Jong Yeoul Park; Jeong Ja Bae

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ON SOLUTIONS OF QUASILINEAR WAVE EQUATIONS WITH NONLINEAR DAMPING TERMS

Jong Yeoul Park and Jeong Ja Bae, Pusan

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Abstract. In this paper we consider the existence and asymptotic behavior of solutions of the following problem:

$$\begin{aligned}
&u_{tt}(t,x) - (\alpha + \beta\|\nabla u(t,x)\|_2^2 + \beta\|\nabla v(t,x)\|_2^2)\Delta u(t,x) + \delta|u_t(t,x)|^{p-1}u_t(t,x)\\
&\quad = \mu|u(t,x)|^{q-1}u(t,x), \quad x \in \Omega, \quad t \geqslant 0,\\
&v_{tt}(t,x) - (\alpha + \beta\|\nabla u(t,x)\|_2^2 + \beta\|\nabla v(t,x)\|_2^2)\Delta v(t,x) + \delta|v_t(t,x)|^{p-1}v_t(t,x)\\
&\quad = \mu|v(t,x)|^{q-1}v(t,x), \quad x \in \Omega, \quad t \geqslant 0,\\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega,\\
&v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad x \in \Omega,\\
&u|_{\partial\Omega} = v|_{\partial\Omega} = 0
\end{aligned}$$

where $q > 1$, $p \geqslant 1$, $\delta > 0$, $\alpha > 0$, $\beta \geqslant 0$, $\mu \in \mathbb{R}$ and Δ is the Laplacian in $\mathbb{R}^N$.

Keywords: quasilinear wave equation, existence and uniqueness, asymptotic behavior, Galerkin method

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In this paper we consider the existence of solutions of the problem

$$
\begin{aligned}
(1.1)\qquad & u_{tt}(t,x) - (\alpha + \beta\|\nabla u(t,x)\|_2^2 + \beta\|\nabla v(t,x)\|_2^2)\Delta u(t,x) + \delta|u_t(t,x)|^{p-1}u_t(t,x) \\
&\quad = \mu|u(t,x)|^{q-1}u(t,x), \quad x \in \Omega, \quad t \geqslant 0, \\
& v_{tt}(t,x) - (\alpha + \beta\|\nabla u(t,x)\|_2^2 + \beta\|\nabla v(t,x)\|_2^2)\Delta v(t,x) + \delta|v_t(t,x)|^{p-1}v_t(t,x) \\
&\quad = \mu|v(t,x)|^{q-1}v(t,x), \quad x \in \Omega, \quad t \geqslant 0, \\
& u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \\
& v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad x \in \Omega, \\
& u|_{\partial\Omega} = v|_{\partial\Omega} = 0
\end{aligned}
$$

where $q > 1$, $p \geqslant 1$, $\delta > 0$, $\mu \in \mathbb{R}$, $\alpha > 0$, $\beta \geqslant 0$ and Δ is the Laplacian in $\mathbb{R}^N$.

Here $\|u\|_2^2 = \int_\Omega |u(t,x)|^2\,\mathrm{d}x$, $\quad u_t = \frac{\partial u}{\partial t}$ and $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$.

Equation (1.1) has its origin in the nonlinear vibrations of an elastic string (cf. R. Narasimha [6]). Many authors have studied the existence and uniqueness of solutions of (1.1) by using various methods.

When $\delta > 0$ and $\mu = 0$, for the degenerate case, Nishihara and Yamada [7] have proved the global existence of a unique solution under the assumptions that the initial data $\{u_0, u_1\}$ are sufficiently small and $u_0 \neq 0$. For the problem with linear damping δu_t, there are works of Brito [1], Ikehata [3], Ono [8] and the references therein. In the present paper we will study the existence and uniqueness of solutions of the unilateral problem (1.1) by using the Galerkin method and we will also investigate its asymptotic behavior.

The content of this paper is as follows: In Section 2, we present the preliminaries and some lemmas. In Section 3, we give the statement of the main theorem. In Section 4, we deal with a priori estimates for solutions of (1.1) and prove our main theorem, while Section 5 deals with the asymptotic behavior of the solutions obtained in Section 4.

2. Preliminaries

We first present the following well known lemmas which will be needed later.

Lemma 2.1 (Sobolev-Poincaré [4]). *If either* $1 \leqslant q < +\infty \quad (N = 1,2)$ *or* $1 \leqslant q \leqslant \frac{N+2}{N-2} \quad (N \geqslant 3)$, *then there is a constant* $C(\Omega, q+1)$ *such that*

$$\|u\|_{q+1} \leqslant C(\Omega, q+1)\|\nabla u\|_2 \quad \textit{for} \quad u \in H_0^1(\Omega).$$

In other words, $C(\Omega, q+1) = \sup\{\frac{\|u\|_{q+1}}{\|\nabla u\|_2}|, u \in H_0^1(\Omega), u \neq 0\}$ *is positive and finite.*

Lemma 2.2 (Gagliardo-Nirenberg [4]). *Let* $1 \leqslant r < q \leqslant +\infty$ *and* $p \leqslant q$. *Then the inequality*

$$\|u\|_{W^{k,q}} \leqslant C\|u\|_{W^{m,p}}^{\theta}\|u\|_r^{1-\theta} \quad \text{for } u \in W^{m,p}(\Omega) \cap L^r(\Omega)$$

holds with some $C > 0$ *and* $\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$ *provided that* $0 < \theta \leqslant 1$ *(we assume* $0 < \theta < 1$ *if* $q = +\infty$*).*

We conclude this section by stating a lemma concerning a difference inequality which will be used later.

Lemma 2.3 (Nakao [5]). *Let* $\varphi(t)$ *be a nonincreasing and nonnegative function on* $[0, T], T > 1$, *such that*

$$\varphi(t)^{1+r} \leqslant k_0(\varphi(t) - \varphi(t+1)) \quad \text{on} \quad [0, T]$$

where k_0 *is a positive constant and* r *a nonnegative constant. Then we have*

(i) *if* $r > 0$, *then*

$$\varphi(t) \leqslant (\varphi(0)^{-r} + k_0^{-1} r[t-1]^+)^{-\frac{1}{r}}, \quad \text{where} \quad [t-1]^+ = \max\{t-1, 0\},$$

(ii) *if* $r = 0$, *then*

$$\varphi(t) \leqslant \varphi(0)\mathrm{e}^{-k_1[t-1]^+} \quad \text{on} \quad [0, T], \quad \text{where} \quad k_1 = \log \frac{k_0}{k_0 - 1}.$$

3. Statement of the result

We consider the initial value problem

$$\begin{aligned}
(3.1) \qquad & u_{tt}(t) - (\alpha + \beta\|\nabla u(t)\|_2^2 + \beta\|\nabla v(t)\|_2^2)\Delta u(t) + \delta|u_t(t)|^{p-1}u_t(t) \\
&\quad = \mu|u(t)|^{q-1}u(t), \quad t \geqslant 0, \\
& v_{tt}(t) - (\alpha + \beta\|\nabla u(t)\|_2^2 + \beta\|\nabla v(t)\|_2^2)\Delta v(t) + \delta|v_t(t)|^{p-1}v_t(t) \\
&\quad = \mu|v(t)|^{q-1}v(t), \quad t \geqslant 0, \\
& u(0) = u_0, \quad u_t(0) = u_1, \\
& v(0) = v_0, \quad v_t(0) = v_1, \quad \text{where} \quad \alpha > 0 \quad \text{and} \quad \beta \geqslant 0.
\end{aligned}$$

Now we set

$$J(u,v)=\frac{\alpha}{2}(\|\nabla u\|_2^2+\|\nabla v\|_2^2)+\frac{\beta}{4}(\|\nabla u\|_2^4+\|\nabla v\|_2^4)-\frac{\mu}{q+1}(\|u\|_{q+1}^{q+1}+\|v\|_{q+1}^{q+1}),$$
$$I(u,v)=\alpha(\|\nabla u\|_2^2+\|\nabla v\|_2^2)-\mu(\|u\|_{q+1}^{q+1}+\|v\|_{q+1}^{q+1})$$

and define the potential as

$$W=\{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)|I(u,v)>0\}\cup\{0\}.$$

Next, by setting

$$E(u(t),v(t))=\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|v_t(t)\|^2+J(u(t),v(t))+\frac{\beta}{2}\|\nabla u(t)\|_2^2\|\nabla v(t)\|_2^2,$$

we can state our main theorem.

