, these binary operations are not defined for ordered sets in general. Nevertheless, many of conditions imposed on lattices can be reformulated also for arbitrary sets if one uses the lower and upper cones of subsets instead of the lattice operations.

Definition. Let $S = (S, \leqslant)$ be an ordered set and let $A \subseteq S$. Then the *upper cone* (*lower cone*) of A in S is the set $U(A)$ ($L(A)$) such that

$$U(A) = \{x \in S;\ a \leqslant x \text{ for each } a \in A\} \text{ and, dually,}$$
$$L(A) = \{x \in S;\ x \leqslant a \text{ for each } a \in A\}.$$

If $A = \{a_1, \dots, a_n\}$ is a finite subset of S, then we will write briefly $U(A) = U(a_1, \dots, a_n)$ and $L(A) = L(a_1, \dots, a_n)$. If $A, B \subseteq S$ then we put $U(A, B) = U(A \cup B)$ and $L(A, B) = L(A \cup B)$. It is reasonable to set $L(\emptyset) = U(\emptyset) = S$.

If, for instance, $B = \{b\}$, then we write $U(A, b)$ instead of $U(A, \{b\})$, etc. To simplify expressions, we use $LU(A)$ instead of $L(U(A))$ and similarly, $UL(A)$ instead of $U(L(A))$.

Using the LU language, the notions of distributive and modular ordered sets have been introduced in [18]. (For distributivity see also [6].)

Definition. An ordered set S is called
a) *distributive* if $\forall\ a, b, c \in S$; $L(U(a, b), c) = LU(L(a, c), L(b, c))$;
b) *modular* if $\forall$, $a, b, c \in S$; $a \leqslant c \Rightarrow L(U(a, b), c) = LU(a, L(b, c))$.

Both notions are self-dual. Moreover, by [14], the distributive law holds not only for any triple a, b, c in S but, more generally, also for any elements $a, b_1, \dots, b_n$ in S ($n \in \mathbb{N}$),

$$L(a, U(b_1, \dots, b_n)) = LU(L(a, b_1), \dots, L(a, b_n))$$

and, dually,

$$U(a, L(b_1, \dots, b_n)) = UL(U(a, b_1), \dots, U(a, b_n))$$

are satisfied, see e.g. [14].

If S is a lattice then S is distributive (modular) as a lattice if and only if it is distributive (modular) as an ordered set.

The distributive and modular ordered sets were characterized in [4] by means of forbidden subsets. The results in this direction were further developed in [20] and [21] for the case of semilattices using forbidden subsemilattices.

Many results formulated in the language of upper and lower cones have been obtained for ordered sets in general, especially, for distributive ordered sets and their classes (Boolean, pseudocomplemented, Stone ordered sets) for example in [8]–[15], [2], [3], [19], [22], [23]. (See also below.)

Definition. A subset $I \subseteq S$ of an ordered set S is called an *ideal* if

$$LU(x,y) \subseteq I \text{ whenever } x, y \in I.$$

Remark. a) If S is a lattice then $\emptyset \neq I \subseteq S$ is an ideal in the ordered set S if and only if I is an ideal in the lattice S.

b) If an ordered set has no least element then the empty subset $\emptyset$ is an ideal in S.

Definition. If S is an ordered set then

a) $I \subseteq S$ is called an *s-ideal* if

$$LU(M) \subseteq I \text{ for every finite subset } M \subseteq I;$$

b) an ideal $I \subseteq S$ is called a *prime ideal* if $S \neq I \neq \emptyset$ and if

$$L(x,y) \subseteq I \text{ implies } x \in I \text{ or } y \in I.$$

Remark. If S is a lattice, then the notions of an ideal and an s-ideal coincide for $\emptyset \neq I \neq S$.

Properties and mutual relations among such types of ideals have been studied in detail in [15].

Example 0.1. Let S be an ordered set with the diagram in Figure 0.1 (see also [15]). Then $I = \{a,b,c\}$ is an ideal of S that is not an s-ideal because of $LU(a,b,c) = S \not\subseteq I$.

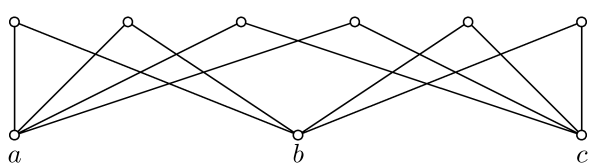

Fig. 0.1

Let us denote by $\mathrm{Id}(S)$ the set of all ideals of S and by $S\,\mathrm{Id}(S)$ the set of all s-ideals of S. Both $(\mathrm{Id}(S), \subseteq)$ and $(S\,\mathrm{Id}(S), \subseteq)$ are complete lattices with the least element $\emptyset$ and the greatest element S in which meets coincide with set intersections. These lattices are algebraic and also constructions of joins are described (see e.g. [15], [19]).

If S is an ordered set and $a, b \in S$, then the set

$$\langle a, b\rangle = \{x \in S;\ UL(a, x) \supseteq U(b)\}$$

is called the *annihilator* in S *defined by the ordered pair* (a, b).

Remark. a) It is evident that an element $x \in S$ belongs to an annihilator $\langle a, b\rangle$ if and only if

$$\forall\ w \in S\colon\ (w \leqslant a, w \leqslant x) \Rightarrow w \leqslant b.$$

This means, if S is a $\wedge$-semilattice then $x \in \langle a, b\rangle$ if and only if $b \geqslant a \wedge x$.

b) If $\langle a_\gamma, b_\gamma\rangle \in S$, $\gamma \in \Gamma \neq \emptyset$, is a family of annihilators in S, then the set intersection of this family need not be an annihilator in S.

