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# $A$-PROJECTIVE RESOLUTIONS AND AN AZUMAYA THEOREM FOR A CLASS OF MIXED ABELIAN GROUPS

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## 1. Introduction

One of the oldest problems in the theory of abelian groups is the question whether a given class of groups is closed with respect to direct summands. Simply presented $p$-groups and completely decomposable groups are examples of classes where this problem has been solved by an Azumaya-style theorem. No similar results can, however, be obtained for direct summands of sums taken over an arbitrary class $\mathcal{A}$ of groups without imposing some immediate restrictions on the elements of $\mathcal{A}$. To simplify our notation while considering such decompositions, the class of $A$-*projective* groups consists of direct summands of direct sums of copies of a fixed group $A$. An abelian group $G$ is $\mathcal{A}$-*decomposable* if it is of the form $G \cong \bigoplus_{A\in\mathcal{A}} P_A$ where each $P_A$ is $A$-projective. Azumaya's original result describes the case that every $A \in \mathcal{A}$ has a local endomorphism ring [4]. Arnold, Hunter, and Richman extended his work in [6] where they showed that the class of $\mathcal{A}$-decomposable groups is closed with respect to direct summands if $\mathcal{A}$ is a pseudo-rigid class of countable groups.

The goal of this paper is to establish the existence of an Azumaya-style theorem for the class of $\mathcal{G}$-decomposable groups where $\mathcal{G}$ is the class of mixed abelian groups which was introduced by Glaz and Wickless in [11]. In order to define $\mathcal{G}$, we first consider the class $\Gamma$ of mixed groups $G$ with the property that $G$ is isomorphic to a pure subgroup of $\prod_p G_p$ containing $\bigoplus_p G_p$. The symbol $\Gamma_\infty$ denotes the groups in $\Gamma$ which have finite torsion-free rank. Every $G \in \Gamma_\infty$ contains a finite independent subset $X$ such that $F = \langle X\rangle$ is a free subgroup of $G$ with $G/F$ torsion. We view $G$ as a pure subgroup of $\prod_p G_p$, and write $X = \{x_i = (x_{ip}) \mid i = 1,\dots,n\}$. Glaz and Wickless investigated the class $\mathcal{G}$ of groups in $\Gamma_\infty$ for which $G_p$ is finite for all $p$ and satisfies $G_p = \langle x_{1p},\dots,x_{np}\rangle$ for all but finitely many $p$. Observe that every element of $\mathcal{G}$ is either an honest mixed group of finite. They showed in [11] that a group $G \in \Gamma_\infty$ such that $G_p$ is finite for all $p$ is in $\mathcal{G}$ if and only if $\operatorname{Hom}(G, tG)$ is a torsion group. In particular, $E(A)/tE(A)$ is a finite dimensional $\mathbb{Q}$-algebra for $A \in \mathcal{G}$. Goeters, Wickless, and the author continued the discussion of [11] in [3] by showing that the elements of $\mathcal{G}$ are the mixed self-small abelian groups which have finite torsion-free rank.

Since $\mathcal{G}$ is not a pseudo-rigid class, we cannot use the results of [6] directly to show that the class of $\mathcal{G}$-decomposable groups is closed with respect to direct summands. However, Corollary 4.4 and Theorem 5.2 yield that every group in $\mathcal{G}$ is the finite direct sum of groups with local Walk-endomorphism ring where Walk is the category with abelian groups as objects, but whose morphisms are defined by $\mathrm{Mor}_{\mathrm{Walk}}(G, H) = \operatorname{Hom}(G, H)/\operatorname{Hom}(G, tH)$ ([13]). Therefore, every direct summand $G$ of a $\mathcal{G}$-decomposable group is Walk-isomorphic to a $\mathcal{G}$-decomposable group. Going back to the category of abelian groups, this only gives us that there exists a torsion group $T$ such that $G \oplus T$ is $\mathcal{G}$-decomposable [13, Theorem 12]. Since our proof that a direct summand $G$ of a $\mathcal{G}$-decomposable group is $\mathcal{G}$-decomposable will not be simpler if $G$ has a torsion complement, we prove the Azumaya-Theorem for $\mathcal{G}$ directly by showing that the Walk-indecomposable groups in $\mathcal{G}$ exhibit a behavior similar to that of modules with local endomorphism ring when they appear as direct summands of $\mathcal{G}$-decomposable groups (Lemma 5.4). This behavior is in stark contrast to the case that $\mathcal{A}$ is the class of torsion-free groups of finite rank, where the class of $\mathcal{A}$-decomposable groups is not closed with respect to direct summands [10, Theorem 91.1].