Theorem 3.1. *Let N be a positive integer. Suppose that $\delta>0$ and $\mu>0$ and $p<\min\{q,\frac{N+4q-Nq}{2}\}$ is such that*

(i) $1\leqslant p<+\infty$ $(N=1,2)$,
(ii) $1\leqslant p\leqslant 3$, $1<q\leqslant 5$ $(N=3)$,
(iii) $1\leqslant p\leqslant\frac{N}{N-2}$, $\frac{N}{N-2}\leqslant q\leqslant\min\{\frac{N+2}{N-2},\frac{N-2}{[N-4]^+}\}$ $(N\geqslant 4)$.

If $u_0,v_0\in W\cap H^2(\Omega)$, $u_1,v_1\in H_0^1(\Omega)$ and

$$\frac{\mu}{\alpha}[C(\Omega,q+1)]^{q+1}\left(\frac{2(q+1)}{\alpha(q-1)}E(u_0,v_0)\right)^{\frac{q-1}{2}}<1,$$

then the problem (3.1) *has solution $(u,v)=(u(t,x),v(t,x))$ satisfying*

$$u,v\in L^\infty(0,\infty;H_0^1(\Omega)\cap H^2(\Omega)),$$
$$u',v'\in L^\infty(0,\infty;H_0^1(\Omega)),$$
$$u'',v''\in L^\infty(0,\infty;L^2(\Omega)).$$

4. Proof of Theorem 3.1

Throughout this section we always assume that $u_0, v_0 \in W \cap H^2(\Omega)$ and $u_1, v_1 \in H_0^1(\Omega)$. We employ the Galerkin method to construct a solution. Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of eigenvalues for $-\Delta w = \lambda w$ in Ω. Let $w_j \in H_0^1(\Omega) \cap H^2(\Omega)$ be the corresponding eigenfunction to λ_j and take $\{w_j\}_{j=1}^{\infty}$ as a complete orthonormal system in $L^2(\Omega)$. We construct approximate solutions $u_m, v_m \quad (m = 1, 2, \ldots)$ in the form

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j, \quad v_m(t) = \sum_{j=1}^{m} h_{jm}(t) w_j$$

which are determined by the ordinary differential equations

$$\begin{aligned}(u_m''(t), w) - ((\alpha + \beta\|\nabla u_m(t)\|_2^2 + \beta\|\nabla v_m(t)\|_2^2)\Delta u_m(t), w)\\ + \delta|u_m'(t)|^{p-1}(u_m'(t), w) = \mu|u_m(t)|^{q-1}(u_m(t), w),\end{aligned} \tag{4.1}$$

$$\begin{aligned}(v_m''(t), w) - ((\alpha + \beta\|\nabla u_m(t)\|_2^2 + \beta\|\nabla v_m(t)\|_2^2)\Delta v_m(t), w)\\ + \delta|v_m'(t)|^{p-1}(v_m'(t), w) = \mu|v_m(t)|^{q-1}(v_m(t), w)\end{aligned} \tag{4.2}$$

($'= \frac{\partial}{\partial t}$ and $'' = \frac{\partial^2}{\partial t^2}$) with the initial conditions

$$\begin{aligned}u_m(0) = u_{0m} &= \sum_{j=1}^{m} (u_0, w_j) w_j \to u_0 \quad \text{in } H_0^1(\Omega) \cap H^2(\Omega),\\ v_m(0) = v_{0m} &= \sum_{j=1}^{m} (v_0, w_j) w_j \to v_0 \quad \text{in } H_0^1(\Omega) \cap H^2(\Omega),\end{aligned} \tag{4.3}$$

$$\begin{aligned}u_m'(0) = u_{1m} &= \sum_{j=1}^{m} (u_1, w_j) w_j \to u_1 \quad \text{strongly in } H_0^1(\Omega),\\ v_m'(0) = v_{1m} &= \sum_{j=1}^{m} (v_1, w_j) w_j \to v_1 \quad \text{strongly in } H_0^1(\Omega).\end{aligned} \tag{4.4}$$

Therefore we can solve the system (4.1)–(4.4) by Picard's iteration method. Hence the system (4.1)–(4.4) has a unique solution on some interval $[0, T_m)$ with $0 < T_m \leqslant +\infty$. Note that $u_m(t)$ is in the C^2-class. We will see that $u_m(t)$ and $v_m(t)$ can be extended to $[0, \infty)$. We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for u_m and v_m. But this procedure allows us to employ the energy method for smooth solution $(u(t), v(t))$ to the problem (4.1)–(4.4) (the results should be in fact applied to the approximate solutions).

Multiplying the equation (4.1) by $u_m'(t)$ and multiplying the equation (4.2) by $v_m'(t)$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u_m'(t)\|_2^2+\frac{\alpha}{2}\|\nabla u_m(t)\|_2^2+\frac{\beta}{4}\|\nabla u_m(t)\|_2^4-\frac{\mu}{q+1}\|u_m(t)\|_{q+1}^{q+1}\right)\\+\frac{\beta}{2}\|\nabla v_m(t)\|_2^2\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_m(t)\|_2^2+\delta\|u_m'(t)\|_{p+1}^{p+1}=0\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|v_m'(t)\|_2^2+\frac{\alpha}{2}\|\nabla v_m(t)\|_2^2+\frac{\beta}{4}\|\nabla v_m(t)\|_2^4-\frac{\mu}{q+1}\|v_m(t)\|_{q+1}^{q+1}\right)\\+\frac{\beta}{2}\|\nabla u_m(t)\|_2^2\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v_m(t)\|_2^2+\delta\|v_m'(t)\|_{p+1}^{p+1}=0.\end{aligned} \tag{4.6}$$

Adding (4.5) and (4.6) and then integrating from 0 to t yields the energy identity

$$E(u_m(t),v_m(t))+\delta\int_0^t(\|u_m'(s)\|_{p+1}^{p+1}+\|v_m'(s)\|_{p+1}^{p+1})\,\mathrm{d}s=E(u_0,v_0) \tag{4.7}$$

where

$$\begin{aligned}E(u_m(t),v_m(t))=\frac{1}{2}\|u_m'(t)\|_2^2+\frac{1}{2}\|v_m'(t)\|_2^2+\frac{\alpha}{2}\|\nabla u_m(t)\|_2^2\\+\frac{\alpha}{2}\|\nabla v_m(t)\|_2^2+\frac{\beta}{4}\|\nabla u_m(t)\|_2^4+\frac{\beta}{4}\|\nabla v_m(t)\|_2^4\\+\frac{\beta}{2}\|\nabla u_m(t)\|_2^2\|\nabla v_m(t)\|_2^2-\frac{\mu}{q+1}\|u_m(t)\|_{q+1}^{q+1}\\-\frac{\mu}{q+1}\|v_m(t)\|_{q+1}^{q+1}.\end{aligned}$$

In particular, $E(u_m(t),v_m(t))$ is nonincreasing on $[0,\infty)$ and

$$E(u_m(t),v_m(t))\leqslant E(u_0,v_0). \tag{4.8}$$

Now, to obtain a priori estimates, we need the following result.