Definition. A subset $C \subseteq S$ is called an *indexed annihilator* in S if C is the intersection of a family of annihilators in S.

Remark. $C \subseteq S$ is an indexed annihilator in S if and only if there exist elements $a_\gamma, b_\gamma \in S$, $\gamma \in \Gamma \neq \emptyset$, such that

$$C = \{z \in S;\ UL(z, a_\gamma) \supseteq U(b_\gamma) \text{ for each } \gamma \in \Gamma\}.$$

Let $IA(S)$ denote the set of all indexed annihilators in S. In [3, Theorem 1] it is proved that $(IA(S), \subseteq)$ is a complete lattice with the greatest element S where meets coincide with set intersections. By [3, Theorem 5], the complete lattice $IA(S)$ is pseudocomplemented with the pseudocomplement

$$A^* = \{x \in S;\ UL(a, x) = S \text{ for each } a \in A\}$$

for every $A \in IA(S)$.

Definition. Let S be an ordered set, $x, y \in S$ and $R \subseteq S$. Then x and y are called *R-orthogonal* (notation $x \perp_R y$) whenever $L(x, y) \subseteq LU(R)$. In a special case $R = L(S)$ we call these two elements *orthogonal* and denote the fact by the symbol $\perp$, see [15].

Remark. a) The definition of orthogonality can be also reformulated as follows:

(i) If S has the least element 0 then $x \perp y$ if and only if $\inf\{x, y\}$ exists and $\inf\{x, y\} = 0$.

(ii) If S is lower unbounded then $x \perp y$ if and only if $L(x, y) = \emptyset$.

b) Many authors have also studied ordered sets with orthogonality as generalizations of ortholattices. That is, a system $S = (S, \leqslant, 1, \delta)$ is called an *ordered set with orthogonality* if $(S, \leqslant)$ is an ordered set, 1 is the greatest element in S and $\delta\colon S \to S$ is a mapping that assigns to any element $s \in S$ an element $s^\delta \in S$ such that

1. $\forall\ s \in S;\ (s^\delta)^\delta = s$,
2. $\forall\ s, t \in S;\ s \leqslant t \Rightarrow s^\delta \geqslant t^\delta$,
3. $\forall\ s \in S\ \exists\ \sup\{s, s^\delta\}$ and $\sup\{s, s^\delta\} = 1$.

In this case there exists also $\inf\{s, s^\delta\}$ and $\inf\{s, s^\delta\} = 0$. Then s^δ is called the *δ-orthogonal complement* of s. Further, an element $z \in S$ is called *δ-orthogonal* to $s \in S$ (notation $z\delta s$) if $z \leqslant s^\delta$. It is obvious that if S is an ordered set with orthogonality then $L(x^\delta) \subseteq x^\perp$ (see definition below). But the notion of orthogonal elements is more general than that of δ-orthogonal elements. It is applicable also to ordered sets without 0 and 1 and the orthogonal elements to s need not form the lower cone of any element.

From now on, let R be an arbitrary but fixed subset of S.

Definition. If S is an ordered set and $X \subseteq S$, then the set $X^\perp_R = \{y \in S;\ y \perp_R x$ for all $x \in X\}$ is called the *R-polar* of X (or the polar of X relative to R) in S. In a special case $R = L(S)$ the R-polar is called simply the *polar*. Properties of polars were in detail investigated in [15].

For $x \in S$ we will write $x^\perp_R$ instead of $\{x\}^\perp_R$.

Definition. A subset $X \subseteq S$ is called an *R-polar* in S if there exists $Y \subseteq S$ such that $X = Y^\perp_R$.

Let us note that for $X \subseteq S$ we have $X^\perp_R = S$ if and only if $X \subseteq LU(R)$.

Let us denote the set of all R-polars in S by $\mathrm{Pol}_R(S)$. It is evident that $X \in \mathrm{Pol}_R(S)$ if and only if $X = (X^\perp_R)^\perp_R = X^{\perp\perp}_R$.

Theorem 1.1. *If S is an ordered set, then $\mathrm{Pol}_R(S)$ forms, with respect to set inclusion, a complete lattice in which meets coincide with set intersections and where joins satisfy: if $X, Y \in \mathrm{Pol}_R(S)$, then*

$$X \vee Y = (X_R^{\perp} \cap Y_R^{\perp})_R^{\perp}.$$

Proof. If $X_\alpha \in \mathrm{Pol}_R(S)$, $\alpha \in \Lambda$, then

$$\bigcap\{X_\alpha;\ \alpha \in \Lambda\} = \Big(\bigcup\{X_\alpha;\ \alpha \in \Lambda\}\Big)_R^{\perp}.$$

Clearly, $S = \emptyset_R^{\perp} \in \mathrm{Pol}_R(S)$. It is easy to prove that $^{\perp\perp}$ is a closure operator on S and that the closed sets are precisely the R-polars. □

If $X \subseteq S$, denote by $A(X)$ the indexed annihilator generated by X. Recall the construction of $A(X)$ shown in [3, Construction]. Let $a \in S$. Consider the set $B_a = \{b_{\gamma_a} \in S;\ UL(a,x) \supseteq U(b_{\gamma_a})$ for each $x \in X\}$ and denote $B_a = \{b_{\gamma_a};\ \gamma_a \in \Gamma_a\}$.