The Azumaya Theorem for $\mathcal{G}$ is a consequence of our discussion of the structure of $A$-generated groups in the first part of this paper. For a fixed $A \in \mathcal{G}$, call an abelian group *(finitely) $A$-generated* if it is an epimorphic image of a group of the form $\bigoplus_I A$ for some (finite) index-set $I$. Theorem 2.2 shows that every reduced $A$-generated group is in $\Gamma$ and isomorphic to a subgroup of $\prod_I A$ for some index-set $I$. Furthermore, if $G \in \Gamma_\infty$ is $A$-generated, then $G = H + tG$ for some $A$-generated group $H \in \mathcal{G}$. Finally, an $A$-generated reduced group is in $\mathcal{G}$ if and only if it is finitely $A$-generated.

Sections 3 discusses finitely $A$-generated groups. Following [1], we say that an abelian group $G$ is *$A$-solvable* if the evaluation map $\theta_G \colon \operatorname{Hom}(A, G) \otimes_{E(A)} A \to G$ is an isomorphism. Theorem 3.2 and Corollary 3.4 characterize the finitely $A$-generated $A$-solvable groups. Applications of these results are given in Section 4 where we investigate when finitely $A$-generated $A$-solvable groups are essentially indecomposable (Proposition 4.1 and Theorem 4.2). We want to remind the reader that a group $A$ is essentially indecomposable if, whenever $A = B \oplus C$, then $B$ or $C$ is bounded. In particular, we show that a group $A \in \mathcal{G}$ is essentially indecomposable if and only if $E(A)/tE(A)$ is a local ring (Corollary 4.4).

The results of Sections 2 and 3 not merely lay the ground-work for the proof of the Azumaya-Theorem. They also provide substantially deeper insight in the structure of $A$-generated groups in $\mathcal{G}$ than the corresponding results did in the case of torsion-free groups. This is primarily due to the fact that we are able to combine the well-developed machinery for the discussion of $A$-solvable groups with a structure theory, which is significantly richer than that for torsion-free abelian groups of finite rank. In addition, we find that these characterizations can be obtained without imposing any immediate restrictions on the $E(A)$-module structure of $A$. In contrast, this was necessary for the discussion of $A$-solvable groups when $A$ is torsion-free.

This demonstrates that there are significant differences between $\mathcal{G}$ and $\mathcal{TF}_\infty$, the category of torsion-free abelian groups of finite rank, in spite of Wickless' results in [18] which establish a high degree of similarity between these categories at the quasi-level. We want to point out that Wickless' results just like our proof of the Azumaya-Theorem does not use the category Walk.

## 2. $A$-generated Abelian groups

We begin this section with a summary of standard properties of reduced groups $G \in \Gamma$ which we will frequently use without reference. We view $G \in \Gamma$ as a pure subgroup of $\prod_p G_p$. For any finite number $p_1, \ldots, p_n$ of primes, $G = G_{p_1} \oplus \ldots \oplus G_{p_n} \oplus G'$ where $G'$ is a fully invariant subgroup of $G$ such that multiplication by $p_i$ is an automorphism of $G'$ for $i = 1, \ldots, n$. Moreover, a reduced group $G$ is in $\Gamma$ if and only if $G_p$ is a direct summand of $G$ for all primes $p$ and $G/tG$ is divisible. In particular, if $G \in \Gamma$ has bounded $p$-primary subgroups for all $p$, then $E(G)$ is a pure subring of $\prod_p E(G_p)$ and $tE(G) = \bigoplus_p E(G_p)$.