Lemma 4.1. *Assume that either*

$$1\leqslant q<+\infty\ (N=1,2)\quad \textit{or}\quad 1\leqslant q\leqslant\frac{N+3}{N-2}\ (N\geqslant 3).$$

Let $(u_m(t), v_m(t))$ *be the solution of (4.1)–(4.4) with* $(u_0, v_0) \in W$ *and* $u_1, v_1 \in H_0^1(\Omega)$*. If*

$$\frac{\mu}{\alpha}[C(\Omega, q+1)]^{q+1}\left(\frac{2(q+1)}{\alpha(q-1)}E(u_0, v_0)\right)^{\frac{q-1}{2}} < 1, \tag{4.9}$$

then $(u_m(t), v_m(t)) \in W$ *on* $[0, +\infty)$*, that is,*

$$\alpha(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2) - \mu(\|u_m\|_{q+1}^{q+1} + \|v_m\|_{q+1}^{q+1}) > 0 \quad \textit{on} \quad [0, +\infty).$$

P r o o f. Since $I(u_0, v_0) > 0$, it follows from the continuity of $u_m(t)$ and $v_m(t)$ that

$$I(u_m(t), v_m(t)) \geqslant 0 \quad \text{for some interval near} \quad t = 0. \tag{4.10}$$

Let $t_{\max}$ be a maximal time (possibly $t_{\max} = T_m$) such that (4.10) holds on $[0, t_{\max})$. Note that

$$\begin{aligned} J(u_m(t), v_m(t)) &= \frac{\alpha}{2}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2) + \frac{\beta}{4}(\|\nabla u_m(t)\|_2^4 + \|\nabla v_m(t)\|_2^4) \\ &\quad - \frac{\mu}{q+1}(\|u_m(t)\|_{q+1}^{q+1} + \|v_m(t)\|_{q+1}^{q+1}), \\ &= \frac{1}{q+1}I(u_m(t), v_m(t)) + \frac{\alpha(q-1)}{2(q+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2) \\ &\quad + \frac{\beta}{4}(\|\nabla u_m(t)\|_2^4 + \|\nabla v_m(t)\|_2^4) \\ &\geqslant \frac{\alpha(q-1)}{2(q+1)}(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2) \quad \text{on} \quad [0, t_{\max}). \end{aligned} \tag{4.11}$$

By the energy identity (4.7), (4.8) and (4.11), we have

$$\begin{aligned} \|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2 &\leqslant \frac{2(q+1)}{\alpha(q-1)}J(u_m(t), v_m(t)) \\ &\leqslant \frac{2(q+1)}{\alpha(q-1)}E(u_m(t), v_m(t)) \\ &\leqslant \frac{2(q+1)}{\alpha(q-1)}E(u_0, v_0) \quad \text{on} \quad [0, t_{\max}). \end{aligned} \tag{4.12}$$

It follows from the Sobolev-Poincaré inequality, (4.9) and (4.12) that

$$\begin{aligned}
\mu\|u_m(t)\|_{q+1}^{q+1} &\leqslant \mu C(\Omega,q+1)^{q+1}\|\nabla u_m(t)\|_2^{q+1} \\
&= \frac{\mu}{\alpha}C(\Omega,q+1)^{q+1}\|\nabla u_m(t)\|_2^{q-1}\cdot\alpha\|\nabla u_m(t)\|_2^2 \\
&\leqslant \frac{\mu}{\alpha}C(\Omega,q+1)^{q+1}\left(\frac{2(q+1)}{\alpha(q-1)}E(u_0,v_0)\right)^{\frac{q-1}{2}}\alpha\|\nabla u_m(t)\|_2^2 \\
&\leqslant \alpha\|\nabla u_m(t)\|_2^2 \quad \text{on} \quad [0,t_{\max}).
\end{aligned} \tag{4.13}$$

Similarly,

$$\mu\|v_m(t)\|_{q+1}^{q+1} \leqslant \alpha\|\nabla v_m(t)\|_2^2 \quad \text{on} \quad [0,t_{\max}). \tag{4.14}$$

Thus from (4.13) and (4.14) we obtain

$$\begin{aligned}
&\mu(\|u_m(t)\|_{q+1}^{q+1}+\|v_m(t)\|_{q+1}^{q+1}) \\
&\leqslant \alpha(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2) \quad \text{on} \quad [0,t_{\max}).
\end{aligned} \tag{4.15}$$

Therefore we get $I(u(t),v(t))>0$ on $[0,t_{\max})$. This implies that we can take $t_{\max}=T_m$. This completes the proof of Lemma 4.1. □

Using Lemma 4.1, we can deduce a priori estimates for u_m and v_m. Lemma 4.1 implies that

$$\begin{aligned}
E(u_m(t),v_m(t)) &\geqslant \frac{1}{2}\|u_m'(t)\|_2^2+\frac{1}{2}\|v_m'(t)\|_2^2+\frac{1}{q+1}I(u_m(t),v_m(t)) \\
&\quad+\frac{\alpha(q-1)}{2(q+1)}(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2) \\
&\geqslant \frac{1}{2}\|u_m'(t)\|_2^2+\frac{1}{2}\|v_m'(t)\|_2^2 \\
&\quad+\frac{\alpha(q-1)}{2(q+1)}(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2).
\end{aligned} \tag{4.16}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}(\|u_m'(t)\|_2^2+\|v_m'(t)\|_2^2)+\frac{\alpha(q-1)}{2(q+1)}(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2) \\
&\quad+\delta\int_0^t(\|u_m'(s)\|_{p+1}^{p+1}+\|v_m'(s)\|_{p+1}^{p+1})\,\mathrm{d}s \\
&\leqslant E(u_0,v_0).
\end{aligned} \tag{4.17}$$

Multiplying the equation (4.1) by $-\Delta u'_m(t)$, multiplying the equation (4.2) by $-\Delta v'_m(t)$ and adding these two equations we obtain

$$\begin{aligned}(4.18)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla u'_m(t)\|_2^2+\|\nabla v'_m(t)\|_2^2+\alpha(\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2)\\ &\quad+\frac{\beta}{2}(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)(\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2)\big)\\ &\quad+p\delta(|u'_m(t)|^{p-1}\nabla u'_m(t),\nabla u'_m(t))+p\delta(|v'_m(t)|^{p-1}\nabla v'_m(t),\nabla v'_m(t))\\ &=\mu\big((\nabla[|u_m(t)|^{q-1}u_m(t)],\nabla u'_m(t))+(\nabla[|v_m(t)|^{q-1}v_m(t)],\nabla v'_m(t))\big)\\ &\quad+\beta(\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2)\big((\nabla u_m(t),\nabla u'_m(t))+(\nabla v_m(t),\nabla v'_m(t))\big).\end{aligned}$$