Lemma 1.2 ([3, Construction])**.** *Let $X \subseteq S$ and let $A_a = \bigcap\{\langle a, b_{\gamma_a}\rangle;\ \gamma_a \in \Gamma_a\}$ for each $a \in S$. Then $A(X) = \bigcap\{A_a;\ a \in X\}$.*

Now, we will show that every R-polar is the R-polar of some indexed annihilator. Namely, we have

Theorem 1.3. *If $X \subseteq S$, then $X_R^{\perp} = (A(X))_R^{\perp}$.*

Proof. Since $X \subseteq (X)$, $X_R^{\perp} \supseteq (A(X))_R^{\perp}$. Let now $w \in X_R^{\perp}$ be an arbitrary element. Then $L(w,x) \subseteq LU(R)$ for each $x \in X$. Consider an arbitrary element $z \in A(X)$. By Lemma 1.2 we have $z \in \langle a, b_{\gamma_a}\rangle$ for each $a \in S$ and each $b_{\gamma_a} \in B_a = \{y \in S;\ L(a,x) \subseteq L(y)$ for each $x \in X\}$. If we put $a = w$, then $B_w = \{y \in S;\ L(w,x) \subseteq L(y)$ for all $x \in X\}$. Consider an arbitrary element $q \in U(R)$. Then $L(w,x) \subseteq LU(R) \subseteq L(q)$ for each $x \in X$, so $B_w \supseteq L(R)$. But this means that $z \in \langle w, q\rangle$ for each $q \in U(R)$, so $L(z,w) \subseteq L(q)$, and thus $L(z,w) \subseteq \bigcap\{L(q);\ q \in U(R)\} = LU(R)$. This yields $w \perp_R z$, and so $w \in (A(X))_R^{\perp}$. □

Lemma 1.4. *Every subset of S of the form $LU(R)$ is an indexed annihilator. If $X \subseteq S$ and $A_R(X) = A(X) \vee LU(R)$ in $IA(S)$, then*

$$X_R^{\perp} = (A_R(X))_R^{\perp}.$$

Proof. If $U(R) = \emptyset$ then $LU(R) = L(\emptyset) = S$ and $LU(R)$ is an indexed annihilator. Evidently, if $U(R) \neq \emptyset$ then $LU(R) = \bigcap\{L(z);\ z \in U(R)\}$ and $L(z) =$

$\bigcap\{\langle s,z\rangle;\ s\in S\}$. This implies $LU(R)=\bigcap\{\langle s,z\rangle;\ s\in S, z\in U(R)\}$, so $LU(R)$ is an indexed annihilator again.

Since $LU(R)$ is an indexed annihilator, $A_R(X)\in IA(S)$. It is clear that $A_R(X)=A(X\cup LU(R))$ and $(LU(R))^{\perp}_R=S$, hence by Theorem 1.3,

$$(A_R(X))^{\perp}_R=(A(X\cup LU(R))^{\perp}_R=(X\cup LU(R))^{\perp}_R=X^{\perp}_R\cap(LU(R))^{\perp}_R=X^{\perp}_R.$$

$\square$

Lemma 1.5. *The interval $[LU(R),S]$ in the lattice $IA(S)$ is a pseudocomplemented lattice with the pseudocomplement $B^{\perp}_R$ for $B\in[LU(R),S]$.*

Proof. By Lemma 1.4 the set $A=LU(R)$ belongs to $IA(S)$. Evidently, $B^{\perp}_R=\{y\in S;\ L(y,b)\subseteq A$ for every $a\in A\}\supseteq A$, and, moreover, $B^{\perp}_R=\bigcap\{\langle b,w\rangle;\ b\in B, w\in U(R)\}$, i.e. $B^{\perp}_R\in IA(S)$. Let us show that $B\cap B^{\perp}_R=A$. If $z\in B$, $z\in B^{\perp}_R=\{y\in S;\ L(y,b)\subseteq A$ for every $b\in B\}$, then for $y=z=b$ we have $L(z)\subseteq LU(R)$, so $z\in LU(R)$. If $B\cap C=A$ holds for some $C\in[A,S]$ then for $b\in B$, $c\in C$ we have $L(b,c)\subseteq A$, i.e. $c\in B^{\perp}_R$, $C\subseteq B^{\perp}_R$. $\square$

The following lemma is a direct consequence of [7, Theorem 1.6.4].

Lemma 1.5. *Let $B([LU(R),S])$ be the set of all Boolean elements of the pseudocomplemented lattice $[LU(R),S]$ (that is, the elements of $B([LU(R),S])$ are precisely X^*, $X\in[LU(R),S]$). Then $B([LU(R),S])$ with the operations $X\wedge Y=X\cap Y$,*

$$X\vee Y=(X^*\cap Y^*)^*$$

is a Boolean lattice.

Now we can compare the lattices $\mathrm{Pol}_R(S)$ and $B([LU(R),S])$.