Given abelian groups $A$ and $G$, composition of maps induces a right $E(A)$-module-structure on $H_A(G) = \mathrm{Hom}(A, G)$. For a right $E(A)$-module $M$, the symbol $T_A(M)$ denotes $M \otimes_{E(A)} A$. Since the functors induced by $H_A$ and $T_A$ between the category of abelian groups, $\mathcal{A}b$, and the category, $\mathcal{M}_{\mathcal{E}(\mathcal{A})}$, of right $E(A)$-modules form an adjoint pair, there exist natural maps $\theta_G \colon T_A H_A(G) \to G$ for $G \in \mathcal{A}b$ and $\varphi_M \colon M \to H_A T_A(M)$ for $M \in \mathcal{M}_{\mathcal{E}(\mathcal{A})}$.

**Lemma 2.1.** *Let $A \in \mathcal{G}$.*

a) *$tA$ is projective as a left $E(A)$-module, and $\mathrm{Tor}^1_{E(A)}(M, A)$ is torsion-free divisible for all right $E(A)$-modules $M$. In particular, $\mathrm{Tor}^1_{E(A)}(M, A) = 0$ whenever the additive group of $M$ is torsion.*

b) *Every reduced $A$-generated torsion group $G$ is $A$-solvable and has the property that $H_A(G) \cong \bigoplus_p H_A(G_p)$ is torsion. Moreover, if $G \in \mathcal{G}$ is $A$-generated, then $tG \subseteq tA^n$ for some $n < \omega$.*

P r o o f. a) Let $p$ be a prime of $\mathbb{Z}$, and write $A = A_p \oplus A^p$ where $A^p$ is p-divisible and $A^p[p] = 0$. Since $E(A) = E(A_p) \times E(A^p)$, every right $E(A)$-module $M$ decomposes as $M = M_p \oplus M^p$ where $M_p$ is an $E_p = E(A_p)$-module and $M^p$ is an $E^p = E(A^p)$-module. In order to show that $A_p$ is a projective $E(A)$-module, it thus is enough to show that it is projective over $E_p$. For this, observe that every subgroup of $A_p^n$ is $A_p$-generated whenever $n < \omega$ since $A_p$ is finite. By Ulmer's Theorem [17], $A_p$ is a flat $E_p$-module. Since $E_p$ is finite, every finite flat $E_p$-module is projective.

Furthermore, $\mathrm{Tor}^1_{E(A)}(M, A/tA)$ is torsion-free divisible, and fits into the induced exact sequence $0 = \mathrm{Tor}^1_{E(A)}(M, tA) \to \mathrm{Tor}^1_{E(A)}(M, A) \to \mathrm{Tor}^1_{E(A)}(M, A/tA) \xrightarrow{\Delta} M \otimes_{E(A)} tA$, in which $M \otimes_{E(A)} tA$ is a torsion group with bounded $p$-components. Thus, $\mathrm{Im}\,\Delta = 0$, and $\mathrm{Tor}^1_{E(A)}(M, A) \cong \mathrm{Tor}^1_{E(A)}(M, A/tA)$ is torsion-free divisible. If the additive group of $M$ is torsion, then so is $\mathrm{Tor}^1_{E(A)}(M, -)$. By what has been shown, $\mathrm{Tor}^1_{E(A)}(M, A) = 0$.

b) Let $G$ be a reduced $A$-generated torsion group. It remains to show that $\theta_G$ is one-to-one. Consider an exact sequence $\bigoplus_I A \to G_p \to 0$. Since $\mathrm{Hom}(A^p, G_p) = 0$, the group $G_p$ is an epimorphic image of $\bigoplus_I A_p$. If $k_p < \omega$ is minimal with $p^{k_p} A_p = 0$, then $A$ has a direct summand $U_p$ isomorphic to $\mathbb{Z}/p^{k_p}\mathbb{Z}$ and $p^{k_p} G = 0$. Therefore, we can find a monomorphism $\alpha\colon\ G_p \to \bigoplus_{I_p} U_p$ for some index-set $I_p$. Consequently, $G$ is isomorphic to a subgroup of the $A$-projective torsion group $P = \bigoplus_p \big[\bigoplus_{I_p} U_p\big]$. Observe that $H_A(P) \cong \bigoplus_p \big[\bigoplus_{I_p} H_A(U_p)\big]$ since $A$ is self-small. Thus, $H_A(P)$ is torsion and $\mathrm{Tor}^1_{E(A)}(H_A(P)/H_A(G), A) = 0$ by a). Therefore, the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T_A H_A(G) & \longrightarrow & T_A H_A(P) \\ & & \Big\downarrow \theta_G & & \wr \Big\downarrow \theta_P \\ 0 & \longrightarrow & G & \longrightarrow & P \end{array}$$

has exact rows, from which it follows that $\theta_G$ is one-to-one.