Now we shall compute the first term on the right hand side of (4.18). In the case $\frac{N}{N-2}\leqslant q\leqslant \min\{\frac{N+2}{N-2},\frac{N-2}{[N-4]^+}\}(N\geqslant 3)$, we also see that

$$\begin{aligned}(4.19)\qquad |(\nabla[|u_m(t)|^{q-1}u_m(t)],\nabla u'_m(t))| &\leqslant q\||u_m(t)|^{q-1}\nabla u_m(t)\|_2\|\nabla u'_m(t)\|_2\\ &\leqslant q\|u_m(t)\|_{(q-1)N}^{q-1}\|\nabla u_m(t)\|_{\frac{2N}{N-2}}\|\nabla u'_m(t)\|_2\\ &\leqslant qC\|u_m(t)\|_{(q-1)N}^{q-1}\|\Delta u_m(t)\|_2\|\nabla u'_m(t)\|_2\end{aligned}$$

where we have used Hölder's inequality and Sobolev-Poincaré's inequality. We observe from the Gagliardo-Nirenberg inequality and Sobolev-Pointcaré's inequality that

$$\begin{aligned}(4.20)\qquad \|u_m(t)\|_{(q-1)N}^{q-1} &\leqslant C\|u_m(t)\|_{\frac{2N}{N-2}}^{(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{(q-1)\theta}\\ &\leqslant C\|\nabla u_m(t)\|_2^{(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{(q-1)\theta}\\ &\quad\text{with}\quad \theta=\frac{N-2}{2}-\frac{1}{q-1}\ (<1).\end{aligned}$$

Thus, (4.17), (4.19) and (4.20) imply

$$\begin{aligned}(4.21)\qquad &|\mu(\nabla[|u_m(t)|^{q-1}u_m(t)],\nabla u'_m(t))|\\ &\leqslant q\mu C\|\nabla u_m(t)\|_2^{(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{1+(q-1)\theta}\|\nabla u'_m(t)\|_2\\ &\leqslant q\mu C\Big(\frac{2(q+1)}{\alpha(q-1)}E(u_0,v_0)\Big)^{\frac{1}{2}(q-1)(1-\theta)}\|\Delta u_m(t)\|_2^{1+(q-1)\theta}\|\nabla u'_m(t)\|_2.\end{aligned}$$

Similarly

$$\begin{aligned}(4.22)\qquad &|\mu(\nabla[|v_m(t)|^{q-1}v_m(t)],\nabla v'_m(t))|\\ &\leqslant q\mu C\Big(\frac{2(q+1)}{\alpha(q-1)}E(u_0,v_0)\Big)^{\frac{1}{2}(q-1)(1-\theta)}\|\Delta v_m(t)\|_2^{1+(q-1)\theta}\|\nabla v'_m(t)\|_2.\end{aligned}$$

Next, we shall compute the second term on the right hand side of (4.18):

$$
\begin{aligned}
&\beta|(\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2)\big((\nabla u_m(t),\nabla u_m'(t))+(\nabla v_m(t),\nabla v_m'(t))\big)|\\
&\leqslant \beta(\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2)\big(\|\nabla u_m(t)\|_2\|\nabla u_m'(t)\|_2+\|\nabla v_m(t)\|_2\|\nabla v_m'(t)\|_2\big)\\
&\leqslant \beta\bigg(\frac{2(q+1)}{\alpha(q-1)}E(u_0,v_0)\bigg)^{\frac12}(\|\Delta u_m(t)\|_2^2+\|\Delta u_m(t)\|_2^2)(\|\nabla u_m'(t)\|_2+\|\nabla v_m'(t)\|_2).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\text{(4.23)}\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\|\nabla u_m'(t)\|_2^2+\|\nabla v_m'(t)\|_2^2+\alpha(\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2)\\
&\qquad+\frac{\beta}{2}(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)(\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2)\bigg)\\
&\leqslant q\mu C(\|\Delta u_m(t)\|_2^{1+(q-1)\theta}\|\nabla u_m'(t)\|_2+\|\Delta v_m(t)\|_2^{1+(q-1)\theta}\|\nabla v_m'(t)\|_2)\\
&\qquad\times\bigg(\frac{2(q+1)}{\alpha(q-1)}E(u_0,v_0)\bigg)^{\frac12(q-1)(1-\theta)}\\
&\quad+\beta(\|\Delta u_m(t)\|_2^2+\|\Delta u_m(t)\|_2^2)(\|\nabla u_m'(t)\|_2+\|\nabla v_m'(t)\|_2)\\
&\qquad\times\bigg(\frac{2(q+1)}{\alpha(q-1)}E(u_0,v_0)\bigg)^{\frac12}.
\end{aligned}
$$

Integrating (4.23) from 0 to t, we obtain

(4.24)

$$
\begin{aligned}
&\frac{1}{2}(\|\nabla u_m'(t)\|_2^2+\|\nabla v_m'(t)\|_2^2)+\frac{\alpha}{2}(\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2)\\
&\quad+\frac{\beta}{2}(\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2)(\|\nabla u_m(t)\|_2^2+\|\nabla v_m(t)\|_2^2)\\
&\leqslant \frac{1}{2}(\|\nabla u_1\|_2^2+\|\nabla v_1\|_2^2)+\frac{\alpha}{2}(\|\Delta u_0\|_2^2+\|\Delta v_0\|_2^2)\\
&\quad+\frac{\beta}{2}(\|\Delta u_0\|_2^2+\|\Delta v_0\|_2^2)(\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2)\\
&\quad+q\mu C\bigg(\frac{2(q+1)}{\alpha(q-1)}E(u_0,v_0)\bigg)^{\frac12(q-1)(1-\theta)}\\
&\qquad\times\int_0^t(\|\Delta u_m(s)\|_2^{1+(q-1)\theta}\|\nabla u_m'(s)\|_2+\|\Delta v_m(s)\|_2^{1+(q-1)\theta}\|\nabla v_m'(s)\|_2)\,\mathrm{d}s\\
&\quad+\beta\bigg(\frac{2(q+1)}{\alpha(q-1)}E(u_0,v_0)\bigg)^{\frac12}\\
&\qquad\times\int_0^t(\|\Delta u_m(s)\|_2^2+\|\Delta v_m(s)\|_2^2)(\|\nabla u_m'(s)\|_2+\|\nabla v_m'(s)\|_2)\,\mathrm{d}s.
\end{aligned}
$$

where we have used the inequality

$$p\delta\int_0^t((|u_m'(s)|^{p-1}\nabla u_m'(s),\nabla u_m'(s))+(|v_m'(s)|^{p-1}\nabla v_m'(s),\nabla v_m'(s)))\,\mathrm{d}s\geqslant 0.$$

Thus

$$\begin{aligned}E^*(u_m(t),v_m(t))&\leqslant C(E^*(u_0,v_0))\\&\quad+C^*(u_0,v_0,q)\int_0^t(E^*(u_m(s),v_m(s))+E^*(u_m(s),v_m(s))^{1+(q-1)\theta}\\&\qquad\qquad+E^*(u_m(s),v_m(s))^2)\,\mathrm{d}s\end{aligned}\tag{4.25}$$

where $C(E^*(u_0,v_0)), C^*(u_0,v_0,q)$ are some constants depending on u_0,v_0 and q and

$$E^*(u_m(t),v_m(t))=\frac{1}{2}(\|\nabla u_m'(t)\|_2^2+\|\nabla v_m'(t)\|_2^2)+\frac{\alpha}{2}(\|\Delta u_m(t)\|_2^2+\|\Delta v_m(t)\|_2^2).$$

We set $g(s)=s+s^{1+(q-1)\theta}+s^2$ on $s\geqslant 0$. Then we have

$$E^*(u_m(t),v_m(t))\leqslant C(E^*(u_0,v_0))+C^*(u_0,v_0,q)\int_0^t g(E^*(u_m(s),v_m(s))\,\mathrm{d}s.\tag{4.26}$$

Note that $g(s)$ is continuous and nondecreasing on $s\geqslant 0$. By applying Bihari-Langenhop's inequality (see [2]), we get

$$E^*(u_m(t),v_m(t))\leqslant M_1\quad\text{for some constant}\quad M_1>0.\tag{4.27}$$