Theorem 1.6. *The lattices $\mathrm{Pol}_R(S)$ and $B([LU(R),S])$ are isomorphic.*

Proof. Define a mapping $f\colon \mathrm{Pol}_R(S)\to B([LU(R),S])$ as follows:

$$\forall\ X\subseteq S;\ f(X^{\perp}_R)=(A_R(X))^*.$$

a) If $X,Y\subseteq S$ and $X^{\perp}_R=Y^{\perp}_R$, then $(A_R(X))^{\perp}_R=(A_R(Y))^{\perp}_R$ by Lemma 1.4 and thus $(A_R(X))^*=(A_R(Y))^*$. Therefore, f is defined correctly.

b) f is evidently surjective.

c) If $X,Y\subseteq S$ and $(A_R(X))^*=(A_R(Y))^*$, then $X^{\perp}_R=(A_R(X))^{\perp}_R=(A_R(Y))^{\perp}_R=Y^{\perp}_R$, hence f is also injective.

d) If $X, Y \subseteq S$, then

$$f(X_R^{\perp} \cap Y_R^{\perp}) = f((X \cup Y)_R^{\perp}) = (A_R(X \cup Y))^* = (A_R(X \cup Y))_R^{\perp} = (X \cup Y)_R^{\perp}$$
$$= (X_R^{\perp} \cap Y_R^{\perp}) = (A_R(X))_R^{\perp} \cap (A_R(Y))_R^{\perp} = f(X_R^{\perp}) \cap f(Y_R^{\perp}).$$

Because the join is defined in both lattices by the meet in the same way, f respects also joins.

Therefore, f is an isomorphism of $\mathrm{Pol}_R(S)$ onto $B([LU(R), S])$. $\square$

Corollary 1.7. *For any ordered set $S, \mathrm{Pol}_R(S)$ is a Boolean lattice.*

2. Polars and prime ideals of distributive ordered sets

In this section we will study R-polars in distributive ordered sets. Nevertheless, although for lattices the distributivities of S and $\mathrm{Id}(S)$ are equivalent, there are distributive ordered sets with non-distributive lattices of ideals (see [14], [15]).

An ordered set S is called *ideal-distributive* if $\mathrm{Id}(S)$ is a distributive lattice. By [14], every ideal-distributive set is distributive. On the other hand, there are distributive sets that are not ideal-distributive.

Example 2.1. Consider a distributive ordered set S with the diagram in Figure 2.1 (see also [15]). Denote $I_1 = L(e')$, $I_2 = \{a, b, c, d\}$, $I_3 = L(d')$. We have $I_1 \supset I_2$, but $I_3 \cap I_1 = \{a, b, c\} = I_3 \cap I_2$, $I_1 \vee I_3 = S = I_2 \vee I_3$, hence S (by [18]) is not even modular, and therefore it is not distributive.

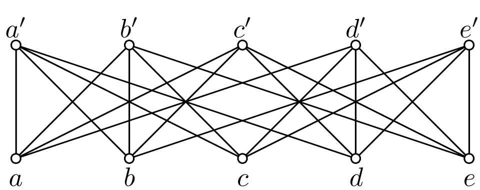

Fig. 2.1

Let us recall the following lemma from [15]:

Lemma 2.1 (see [15]). *Let S be an ideal-distributive set. Then the proper ideal $I \subset S$ is prime if and only if I is a meet-irreducible element of* $\mathrm{Id}(S)$.

Theorem 2.2. *Let S be a distributive set and let $A \subseteq S$. If S is ideal-distributive and $A_R^{\perp} \neq S$ then $A_R^{\perp}$ is equal to the intersection of all prime ideals in S containing $LU(R)$ and not containing A.*

P r o o f. Let $x \in A^{\perp}_{R}$, that is $L(a,x) = LU(R)$ for each $a \in A$. Let $P \supseteq LU(R)$ be a prime ideal in S such that $A \nsubseteq P$. Then there exists an element $a \in A \setminus P$. For a we have $L(a,x) = LU(R) \subseteq P$, and since P is a prime ideal, $x \in P$. Hence $A^{\perp}_{R} \subseteq P$.

Conversely, let $x \notin A^{\perp}_{R}$. Let us show that there exists a prime ideal P_x with $P_x \supseteq LU(R)$, $x \notin P_x$ and $A \nsubseteq P_x$. Since $x \notin A^{\perp}_{R}$, there exists an element $a \in A$ such that $L(a,x) \nsubseteq LU(R)$, which implies the existence of $b \in L(a,x) \setminus LU(R)$ (evidently, $L(b) \neq L(S)$). By Zorn's lemma there exists a maximal ideal I containing $LU(R)$ and not containing b. Let us show that I is a prime ideal. If not, then $I = I_1 \cap I_2$ for some $I_1, I_2 \in \mathrm{Id}(S)$, $I_1, I_2 \supset I$. But then $b \notin I_1$, $b \notin I_2$. By the maximality of I we infer $I_1 = I_2 = S$, so $I = S$, a contradiction. Now, because $b \notin I$, we have $a \notin I$, $x \notin I$, so $I = P_x$ is a prime ideal not containing x and A. $\square$

Theorem 2.3. *Any R-polar in a distributive ordered set is an s-ideal.*

P r o o f. Let $A \subseteq S$, $x_1, \dots, x_n \in A^{\perp}_{R}$, and let $z \in LU(x_1, \dots, x_n)$. Then $L(x_i, a) \subseteq LU(R)$ for each $i \in \{1, \dots, n\}$. Hence the distributivity of S yields

$$\begin{aligned} UL(a,z) &= U(L(a) \cap L(z)) \supseteq U(L(a) \cap LU(x_1, \dots, x_n)) = UL(a, U(x_1, \dots, x_n)) \\ &= ULU(L(a,x_1), \dots, L(a,x_n)) \supseteq ULU(R) = U(R), \end{aligned}$$

thus $L(a,z) = LU(R)$, i.e. $z \in A^{\perp}_{R}$. $\square$

Now we will characterize minimal elements in the set of all prime ideals containing the set $LU(R) \neq S$ in finite ideal-distributive sets. By a minimal prime ideal containing $LU(R)$ we mean the minimal element in the set of all prime ideals containing $LU(R)$.