If $G \in \mathcal{G}$ is $A$-generated, then every cyclic summand of $G_p$ is isomorphic to a subgroup of $A_p$ by what has been established. The statement follows once we have shown that there exists $m < \omega$ such that $G_p$ is the direct sum of at most $m$ cyclic groups. To see this, observe that, for all but finitely many primes, $G_p$ is generated by at most $n$ elements where $n = r_0(G)$. But then, $G_p$ cannot be the direct sum of more than $n$ non-zero cyclic subgroups. Since $G_p$ is finite for all primes, the last statement in b) follows. $\square$

In the following, the *A-radical of a group G* is denoted by $R_A(G) = \bigcap\{\ker\varphi \mid \varphi \in \mathrm{Hom}(G, A)\}$. Clearly, $R_A(G) = 0$ if and only if $G \subseteq A^I$ for some index-set $I$.

**Theorem 2.2.** *Let $A \in \mathcal{G}$.*

a) *The following conditions are equivalent for an $A$-generated group $G$:*

   i) *$G$ is reduced.*

   ii) *$\ker \theta_G$ is torsion-free divisible, and $\mathbb{Q} \not\subseteq G$.*

   iii) *$R_A(G) = 0$.*

   iv) *$\mathbb{Q} \not\subseteq G$; and if $p^k A_p = 0$ for some $k < \omega$, then $p^k G_p = 0$.*

b) *A reduced $A$-generated group $G$ is finitely $A$-generated if and only if $G \in \mathcal{G}$.*

c) *Let $G$ be a reduced $A$-generated group which has finite torsion-free rank. Then, $G$ contains a finitely $A$-generated subgroup $H$ such that $\ker \theta_H \cong \ker \theta_G$ and $G = H + tG$. In particular, $G/H$ is $A$-solvable.*

P r o o f. For the sake of an easier reference, we first show that every reduced $A$-generated group $G$ is in $\Gamma$: Observe that $G/tG$ is an epimorphic image of $\bigoplus_I A$ for some index-set $I$. Under this isomorphism, $\bigoplus_I tA$ is mapped to zero. Hence $G/tG$ is divisible as an epimorphic image of the divisible group $\bigoplus_I A/tA$.

Let $k_p$ be the smallest positive integer such that $p^{k_p} A_p = 0$. If $p^{k_p} G_p \neq 0$, then $G$ has a direct summand $U \cong \mathbb{Z}/p^n\mathbb{Z}$ for some integer $n > k_p$ because $G_p$ is reduced. Since $U$ is $A$-generated, there is an epimorphism $\varphi\colon A \to U$ whose kernel contains $p^n A$. If we write $A = A_p \oplus A^p$ with $A^p$ is p-divisible and $A[p] = 0$, then $n > k_p$ yields $p^n A = A^p$, and $A/p^n A$ is bounded by $p^{k_p}$. Hence, $U$ cannot be an image of $A$, a contradiction. Therefore, $G_p$ is a bounded direct summand of $G$, and $G \in \Gamma$. Furthermore, $tG$ is $A$-generated. We now prove the equivalences in part a) of the theorem:

iv) $\Rightarrow$ ii): Since $G$ has to be reduced, $G \in \Gamma$, and $tG$ is an $A$-solvable group such that $H_A(tG) \cong \bigoplus_p H_A(G_p)$ is torsion (Lemma 2.1). The fact that $H_A(G)/H_A(tG)$ is isomorphic to a subgroup of the torsion-free divisible group $H_A(G/tG)$ yields $tH_A(G) = H_A(tG)$. On the other hand, if we view $G$ as a pure subgroup of $\prod_p G_p$, then $H_A(G)$ is a pure subgroup of $H_A\left(\prod_p G_p\right)$, and the torsion subgroup of the latter group is $H_A(tG)$ too. Consequently, the right $E(A)$-module $M = H_A(G)/H_A(tG)$ has a torsion-free divisible additive group. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & \ker\theta_G \arrow[r, "\lambda"] \arrow[d, "\tau"] & \ker\theta \arrow[r] \arrow[d, "\varrho"] & 0 \\
 & T_A H_A(tG) \arrow[r, "T_A H_A(\alpha)"] \arrow[d, "\theta_{tG}", "\wr"'] & T_A H_A(G) \arrow[r, "T_A H_A(\beta)"] \arrow[d, "\theta_G"] & T_A(M) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & tG \arrow[r, "\alpha"] & G \arrow[r, "\beta"] & G/tG \arrow[r] & 0
\end{tikzcd}$$