Hence

$$\begin{aligned}\|\nabla u_m'(t)\|_2^2+\|\Delta u_m(t)\|_2^2+\|\nabla v_m'(t)\|_2^2+\|\Delta v_m(t)\|_2^2&\leqslant M_2\\ \text{for some constant}\quad M_2&>0.\end{aligned}\tag{4.28}$$

Finally, by multiplying the equation (4.1) by $u_m''(t)$ we have

$$\|u_m''(t)\|_2^2 \leqslant \Big(\alpha\|\Delta u_m(t)\|_2 + \beta(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)\|\Delta u_m(t)\|_2\Big)\|u_m''(t)\|_2 \\ + |\delta|u_m'(t)|^{p-1}(u_m'(t), u_m''(t))| + |\mu|u_m(t)|^{q-1}(u_m(t), u_m''(t))|.$$

Note that

$$\begin{aligned}\delta|u_m'(t)|^{p-1}(u_m'(t)), u_m''(t)) &\leqslant \delta\int_\Omega |u_m'(t)|^p||u_m''(t)|\,\mathrm{d}x \\ &\leqslant \delta\Big(\int_\Omega |u_m'(t)|^{2p}\,\mathrm{d}x\Big)^{\frac12}\Big(\int_\Omega |u_m''(t)|^2\,\mathrm{d}x\Big)^{\frac12} \\ &= \delta\|u_m'(t)\|_{2p}^p\|u_m''(t)\|_2\end{aligned}$$

and similarly

$$\mu|u_m(t)|^{q-1}(u_m(t)), u_m''(t)) \leqslant \mu\|u_m(t)\|_{2q}^q\|u_m''(t)\|_2.$$

Thus, we get

$$\|u_m''(t)\|_2 \leqslant \alpha\|\Delta u_m(t)\|_2 + \beta(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)\|\Delta u_m(t)\|_2 \\ + \delta\|u_m'(t)\|_{2p}^p + \mu\|u_m(t)\|_{2q}^q.$$

Now it follows from the Gagliardo-Nirenberg inequality that

$$\begin{aligned}\|u_m'(t)\|_{2p}^p &\leqslant C_1\|\nabla u_m'(t)\|_2^{p\theta_1}\|u_m'(t)\|_2^{p(1-\theta_1)} \\ &\leqslant C_2\|\nabla u_m'(t)\|_2^{p\theta_1} \quad \text{with} \quad \theta_1 = \frac{(p-1)N}{2p}, \\ \|u_m(t)\|_{2q}^q &\leqslant C_3\|\nabla u_m(t)\|_2^{q\theta_2}\|u_m(t)\|_2^{q(1-\theta_2)} \\ &\leqslant C_4\|\nabla u_m(t)\|_2^{q\theta_2} \quad \text{with} \quad \theta_2 = \frac{(q-1)N}{2q}.\end{aligned}$$

Thus,

$$\begin{aligned}\|u_m''(t)\|_2 &\leqslant \alpha\|\Delta u_m(t)\|_2 + \beta(\|\nabla u_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2)\|\Delta u_m(t)\|_2 \\ &\quad + C_2\|\nabla u_m'(t)\|_2^{p\theta_1} + C_4\|\nabla u_m(t)\|_2^{q\theta_2} \\ &\leqslant M_3 \quad \text{for some constant} \quad M_3 > 0.\end{aligned} \tag{4.29}$$

By applying similar method as that used for u_m, we get

$$\|v_m''(t)\|_2 \leqslant M_4 \quad \text{for some constant} \quad M_4 > 0. \tag{4.30}$$

By the above estimates (4.17), (4.28), (4.29) and (4.30), $\{u_m\}, \{v_m\}$ have subsequences still denoted by $\{u_m\}, \{v_m\}$ such that

$$u_m \to u, \quad v_m \to v \quad \text{in} \quad L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \quad \text{weak}^*, \tag{4.31}$$

$$u'_m \to u', \quad v'_m \to v' \quad \text{in} \quad L^\infty(0,T; H_0^1(\Omega)) \quad \text{weak}^*, \tag{4.32}$$

$$u''_m \to u'', \quad v''_m \to v'' \quad \text{in} \quad L^\infty(0,T; L^2(\Omega)) \quad \text{weak}^*, \tag{4.33}$$

$$u'_m \to u', \quad v'_m \to v' \quad \text{in} \quad L^2(0,T; H_0^1(\Omega)) \quad \text{weak}, \tag{4.34}$$

$$-\Delta u_m \to -\Delta u, \quad -\Delta v_m \to -\Delta v \quad \text{in} \quad L^\infty(0,T; H^{-1}(\Omega)) \quad \text{weak}^*. \tag{4.35}$$

Using Aubin-Lions's compactness lemma, we can extract from $\{u_m\}, \{v_m\}$ subsequences still denoted by $\{u_m\}, \{v_m\}$ such that

$$u_m \to u, \quad v_m \to v \quad \text{strongly in} \quad L^2(0,T; H_0^1(\Omega)). \tag{4.36}$$

It follows from (4.36) that for each $t \in [0,T]$,

$$u_m(t) \to u(t), \quad v_m(t) \to v(t) \quad \text{strongly in} \quad H_0^1(\Omega). \tag{4.37}$$

By letting $m \to \infty$ in (4.1) and (4.2), we can find that u and v satisfy the equations

$$\begin{aligned} &(u''(t), w) - ((\alpha + \beta\|\nabla u(t)\|_2^2 + \beta\|\nabla v(t)\|_2^2)\Delta u(t), w) \\ &\quad + \delta|u'(t)|^{p-1}(u'(t), w) = \mu|u(t)|^{q-1}(u(t), w) \quad \text{for all} \quad w \in H_0^1(\Omega), \end{aligned} \tag{4.38}$$

$$\begin{aligned} &(v''(t), w) - ((\alpha + \beta\|\nabla u(t)\|_2^2 + \beta\|\nabla v(t)\|_2^2)\Delta v(t), w) \\ &\quad + \delta|v'(t)|^{p-1}(v'(t), w) = \mu|v(t)|^{q-1}(v(t), w) \quad \text{for all} \quad w \in H_0^1(\Omega). \end{aligned} \tag{4.39}$$

Now, (4.37) implies

$$u_m(0) = u_{0m} \to u_0 \quad \text{strongly in} \quad H_0^1(\Omega). \tag{4.40}$$

Thus, from (4.3) and (4.40), we conclude $u(0) = u_0$. Also, from (4.34) we obtain

$$(u'_m(0) - u'(0), w) \to 0 \quad \text{as} \quad m \to \infty \quad \text{for each} \quad w \in H_0^1(\Omega). \tag{4.41}$$

Thus, (4.4) and (4.41) imply

$$u'(0) = u_1.$$

Similarly, we obtain $v(0) = v_0$ and $v'(0) = v_1$. This completes the proof of Theorem 3.1.

5. Asymptotic behavior of solutions

Theorem 5.1. *Let $u(t), v(t)$ and q be as in Theorem 3.1. Assume that either $1 \leqslant p < \infty \quad (N = 1, 2)$ or $1 \leqslant p \leqslant \frac{N}{N-2} \quad (N \geqslant 3)$ holds. Then we have the decay estimates if $p = 1$, then*

$$E(u(t), v(t)) \leqslant C_0 \mathrm{e}^{-kt} \quad \text{on} \quad [0, \infty)$$

and if $p > 1$, then

$$E(u(t), v(t)) \leqslant C_1 (1+t)^{-\frac{2}{p-1}} \quad \text{on} \quad [0, +\infty)$$

where k, C_0 and C_1 are certain positive constants depending on $\|\nabla u_0\|_2$ and $\|u_1\|_2$.