Lemma 2.4. *Let S be a finite ideal-distributive ordered set. If P is a minimal prime ideal in S containing $LU(R)$, then for any $y \in S$ we have*

$$y \in P \Rightarrow y^{\perp}_{R} \nsubseteq P.$$

P r o o f. Let $y \in P$ and let $y^{\perp}_{R} \subseteq P$. Since S is finite, by Theorem 2.2, $y^{\perp}_{R}$ is the intersection of all prime ideals not containing y and containing $LU(R)$, i.e.

$$y^{\perp}_{R} = \bigcap \{P_i;\ P_i \supseteq LU(R),\ i \in \{1, \dots, n\}\},$$

where P_i, $i \in I$, are all prime ideals in S that do not contain y and contain $LU(R)$.

Hence clearly $\bigcap\{P_i;\ i \in I\} \subseteq P$. The ideal-distributivity implies

$$P = P \vee \Big(\bigcap\{P_i;\ i \in I\}\Big) = \bigcap\{(P \vee P_i;\ i \in I)\}.$$

By Lemma 2.1 any prime ideal is meet-irreducible in $\mathrm{Id}(S)$, thus $P = P \vee P_i$ for some i, therefore $P \supseteq P_i$. But, by assumption, $y \in P$, $y \notin P_i$, hence $P \neq P_i$, a contradiction with the minimality of P. $\square$

Lemma 2.5. *If $P \supseteq LU(R)$ is a prime ideal in an ordered set S with the property $y \in P \Rightarrow y_R^{\perp} \nsubseteq P$, then $P = \bigcup\{x_R^{\perp};\ x \notin P\}$.*

Proof. If $z \in x_R^{\perp}$, where $x \notin P$, then $L(x,z) \subseteq LU(R) \subseteq P$, and since P is a prime ideal, $z \in P$. Therefore $\bigcup\{x_R^{\perp};\ x \notin P\} \subseteq P$.

Conversely, let $p \in P$. Then, by assumption, $p_R^{\perp} \nsubseteq P$. Hence there exists $z \in p_R^{\perp}$ with $z \notin P$. This implies $p \in z_R^{\perp}$ and $z \notin P$, therefore $P \subseteq \{x_R^{\perp};\ x \notin P\}$. $\square$

Lemma 2.6. *Let $P \supseteq LU(R)$ be a prime ideal in an ordered set S. If $P = \bigcup\{x_R^{\perp};\ x \notin P\}$, then P is a minimal prime ideal in S containing $LU(R)$.*

Proof. Suppose that there exist a prime ideal $P_1 \subseteq P$, $P_1 \supseteq LU(R)$, and an element $p \in P \setminus P_1$. Then $p \in x_R^{\perp}$, i.e. $x \in p_R^{\perp}$, for some element $x \notin P$. But P_1 is a prime ideal and $p \notin P_1$, hence $p_R^{\perp} \subseteq P_1$. Thus $x \in p_R^{\perp} \subseteq P_1 \subseteq P$, a contradiction. $\square$

As a direct consequence of Lemmas 2.4, 2.5 and 2.6 we get

Theorem 2.7. *Let S be a finite ideal-distributive ordered set and let $P \supseteq LU(R)$ be a prime ideal in S. Then the following conditions are equivalent:*

(i) *P is a minimal prime ideal containing $LU(R)$;*

(ii) *$P = \bigcup\{x_R^{\perp};\ x \notin P\}$;*

(iii) *if $y \in P$, then $y_R^{\perp} \nsubseteq P$.*

By Theorem 1.3, we know, that $X_R^{\perp} = (A(X))_R^{\perp}$ for any ordered set S and for any $X \subseteq S$, that means any R-polar in S is the polar of an appropriate indexed annihilator. Now, let us show that in the case of distributive ordered sets this result can be simplified.

Lemma 2.8. *If S is a distributive ordered set and $X \subseteq S$, then*

$$X_R^{\perp} = (\mathrm{Id}(X))_R^{\perp} = (S\,\mathrm{Id}(X))_R^{\perp}.$$

P r o o f. By [3, Theorem 2], an ordered set S is distributive if and only if any indexed annihilator in S is an ideal. Hence in our case $A(X)$ is an ideal and clearly $A(X) \supseteq \mathrm{Id}(X) \supseteq X$. This implies

$$X_R^{\perp} \subseteq (\mathrm{Id}(X))_R^{\perp} \subseteq (A(X))_R^{\perp} = X_R^{\perp},$$

so $X_R^{\perp} = (\mathrm{Id}(X))_R^{\perp}$. But by [15], every annihilator is an s-ideal, thus, in the same way, we get $X_R^{\perp} = (S\,\mathrm{Id}(X))_R^{\perp}$. $\square$

Remark. The assertion of Lemma 2.8 need not by valid in any non-distributive ordered set. For instance, an ordered set S depicted in Figure 2.2 is non-distributive and for $X = \{a, b, c\} \subseteq S$ we have $X^{\perp} = \{x\}$, but $(S\,\mathrm{Id}(X))^{\perp} = S^{\perp} = \emptyset$.