where $\lambda$ is onto by the Snake-Lemma. Since $T_A(M)$ and $G/tG$ are torsion-free divisible, $\ker\theta_G \cong \ker\theta$ is torsion-free divisible.

ii) $\Rightarrow$ i): Observe that $G$ is isomorphic to a direct summand of the group $H = T_AH_A(G)$. Any free resolution of $H_A(G)$ induces an exact sequence $0 \to U \xrightarrow{\alpha} \bigoplus_I A \xrightarrow{\beta} H \to 0$ with $S_A(U) = U$. Let $p$ be a prime of $\mathbb{Z}$. For $x \in H_p$, we can find $y \in \bigoplus_I A$ with $\beta(y) = x$. If $p^m x = 0$, then there are $a_1, \dots, a_n \in A$ and $\varphi_1, \dots, \varphi_n \in H_A(U)$ with $p^m y = \sum_{i=1}^n \alpha\varphi_i(a_i)$. Since $A/tA$ is divisible, we can find $b_1, \dots, b_n \in tA$ and $c_1, \dots, c_n \in A$ with $a_i = p^m c_i + b_i$. Thus $p^m\big[y - \sum_{i=1}^n \alpha\varphi_i(c_i)\big] = \sum_{i=1}^n \alpha\varphi_i(b_i) \in \bigoplus_I tA$. Write $o\big(y - \sum_{i=1}^n \alpha\varphi_i(c_i)\big) = p^t q$ where $t \leqslant k_p$ and $(p,q) = 1$. Then, $p^t q x = \beta\big(p^t q\big(y - \sum_{i=1}^n \alpha\varphi_i(c_i)\big)\big) = 0$. Since $x \in H_p$, we obtain $p^t x = 0$. Thus, $H_p$ is bounded by $p^{k_p}$, and $G_p$ is reduced.

i) $\Rightarrow$ iii): Since $G_p$ is bounded by $p^{k_p}$, we have $G_p \subseteq \prod_{I_p} A_p$ for some index-set $I_p$. But $G \in \Gamma$ implies $G \subseteq \prod_p G_p$.

iii) $\Rightarrow$ iv) follows directly from the initial remarks of the proof.

b) Let $G$ be a reduced $A$-generated group. If $G$ admits an exact sequence $A^n \to G \to 0$, then $r_0(G) < \infty$. Using the arguments in the initial part of the proof of a), we obtain that $G \in \Gamma_\infty$, $tG$ is $A$-generated, and $|G_p| < \infty$ for all $p$. It remains to show that $\mathrm{Hom}(G, tG)$ is torsion. For this, observe that there is an induced exact sequence $0 \to \mathrm{Hom}(G, tG) \to \bigoplus_n H_A(tG)$, in which $H_A(tG)$ is torsion by Lemma 2.1b. Conversely, assume that $G$ is an $A$-generated group in $\mathcal{G}$. We can find $n < \omega$ and a map $\alpha\colon A^n \to G$ such that $G/\alpha(A^n)$ is torsion. We show that $G = tG + \alpha(A^n)$. For this observe that $\big[tG + \alpha(A^n)\big]/tG \cong \alpha(A^n)/\big[tG \cap \alpha(A^n)\big]$ is a torsion-free image of $A^n$, and hence an image of the divisible group $A^n/tA^n$. Therefore, $\big[tG + \alpha(A^n)\big]/tG$ is a direct summand of the torsion-free divisible group $G/tG$. Since $G/\big[tG + \alpha(A^n)\big]$ is torsion, we have $G = tG + \alpha(A^n)$.