To prove our theorem, we need the following lemma.

Lemma 5.2. *Let $u(t)$ and q be as in Lemma 4.1. Then there is a certain number η_0 with $0 < \eta_0 < 1$ such that*

$$\mu(\|u(t)\|_{q+1}^{q+1} + \|v(t)\|_{q+1}^{q+1}) \leqslant (1-\eta_0)\alpha(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) \quad \text{on} \quad [0, \infty)$$

where

$$\eta_0 \equiv 1 - \frac{\mu}{\alpha} C(\Omega, q+1)^{q+1} \left(\frac{2(q+1)}{\alpha(q-1)} E(u_0, v_0) \right)^{\frac{q-1}{2}}.$$

Proof. It follows from the Sobolev-Poincaré's inequality and (4.17) that

$$\begin{aligned}
\mu\|u(t)\|_{q+1}^{q+1} &\leqslant \mu C(\Omega, q+1)^{q+1} \|\nabla u(t)\|_2^{q+1} \\
&= \frac{\mu}{\alpha} C(\Omega, q+1)^{q+1} \|\nabla u(t)\|_2^{q-1} \alpha \|\nabla u(t)\|_2^2 \\
&\leqslant \frac{\mu}{\alpha} C(\Omega, q+1)^{q+1} \left(\frac{2(q+1)}{\alpha(q-1)} E(u_0, v_0) \right)^{\frac{q-1}{2}} \alpha \|\nabla u(t)\|_2^2 \\
&= (1-\eta_0)\alpha\|\nabla u(t)\|_2^2 \quad \text{on} \quad [0, \infty)
\end{aligned}$$

and

$$\mu\|v(t)\|_{q+1}^{q+1} \leqslant (1-\eta_0)\alpha\|\nabla v(t)\|_2^2 \quad \text{on} \quad [0, \infty).$$

Thus

$$\mu(\|u(t)\|_{q+1}^{q+1} + \|v(t)\|_{q+1}^{q+1}) \leqslant (1-\eta_0)\alpha(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) \quad \text{on} \quad [0, \infty).$$

This completes the proof of Lemma 5.2. $\square$

Proof of Theorem 5.1. We denote $E(u(t),v(t))$ by $E(t)$ and $E(u_0,v_0)$ by $E(0)$. Let $(u(t),v(t))$ be solutions of the problems

$$u''(t)-\big(\alpha+\beta(\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2)\big)\Delta u(t)+\delta|u'(t)|^{p-1}u'(t) = \mu|u(t)|^{q-1}u(t), \tag{5.1}$$

$$v''(t)-\big(\alpha+\beta(\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2)\big)\Delta v(t)+\delta|v'(t)|^{p-1}v'(t) = \mu|v(t)|^{q-1}v(t), \tag{5.2}$$

$$\begin{aligned} u(0)&=u_0, \quad u'(0)=u_1,\\ v(0)&=v_0, \quad v'(0)=v_1. \end{aligned} \tag{5.3}$$

Multiplying the equation (5.1) by $u'(t)$, multiplying the equation (5.2) by $v'(t)$, adding these two equations and then integrating over $[t,t+1]\times\Omega$, we get

$$\begin{aligned} \delta\int_t^{t+1}(\|u'(s)\|_{p+1}^{p+1}+\|v'(s)\|_{p+1}^{p+1})\,\mathrm{d}s &= E(t)-E(t+1)\\ &\equiv \delta F(t)^{p+1} \end{aligned} \tag{5.4}$$

where

$$E(t)=\frac12\|u'(t)\|_2^2+\frac12\|v'(t)\|_2^2+J(u(t),v(t))+\frac{\beta}{2}\|\nabla u(t)\|_2^2\|\nabla v(t)\|_2^2.$$

It follows from Hölder's inequality and (5.4) that

$$\begin{aligned} \int_t^{t+1}\|u'(s)\|_2^2\,\mathrm{d}s &= \int_t^{t+1}\int_\Omega |u'(s)|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant m(\Omega)^{\frac{p-1}{p+1}}\int_t^{t+1}\Big(\int_\Omega |u'(s)|^{p+1}\,\mathrm{d}x\Big)^{\frac{2}{p+1}}\mathrm{d}s\\ &\leqslant m(\Omega)^{\frac{p-1}{p+1}}\int_t^{t+1}\|u'(s)\|_{p+1}^2\,\mathrm{d}s\\ &\leqslant m(\Omega)^{\frac{p-1}{p+1}}\Big(\int_t^{t+1}\|u'(s)\|_{p+1}^{p+1}\,\mathrm{d}s\Big)^{\frac{2}{p+1}}\Big(\int_t^{t+1}\mathrm{d}s\Big)^{\frac{p-1}{p+1}}\\ &\leqslant m(\Omega)^{\frac{p-1}{p+1}}F(t)^2. \end{aligned} \tag{5.5}$$

Similarly, we obtain

$$\int_t^{t+1}\|v'(s)\|_2^2\,\mathrm{d}s\leqslant m(\Omega)^{\frac{p-1}{p+1}}F(t)^2. \tag{5.6}$$

Applying the mean value theorem to the left hand sides of (5.5)–(5.6), we find two points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u'(t_i)\|_2 \leqslant 2m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \quad i=1,2, \tag{5.7}$$

$$\|v'(t_i)\|_2 \leqslant 2m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \quad i=1,2. \tag{5.8}$$

Next, multiplying (5.1) by $u(t)$, multiplying (5.2) by $v(t)$, adding these two equations and integrating over $[t_1, t_2] \times \Omega$ we obtain (cf. (5.7), (5.8))

$$\begin{aligned}
(5.9)\ & \int_{t_1}^{t_2} I(u(s), v(s))\,\mathrm{d}s \\
&= \int_{t_1}^{t_2} (\alpha(\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2) - \mu(\|u(s)\|_{q+1}^{q+1} + \|v(s)\|_{q+1}^{q+1}))\,\mathrm{d}s \\
&\leqslant \|u'(t_1)\|_2\|u(t_1)\|_2 + \|u'(t_2)\|_2\|u(t_2)\|_2 + \|v'(t_1)\|_2\|v(t_1)\|_2 \\
&\quad + \|v'(t_2)\|_2\|v(t_2)\|_2 + \int_{t_1}^{t_2} (\|u'(s)\|_2^2 + \|v'(s)\|_2^2)\,\mathrm{d}s \\
&\quad + \delta\left|\int_{t_1}^{t_2} |u'(s)|^{p-1}(u'(s), u(s))\,\mathrm{d}s\right| + \delta\left|\int_{t_1}^{t_2} |v'(s)|^{p-1}(v'(s), v(s))\,\mathrm{d}s\right| \\
&\leqslant 4m(\Omega)^{\frac{p-1}{2(p+1)}} F(t)\Big(\max_{t_1\leqslant s\leqslant t_2} \|u(s)\|_2 + \max_{t_1\leqslant s\leqslant t_2} \|v(s)\|_2\Big) + 2m(\Omega)^{\frac{p-1}{(p+1)}} F(t)^2 \\
&\quad + \delta\int_{t_1}^{t_2}\int_\Omega |u'(s)|^p|u(s)|\,\mathrm{d}x\,\mathrm{d}s + \delta\left|\int_{t_1}^{t_2}\int_\Omega |v'(s)|^p|v(s)|\,\mathrm{d}x\,\mathrm{d}s\right| \\
&\leqslant 8m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \max_{t_1\leqslant s\leqslant t_2} E(s)^{\frac{1}{2}} + 2m(\Omega)^{\frac{p-1}{(p+1)}} F(t)^2 \\
&\quad + \delta\int_{t_1}^{t_2}\int_\Omega |u'(s)|^p|u(s)|\,\mathrm{d}x\,\mathrm{d}s + \delta\left|\int_{t_1}^{t_2}\int_\Omega |v'(s)|^p|v(s)|\,\mathrm{d}x\,\mathrm{d}s\right|.
\end{aligned}$$