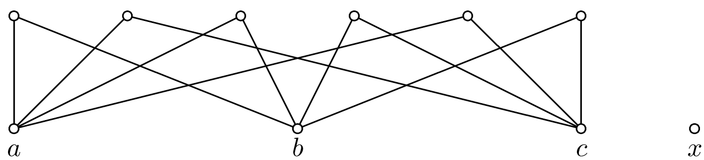

Fig. 2.2

Theorem 2.9. *Let I and J be ideals of an ordered set S. Then*
(i) $(I \cap J)_R^{\perp\perp} = I_R^{\perp\perp} \cap J_R^{\perp\perp}$;
(ii) *if S is distributive, then* $(I \vee_{\mathrm{Id}} J)_R^{\perp\perp} = I_R^{\perp\perp} \vee_{\mathrm{Pol}} J_R^{\perp\perp}$.

P r o o f. (i) Since $I, J \supseteq I \cap J$, we have $(I \cap J)_R^{\perp\perp} \subseteq I_R^{\perp\perp} \cap J_R^{\perp\perp}$. Conversely, let $z \in I_R^{\perp\perp} \cap J_R^{\perp\perp}$, $q \in (I \cap J)_R^{\perp}$, $i \in I$, $j \in J$. Clearly, $L(z, q) \subseteq I_R^{\perp\perp} \cap J_R^{\perp\perp} \cap (I \cap J)_R^{\perp}$ and $L(i, j) \subseteq I \cap J$. Hence we obtain

$$L(z, q, i, j) \subseteq (I \cap J) \cap (I \cap J)_R^{\perp} \subseteq LU(R),$$

so $L(z, q, i, j) \subseteq LU(R)$.

Let r be an arbitrary element in $L(z, q, i)$. Then $L(r) \subseteq L(z, q, i)$ and thus

$$L(r, j) \subseteq L(z, q, i, j) \subseteq LU(R).$$

This means $r \perp_R j$ for any $j \in J$, therefore $r \in J_R^{\perp}$. Further, $r \leqslant z \in J_R^{\perp\perp}$ implies $r \in J_R^{\perp\perp}$, hence $r \in J_R^{\perp} \cap J_R^{\perp\perp}$, so $L(r) \subseteq LU(R)$. This yields

$$L(z, q, i) \subseteq LU(R).$$

Let $m \in L(z,q)$. Then $L(m,i) \subseteq L(z,q,i) \subseteq LU(R)$, thus $m \perp_R i$ and therefore $m \in I_R^{\perp}$. But $m \leqslant z \in I_R^{\perp\perp}$, hence $m \in I_R^{\perp} \cap I_R^{\perp\perp}$, i.e. $L(m) \subseteq LU(R)$. This implies $L(z,q) \subseteq LU(R)$, so $z \perp_R q$. Since q and z are arbitrary, we have

$$(I \cap J)_R^{\perp\perp} \supseteq I_R^{\perp\perp} \cap J_R^{\perp\perp}.$$

(ii) From Lemmas 1.1 and 2.8 and from the fact that $X_R^{\perp\perp\perp} = X_R$ for any $X \subseteq S$ we get

$$I_R^{\perp\perp} \vee_{\mathrm{Pol}} J_R^{\perp\perp} = (I_R^{\perp\perp\perp} \cap J_R^{\perp\perp\perp})_R^{\perp} = (I_R^{\perp} \cap J_R^{\perp})_R^{\perp} = (I \cup J)_R^{\perp\perp} = (I \vee_{\mathrm{Id}} J)_R^{\perp\perp}.$$

□

Corollary 2.10. *If S is a distributive ordered set then the mapping which to any $I \in \mathrm{Id}(S)$ assigns $I_R^{\perp\perp} \in \mathrm{Pol}_R(S)$ is a surjective lattice homomorphism of $\mathrm{Id}(S)$ onto $\mathrm{Pol}_R(S)$.*

Corollary 2.11. *Let S be an ordered set and $a, b \in S$. Then*
(i) $a_R^{\perp\perp} \cap b_R^{\perp\perp} = (L(a,b))_R^{\perp\perp}$;
(ii) *if S is distributive, then* $a_R^{\perp\perp} \vee_{\mathrm{Pol}} b_R^{\perp\perp} = (LU(a,b))_R^{\perp\perp}$.

P r o o f. (i) By Lemma 2.8, $a_R^{\perp\perp} \cap b_R^{\perp\perp} = (L(a))_R^{\perp\perp} \cap L(b))_R^{\perp\perp}$. Hence, by Theorem 2.9,

$$(L(a))_R^{\perp\perp} \cap (L(b))_R^{\perp\perp} = (L(a) \cap L(b))_R^{\perp\perp} = (L(a,b))_R^{\perp\perp}.$$

(ii) By Lemma 2.8, $a_R^{\perp\perp} \vee_{\mathrm{Pol}} b_R^{\perp\perp} = (L(a))_R^{\perp\perp} \vee_{\mathrm{Pol}} (L(b))_R^{\perp\perp}$ and then by Theorem 2.9 we have

$$(L(a))_R^{\perp\perp} \vee_{\mathrm{Pol}} (L(b))_R^{\perp\perp} = (L(a) \vee_{\mathrm{Id}} L(b))_R^{\perp\perp} = (LU(a,b))_R^{\perp\perp}.$$

□

Now, we will examine maximal and minimal R-polars in ideal-distributive ordered sets and their connections with prime ideals.