Consider the exact sequence $0 \to \alpha(A^n) \to G \to T \to 0$ for some torsion group $T$. Since $T \cong \big[tG + \alpha(A^n)\big]/\alpha(A^n) \cong tG/\big[tG \cap \alpha(A^n)\big]$, we have that $T_p$ is finite for all $p$. Moreover, $G[p] = 0$ implies $T[p] = 0$. If $T[p] \neq 0$ for all primes $p$ in an infinite set $P_1$ of primes, then $\mathrm{Hom}(T, tG) \supseteq \prod_{P_1} \mathrm{Hom}(T_p, G_p)$, which cannot be torsion since $T_p$ and $G_p$ are non-zero finite $p$-groups for $p \in P_1$. On the other hand, the exact sequence $0 \to \mathrm{Hom}(T, tG) \to \mathrm{Hom}(G, tG)$ shows that this group is torsion. The resulting contradiction establishes that $T$ is finite, from which it immediately follows that $G$ is finitely $A$-generated.

c) There is nothing to show if $G$ is torsion. Thus, we may assume that $G$ contains a non-empty independent subset $X = \{x_1, \dots, x_n\}$ such that $\langle X\rangle$ is free and $G/\langle X\rangle$ is torsion. Since we can view $G$ as a pure subgroup of $\prod_p G_p$, we set $X_p = \langle x_{1p}, \dots, x_{np}\rangle$ for each prime $p$. The inclusions $X_p \subseteq G_p$ coordinatewise induce a monomorphism $\lambda\colon \prod_p X_p \to \prod_p G_p$. Clearly, $\lambda$ operates like the identity on $\bigoplus_p X_p$ and satisfies $\lambda(x_i) = x_i$ for all $i$. Observe that $x_1, \dots, x_n$ generate a free subgroup of $\prod_p X_p$.

We set $tH = \bigoplus_p X_p$, and let $H$ be the subgroup of $\prod_p X_p$ containing $tH$ such that $H/tH = \bigoplus_{i=1}^n \mathbb{Q}(x_i + tH)$. By definition, $H \in \mathcal{G}$. For $h \in H$, we can find $r_1, \ldots, r_n \in \mathbb{Z}$ and $y \in tH$ such that $mh = r_1x_1 + \ldots + r_nx_n + y$ for some non-zero integer $m$. Then, $m\lambda(h) \in G$ since $tH \subseteq tG$. Since $\left(\prod_p G_p\right)/G$ is torsion-free, we have $\lambda(h) \in G$. We shall identify $H$ with its image under $\lambda$ in $G$. Since $x_1, \ldots, x_n \in H + tG$, we have that $G/(H + tG)$ is torsion. On the other hand, $(H + tG)/tG \cong H/(H \cap tG) = H/tH$ is torsion-free divisible. Thus, $(H + tG)/tG$ is a direct summand of the torsion-free group $G/tG$ whose complement is isomorphic to $G/(H + tG)$. By what has been shown, this is only possible if $G = H + tG$. In particular, $G/H = (H + tG)/H \cong tG/tH$ is a reduced $A$-generated torsion group which is $A$-solvable by Lemma 2.1b.

In the exact sequence $0 \to H_A(H) \to H_A(G) \to M \to 0$, the additive group of $M$ is torsion as a subgroup of $H_A(G/H)$ by Lemma 2.1b. Because of Lemma 2.1a, $\mathrm{Tor}^1_{E(A)}(M, A) = 0$, and the top-row of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_AH_A(H) & \longrightarrow & T_AH_A(G) & \longrightarrow & T_A(M) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \theta_H} & & \big\downarrow{\scriptstyle \theta_G} & & \big\downarrow{\scriptstyle \theta} & & \\
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 0
\end{array}$$

is exact. The induced map $\theta$ satisfies $\theta = \theta_{G/H}T_A(\iota)$ where $\iota\colon M \to H_A(G/H)$ is the inclusion map. Observe that $T_A(\iota)$ is one-to-one by Lemma 2.1a, from which we obtain that $\theta$ is an isomorphism. By the Snake-Lemma, $\theta_H$ is an epimorphism, and $\ker\theta_H \cong \ker\theta_G$. $\square$