Here we note that

$$\begin{aligned}
(5.10)\qquad & \delta\int_{t_1}^{t_2}\int_\Omega |u'(s)|^p|u(s)|\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant \delta\int_{t_1}^{t_2}\Big(\int_\Omega |u'(s)|^{p+1}\,\mathrm{d}x\Big)^{\frac{p}{p+1}}\Big(\int_\Omega |u(s)|^{p+1}\,\mathrm{d}x\Big)^{\frac{1}{p+1}}\,\mathrm{d}s \\
&= \delta\int_{t_1}^{t_2} \|u'(s)\|_{p+1}^p\|u(s)\|_{p+1}\,\mathrm{d}s \\
&\leqslant \delta C(\Omega, p+1)\int_{t_1}^{t_2} \|u'(s)\|_{p+1}^p\|\nabla u(s)\|_2\,\mathrm{d}s
\end{aligned}$$

where we have used Hölder's inequality and Sobolev-Poincaré's inequality. Since $I(u(t),v(t)) \geqslant 0$ on $[0,\infty)$, we see that

$$
\begin{aligned}
E(t) &\geqslant J(u(t),v(t)) \\
&= \frac{1}{q+1} I(u(t),v(t)) + \frac{\alpha(q-1)}{2(q+1)}(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) \\
&\quad + \frac{\beta}{4}(\|\nabla u(t)\|_2^4 + \|\nabla v(t)\|_2^4) + \frac{\beta}{2}\|\nabla u(t)\|_2^2 \|\nabla v(t)\|_2^2 \\
&\geqslant \frac{\alpha(q-1)}{2(q+1)}(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2).
\end{aligned}
\tag{5.11}
$$

From (5.4), (5.10) and (5.11) we get

$$
\begin{aligned}
&\delta \int_{t_1}^{t_2} \int_\Omega |u'(s)|^p |u(s)| \,\mathrm{d}x \,\mathrm{d}s \\
&\leqslant \delta C(\Omega, p+1) \bigg(\int_{t_1}^{t_2} \|u'(s)\|_{p+1}^{p+1} \,\mathrm{d}s\bigg)^{\frac{p}{p+1}} \bigg(\int_{t_1}^{t_2} \mathrm{d}s\bigg)^{\frac{1}{p+1}} \\
&\qquad \times \bigg(\frac{2(q+1)}{\alpha(q-1)}\bigg)^{\frac{1}{2}} \sup_{t_1 \leqslant s \leqslant t_2} E(s)^{\frac{1}{2}} \\
&\leqslant \delta C(\Omega, p+1) \bigg(\frac{2(q+1)}{\alpha(q-1)}\bigg)^{\frac{1}{2}} F(t)^p \sup_{t_1 \leqslant s \leqslant t_2} E(s)^{\frac{1}{2}}.
\end{aligned}
\tag{5.12}
$$

Similary,

$$
\begin{aligned}
&\delta \int_{t_1}^{t_2} \int_\Omega |v'(s)|^p |v(s)| \,\mathrm{d}x \,\mathrm{d}s \\
&\leqslant \delta C(\Omega, p+1) \bigg(\frac{2(q+1)}{\alpha(q-1)}\bigg)^{\frac{1}{2}} F(t)^p \sup_{t_1 \leqslant s \leqslant t_2} E(s)^{\frac{1}{2}}.
\end{aligned}
\tag{5.13}
$$

From (5.9), (5.12) and (5.13) we have

$$
\begin{aligned}
&\int_{t_1}^{t_2} I(u(s),v(s)) \,\mathrm{d}s \\
&\leqslant 8m(\Omega)^{\frac{p-1}{2(p+1)}} F(t) \max_{t_1 \leqslant s \leqslant t_2} E(s)^{\frac{1}{2}} + 2m(\Omega)^{\frac{p-1}{p+1}} F(t)^2 \\
&\quad + 2\delta C(\Omega, p+1) \bigg(\frac{2(q+1)}{\alpha(q-1)}\bigg)^{\frac{1}{2}} F(t)^p \sup_{t_1 \leqslant s \leqslant t_2} E(s)^{\frac{1}{2}}.
\end{aligned}
\tag{5.14}
$$

From Lemma 5.2 and the definition of $I(u(t),v(t))$ we have

$$
\alpha\eta_0(\|\nabla u(t)\|_2^2 + \|\nabla u(t)\|_2^2) \leqslant I(u(t),v(t)). \tag{5.15}
$$

From (5.15) we have

(5.17)

$$\begin{aligned}\int_{t_1}^{t_2} E(s)\,\mathrm{d}s &= \frac{1}{2}\int_{t_1}^{t_2}(\|u'(s)\|_2^2+\|v'(s)\|_2^2)\,\mathrm{d}s + \int_{t_1}^{t_2}(J(u(s),v(s)))\,\mathrm{d}s\\
&\quad+\frac{\beta}{2}\int_{t_1}^{t_2}\|\nabla u(s)\|_2^2\|\nabla v(s)\|_2^2\,\mathrm{d}s\\
&=\frac{1}{2}\int_{t_1}^{t_2}(\|u'(s)\|_2^2+\|v'(s)\|_2^2)\,\mathrm{d}s+\frac{1}{q+1}\int_{t_1}^{t_2} I(u(s),v(s))\,\mathrm{d}s\\
&\quad+\frac{\alpha(q-1)}{2(q+1)}\int_{t_1}^{t_2}(\|\nabla u(s)\|_2^2+\|\nabla v(s)\|_2^2)\,\mathrm{d}s\\
&\quad+\frac{\beta}{4}\int_{t_1}^{t_2}(\|\nabla u(s)\|_2^4+\|\nabla v(s)\|_2^4)\,\mathrm{d}s+\frac{\beta}{2}\int_{t_1}^{t_2}\|\nabla u(s)\|_2^2\|\nabla v(s)\|_2^2\,\mathrm{d}s\\
&\leqslant m(\Omega)^{\frac{p-1}{p+1}}F(t)^2\\
&\quad+\frac{2\eta_0\alpha+\alpha(q-1)}{2\eta_0\alpha(q+1)}\int_{t_1}^{t_2} I(u(s),v(s))\,\mathrm{d}s\\
&\quad+\frac{\beta}{4}\int_{t_1}^{t_2}(\|\nabla u(s)\|_2^4+\|\nabla v(s)\|_2^4)\,\mathrm{d}s+\frac{\beta}{2}\int_{t_1}^{t_2}\|\nabla u(s)\|_2^2\|\nabla v(s)\|_2^2\,\mathrm{d}s.\end{aligned}$$

Note that

$$\begin{aligned}(5.18)\quad \frac{\beta}{2}\int_{t_1}^{t_2}\|\nabla u(s)\|_2^2\|\nabla v(s)\|_2^2\,\mathrm{d}s &\leqslant \beta\int_{t_1}^{t_2}\|\nabla u(s)\|_2^4\,\mathrm{d}s+\beta\int_{t_1}^{t_2}\|\nabla v(s)\|_2^4\,\mathrm{d}s\\
&\leqslant \frac{2\beta(q+1)}{\alpha(q-1)}\int_{t_1}^{t_2}(\|\nabla u(s)\|_2^2+\|\nabla v(s)\|_2^2)\,\mathrm{d}s\\
&\leqslant \frac{2\beta(q+1)}{\alpha^2\eta_0(q-1)}\int_{t_1}^{t_2} I(u(s),v(s))\,\mathrm{d}s\end{aligned}$$

and

$$(5.19)\qquad \begin{aligned}&\frac{\beta}{4}\int_{t_1}^{t_2}(\|\nabla u(s)\|_2^4+\|\nabla v(s)\|_2^4)\,\mathrm{d}s\\
&\quad\leqslant \frac{\beta(q+1)}{2\alpha^2\eta_0(q-1)}\int_{t_1}^{t_2} I(u(s),v(s))\,\mathrm{d}s.\end{aligned}$$