Theorem 3.1. *Let $I \neq \emptyset$ be a linearly ordered ideal in an ordered set S. Then for every element $a \in I$ we have*

$$a_R^{\perp} \neq S \Rightarrow a_R^{\perp} = I_R^{\perp}.$$

Proof. Clearly, $a_R^{\perp} \supseteq I_R^{\perp}$ for every $a \in I$. Let $x \in a_R^{\perp} \setminus I_R^{\perp}$. Then $L(a, x) \subseteq LU(R)$ and there exists an element $b \in I$ such that $L(b, x) \nsubseteq LU(R)$. Since I is a chain, we have $a < b$ or $b < a$.

For $b < a$ we have $LU(R) = L(a, x) \supseteq L(b, x)$, a contradiction. Hence $a < b$.

Further, there exists $y \in L(b, x)$ such that $y \notin LU(R)$. By assumption $b \in I$, hence also $y \in I$. We have $L(a, y) \subseteq L(a, x) \subseteq LU(R)$. Both a and y belong to I, therefore a and y are comparable, hence $L(a, y) = L(a)$ or $L(a, y) = L(y)$. The first case means $a \in LU(R)$ (i.e. $a_R^{\perp} = I_R^{\perp}$) and the other $y \in LU(R)$, so in both cases we obtain a contradiction. □

Theorem 3.2. *Let S be a distributive ordered set and I an ideal in S such that $a \in I$ and $a_R^{\perp} \neq S$ imply $a_R^{\perp} = I_R^{\perp}$. Then if $I_R^{\perp} \neq S$, $I_R^{\perp}$ is a prime ideal containing $LU(R)$.*

Proof. S is distributive, hence, by Theorem 2.3, $I_R^{\perp}$ is an ideal. Let $x, y \in S$, $L(x, y) \subseteq I_R^{\perp}$ and let $x \notin I_R^{\perp}$. Since $a_R^{\perp} = I_R^{\perp}$, for any $a \in I$ with $a_R^{\perp} \neq S$, we get $x \notin a_R^{\perp}$, that is $L(a, x) \nsubseteq LU(R)$. Hence for every such $a \in I$ there exists $x_a \in L(a, x)$ with $x_a \notin LU(R)$. Evidently $x_a \in I$ and at the same time $x_a \notin I_R^{\perp} = (x_a)_R^{\perp}$. Therefore, since $x_a \leqslant x$ and $L(x, y) \subseteq I_R^{\perp}$, we get $x_a \notin L(y)$.

Now, suppose that $y \notin I_R^{\perp}$. Then $y \notin (x_a)_R^{\perp}$, thus there exist elements $b_a \in L(x_a, y)$ such that $b_a \notin LU(R)$. Hence $b_a \leqslant x_a \leqslant x$, $b_a \leqslant y$, thus $b_a \in L(x, y) \subseteq I_R^{\perp}$. Since $x_a \in I$, we also have $b_a \in I$ and so $b_a \in (b_a)_R^{\perp}$, a contradiction. Therefore $x \in I_R^{\perp}$ or $y \in I_R^{\perp}$. □

Lemma 3.3. *Let S be an ideal-distributive set. If $I \in \mathrm{Id}(S)$ is such that $I_R^{\perp}$ is a prime ideal, then $I_R^{\perp}$ is a minimal prime ideal containing $LU(R)$.*

Proof. By Theorem 2.2, $I_R^{\perp}$ is the intersection of all prime ideals not containing I and containing $LU(R)$. Clearly, $I_R^{\perp}$ is a prime ideal which does not contain I because in this case $I \subseteq LU(R)$ and so $I_R^{\perp} = S$. If J is a prime ideal such that $J \subseteq I_R^{\perp}$ and $J \supseteq LU(R)$, then $I \nsubseteq J$ (otherwise $I \subseteq J \subseteq I_R^{\perp}$), hence $I_R^{\perp} \subseteq J$. This implies $J = I_R^{\perp}$, and so $I_R^{\perp}$ is a minimal prime ideal containing $LU(R)$. □

Lemma 3.4. *If S is an ordered set and $I \in \mathrm{Id}(S)$ is such that $I_R^{\perp}$ is a prime ideal, then $I_R^{\perp}$ is a maximal R-polar.*

P r o o f. Let $I_R^{\perp} \subseteq J_R^{\perp} \neq S$ for some $J \subseteq S$. Let $c \in J_R^{\perp} \setminus I_R^{\perp}$. Then $L(c,b) \subseteq LU(R)$ for every $b \in J$, hence $L(c,b) \subseteq LU(R) \subseteq I_R^{\perp}$. Since $I_R^{\perp}$ is a prime ideal and $c \notin I_R^{\perp}$, we have $b \in I_R^{\perp}$. This yields $J \subseteq I_R^{\perp} \subseteq J_R^{\perp}$, so $J_R^{\perp} = S$, a contradiction. □

Lemma 3.5. *An R-polar $I_R^{\perp}$ is maximal in $\mathrm{Pol}_R(S)$ if and only if $I_R^{\perp\perp}$ is minimal in $\mathrm{Pol}_R(S)$.*

P r o o f. Suppose $I_R^{\perp\perp}$ is not minimal. Let $J_R^{\perp} \in \mathrm{Pol}(S)$, $J_R^{\perp} \neq LU(R)$, be such that $J_R^{\perp} \subseteq I_R^{\perp\perp}$, $J_R^{\perp} \neq I_R^{\perp\perp}$. Let $z \in I_R^{\perp\perp} \setminus J_R^{\perp}$. Then $z \perp_R k$ for every $k \in I_R^{\perp}$, and there exists $j \in J$ with $z \notin j_R^{\perp}$. Clearly, $j \in J_R^{\perp\perp} \setminus I_R^{\perp}$, hence $J_R^{\perp\perp} \supseteq I_R^{\perp}$, $J_R^{\perp\perp} \neq I_R^{\perp}$. Furthermore, $J_R^{\perp\perp} \neq S$, because in the opposite case $J_R^{\perp} = S_R^{\perp} = LU(R)$. Therefore the R-polar $I_R^{\perp\perp}$ is minimal.

The proof of the converse implication is similar. □

The following theorem is a consequence of Theorems 3.1 and 3.2 and Lemmas 3.3, 3.4, 3.5.

Theorem 3.6. *Let S be an ideal-distributive ordered set S and let $I \in \mathrm{Id}(S)$. Let us consider the following conditions:*

(1) *There exists a linearly ordered ideal $J \in \mathrm{Id}(S)$ such that $J_R^{\perp} \neq S$ (i.e. $J \nsubseteq LU(R)$) and $J_R^{\perp} = I_R^{\perp}$.*
(2) *$i_R^{\perp} = I_R^{\perp}$ for any $i \in I$ such that $i_R^{\perp} \neq S$.*
(3) *$I_R^{\perp}$ is a prime ideal.*
(4) *$I_R^{\perp}$ is a minimal prime ideal containing $LU(R)$.*
(5) *$I_R^{\perp}$ is a maximal R-polar.*
(6) *$I_R^{\perp\perp}$ is a minimal R-polar.*

Then (1) *implies* (2) *and the conditions* (2)–(6) *are equivalent.*

References

[1] *Anderson, M., Feil, T.*: Lattice-Ordered Groups (An Introduction). Dordrecht, Reidel, 1987.
[2] *Chajda, I.*: Complemented ordered sets. Arch. Math. (Brno) *28* (1992), 25–34.
[3] *Chajda, I., Halaš, R.*: Indexed annihilators in ordered sets. Math. Slovaca *45* (1995), 501–508.
[4] *Chajda, I., Rachůnek, J.*: Forbidden configurations for distributive and modular ordered sets. Order *5* (1989), 407–423.
[5] *Duffus, D., Rival, I.*: A structure theory for ordered sets. Discrete Math. *35* (1981), 53–110.
[6] *Erné, M.*: Distributivgesetze und die Dedekindsche Schnittverwollständigung. Abh. Braunschweig. Wiss. Ges. *33* (1982), 117–145.

[7] *Grätzer, G.*: Lattice Theory. Akademie Verlag, Berlin, 1978.
[8] *Halaš, R.*: Pseudocomplemented ordered sets. Arch. Math. (Brno) *29* (1993), 153–160.
[9] *Halaš, R.*: Characterization of distributive sets by generalized annihilators. Arch. Math. (Brno) *30* (1994), 25–27.
[10] *Halaš, R.*: Decompositions of directed sets with zero. Math. Slovaca *45* (1995), 9–17.
[11] *Halaš, R.*: Ideals and annihilators in ordered sets. Czechoslovak Math. J. *45(120)* (1995), 127–134.
[12] *Halaš, R.*: Some properties of Boolean ordered sets. Czechoslovak Math. J. *46(121)* (1996), 93–98.
[13] *Halaš, R.*: A characterization of finite Stone PC-ordered sets. Math. Bohem. *121* (1996), 117–120.
[14] *Halaš, R.*: Annihilators and ideals in distributive and modular ordered sets. Acta Univ. Palack. Olomouc. Fac. Rerum Natur. Math. *34* (1995), 31–37.
[15] *Halaš, R., Rachůnek, J.*: Polars and prime ideals in ordered sets. Discuss. Math., Algebra and Stochastic Methods *15* (1995), 43–59.
[16] *Jakubík, J.*: M-polars in lattices. Čas. Pěst. Mat. *95* (1970), 252–255.
[17] *Katriňák, T.*: M-Polaren in halbgeordneten Mengen. Čas. Pěst. Mat. *95* (1970), 416–419.
[18] *Larmerová, J., Rachůnek, J.*: Translations of modular and distributive ordered sets. Acta Univ. Palack. Olomouc. Fac. Rerum Natur. Math. *27(91)* (1988), 13–23.
[19] *Niederle, J.*: Boolean and distributive ordered sets: Characterization and representation by sets (preprint).
[20] *Rachůnek, J.*: A characterization of o-distributive semilattices. Acta Sci. Math. (Szeged) *54* (1990), 241–246.
[21] *Rachůnek, J.*: On o-modular and o-distributive semilattices. Math. Slovaca *42* (1992), 3–13.
[22] *Rachůnek, J.*: The ordinal variety of distributive ordered sets of width two. Acta Univ. Palack. Olomouc. Fac. Rerum Natur. Math. *30(100)* (1991), 17–32.
[23] *Rachůnek, J.*: Non-modular and non-distributive ordered sets of lattices. Acta Univ. Palack. Olomouc. Fac. Rerum Natur. Math. *32(110)* (1993), 141–149.
[24] *Šik, F.*: A characterization of polarities the lattice of polars of which is Boolean. Czechoslovak Math. J. *91(106)* (1981), 98–102.

Author's address: Katedra algebry a geometrie PF UP, Tomkova 40, 779 00 Olomouc, Czech Republic.