Thus, (5.18)–(5.19) imply

$$(5.20)\qquad \int_{t_1}^{t_2} E(s)\,\mathrm{d}s \leqslant m(\Omega)^{\frac{p-1}{p+1}}F(t)^2 + C(\alpha,\beta,\eta_0,q)\int_{t_1}^{t_2} I(u(s),v(s))\,\mathrm{d}s$$

$$\text{where}\quad C(\alpha,\beta,\eta_0,q)=\frac{2\eta_0+q-1}{2\eta_0(q+1)}+\frac{5\beta(q+1)}{2\alpha^2\eta_0(q-1)}.$$

From (5.13), (5.14) and (5.20) we have

$$\int_{t_1}^{t_2} E(s)\,\mathrm{d}s \leqslant C_1\Bigl(F(t)\sup_{t_1\leqslant s\leqslant t_2} E(s)^{\frac12} + F(t)^2 + F(t)^p \sup_{t_1\leqslant s\leqslant t_2} E(s)^{\frac12}\Bigr) \tag{5.21}$$
$$\text{for some constant} \quad C_1 > 0.$$

Hence

$$\int_{t_1}^{t_2} E(s)\,\mathrm{d}s \leqslant C_2(E(t)^{\frac12}F(t) + F(t)^2 + E(t)^{\frac12}F(t)^p). \tag{5.22}$$

Again multiplying (5.1) by $u'(t)$, multiplying (5.2) by $v'(t)$, adding these two equations and integrating over $[t,t_2]\times\Omega$ we obtain

$$E(t) = E(t_2) + \delta\int_t^{t_2} (\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1})\,\mathrm{d}s.$$

Since $t_2 - t_1 \geqslant \frac12$, we get

$$\begin{aligned}\int_{t_1}^{t_2} E(s)\,\mathrm{d}s &\geqslant \int_{t_1}^{t_2} E(t_2)\,\mathrm{d}s = (t_2-t_1)E(t_2)\\ &\geqslant \frac12 E(t_2),\end{aligned}$$

that is,

$$E(t_2) \leqslant 2\int_{t_1}^{t_2} E(s)\,\mathrm{d}s.$$

From (5.4) and (5.22) we have

$$\begin{aligned}E(t) &= E(t_2) + \delta\int_t^{t_2} (\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1})\,\mathrm{d}s\\ &\leqslant 2\int_{t_1}^{t_2} E(s)\,\mathrm{d}s + \delta\int_t^{t_2} (\|u'(s)\|_{p+1}^{p+1} + \|v'(s)\|_{p+1}^{p+1})\,\mathrm{d}s\\ &\leqslant 2C_2(E(t)^{\frac12}F(t) + F(t)^2 + E(t)^{\frac12}F(t)^p) + \delta F(t)^{p+1}\\ &\leqslant C_3(E(t)^{\frac12}F(t) + F(t)^2 + E(t)^{\frac12}F(t)^p + F(t)^{p+1})\\ &\quad\text{for some constant} \quad C_3 > 0.\end{aligned}$$

Thus, we have

$$E(t) \leqslant C_4(F(t)^2 + F(t)^{2p} + F(t)^{p+1}) \tag{5.23}$$

for some constant $C_5 > 0$. When $p = 1$, we have

$$E(t) \leqslant C_4(F(t)^2) = C_4(E(t) - E(t+1)). \tag{5.24}$$

Applying Nakao's inequality (cf. Lemma 2.3) to (5.24) yields

$$E(t) \leqslant E(0)\mathrm{e}^{-kt} \quad \text{where} \quad k = \log \frac{C_4}{C_4 - 1}.$$

Note that since $E(t)$ is decreasing and $E(t) \geqslant 0$ on $[0, \infty)$, we have

$$\delta F(t)^{p+1} = E(t) - E(t+1) \leqslant E(0).$$

Hence, we get

$$F(t) \leqslant \Big(\frac{1}{\delta} E(0)\Big)^{\frac{1}{p+1}}. \tag{5.25}$$

On the other hand, when $p > 1$, it follows from (5.23) and (5.25) that

$$\begin{aligned} E(t) &\leqslant C_4(1 + F(t)^{2p-2} + F(t)^{p-1})F(t)^2 \\ &\leqslant C_5\big(1 + E(0)^{\frac{2p-2}{p+1}} + E(0)^{\frac{p-1}{p+1}}\big)F(t)^2 \\ &\equiv C_6(E(0))F(t)^2 \end{aligned}$$

with $\lim\limits_{E(0)\to 0} C_6(E(0)) = C_7 > 0$. Thus we have

$$\begin{aligned} E(t)^{1+\frac{p-1}{2}} &\leqslant C_6(E(0))^{\frac{p+1}{2}} F(t)^{p+1} \\ &\leqslant \frac{1}{\delta} C_6(E(0))^{\frac{p+1}{2}} (E(t) - E(t+1)). \end{aligned} \tag{5.26}$$

Setting $C(E(0)) \equiv \delta C_6(E(0))^{-\frac{p+1}{2}}$, applying Nakao's inequality to (5.26) we conclude that

$$E(t) \leqslant \left(E(0)^{-\frac{p-1}{2}} + \frac{(p-1)C(E(0))}{2}[t-1]^+\right)^{-\frac{2}{p-1}}.$$

This completes the proof of Theorem 5.1. $\square$

References

[1] *E. H. Brito*: Nonlinear Initial Boundary Value Problems. Nonlinear Anal. *11* (1987), 125–137.

[2] *C. Corduneanu*: Principles of Differential and Integral Equations. Chelsea Publishing Company, The Bronx, New York, 1977.

[3] *R. Ikehata*: On the Existence of Global Solutions for some Nonlinear Hyperbolic Equations with Neumann Conditions. T R U Math. *24* (1988), 1–17.

[4] *T. Matsuyama, R. Ikehata*: On Global Solutions and Energy Decay for the Wave Equations of Kirchhoff type with Nonlinear Damping terms. J. Math. Anal. Appl. *204* (1996), 729–753.

[5] *M. Nakao*: Asymptotic Stability of the Bounded or Almost Periodic Solutions of the Wave Equations with Nonlinear Damping terms. J. Math. Anal. Appl. *58* (1977), 336–343.

[6] *K. Narasimha*: Nonlinear Vibration of an Elastic String. J. Sound Vibration *8* (1968), 134–146.

[7] *K. Nishihara, Y. Yamada*: On Global Solutions of some Degenerate Quasilinear Hyperbolic Equation with Dissipative Damping terms. Funkcial. Ekvac. *33* (1990), 151–159.

[8] *K. Ono*: Global Existence, Decay and Blowup of Solutions for some Mildly Degenerate Nonlinear Kirchhoff Strings. J. Differential Equations *137* (1997), 273–301.

[9] *M. D. Silva Alves*: Variational Inequality for a Nonlinear Model of the Oscillations of Beams. Nonlinear Anal. *28* (1997), 1101–1108.

Author's address: Department of Mathematics, Pusan National University, Pusan 609-735, Korea.