**Corollary 2.3.** *The class of reduced $A$-solvable groups is closed with respect to direct sums whenever $A \in \mathcal{G}$.*

P r o o f. Let $\{G_i\}_{i\in I}$ be a family of reduced $A$-solvable groups. Suppose $\varphi \in H_A\left(\bigoplus_I G_i\right)$ satisfies $\pi_i\varphi \neq 0$ for infinitely many $i \in I$ where $\pi_j\colon \bigoplus_{i\in I} G_i \to G_j$ denotes the projection onto the $j^{th}-$coordinate. For each such $i$, there is a map $\alpha_i\colon G_i \to A$ with $\alpha_i\pi_i\varphi \neq 0$ by Theorem 2.2a. Coordinatewise, the maps $\alpha_i$ induce a map $\alpha\colon \bigoplus_I G_i \to \bigoplus_I A$. If $\delta_i\colon \bigoplus_I A \to A$ denotes the projection onto the $i^{th}$-coordinate, then $\delta_i\alpha\varphi = \alpha_i\varphi$ is non-zero for infinitely many i, which contradicts the fact that the groups in $\mathcal{G}$ are self-small [3]. Thus, $\{G_i\}_{i\in I}$ is $A$-small. Since the class of $A$-solvable groups is closed with respect to $A$-small direct sums by [2], the proof is complete. $\square$

Furthermore, Theorem 2.2 allows to answer the question for which groups $A$ there may exist $A$-solvable groups $G$ with $R_A(G) \neq 0$:

**Corollary 2.4.** *Let $A$ be a self-small abelian group of finite torsion-free rank such that $A/tA$ is a faithfully flat $E(A)/tE(A)$-module. Then, there exists a reduced $A$-solvable group $G$ with $R_A(G) \neq 0$ if and only if $A$ is torsion-free and reduced.*

Proof. Suppose that there exists a reduced $A$-solvable group $G$ with $R_A(G) \neq 0$. By Theorem 2.2 together with [3], we obtain that $A$ is either torsion or torsion-free. In the first case, the group $A$ has to be finite, and every $A$-solvable group has a zero $A$-radical, which is not possible. In the second case, it remains to show that $A$ is reduced. If $\mathbb{Q} \subseteq A$, then every torsion-free $A$-generated group is $A$-solvable as in [1]. But this yields that $A$ is homogeneous completely decomposable by [1]. Therefore, $A \cong \mathbb{Q}^n$, and all $A$-solvable groups are $A$-projective. On the other hand, if $A$ is torsion-free reduced, then there exists a right $E(A)$-module $M$ which is $\aleph_1$-free and has $E(A)$ as its $\mathbb{Z}$-endomorphism-ring using the construction of [9] (e.g. see [2]). Let $G = T_A(M)$. Once we have shown that $G$ is $A$-solvable, it will follow as in [2] that $E(G) = \mathrm{Center}\big(E(A)\big)$ and $\mathrm{Hom}(G, A) = 0$. The $A$-solvability of $G$, however, follows from the fact that, for $\varphi_1, \ldots, \varphi_k \in H_A(G)$, there is a free submodule $U$ of $M$ with $\sum_{i=1}^k \varphi_i(A) \subseteq T_A(U)$. □

## 3. $\mathcal{G}_A$-presented Abelian groups

A sequence $0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} G \to 0$ of abelian groups is *almost $A$-balanced* if $M = H_A(G)/\operatorname{Im} H_A(\beta)$ is torsion.

**Lemma 3.1.** *Let $A \in \mathcal{G}$ and $0 \to B \to C \xrightarrow{\pi} G \to 0$ be an almost $A$-balanced sequence in which $C$ $A$-solvable and $G \in \mathcal{G}$. Then, $G$ is $A$-solvable if and only if $B$ is $A$-generated.*

Proof. By [1, Lemma 2.1], $M = \operatorname{Im} H_A(\pi)$ fits into an exact sequence $T_A H_A(B) \xrightarrow{\theta_B} B \to T_A(M) \xrightarrow{\theta} G \to 0$, and it is enough to show that $G$ is $A$-solvable iff $\theta$ is a monomorphism. If $\iota\colon M \to H_A(G)$ denotes the inclusion map, then $\theta = \theta_G T_A(\iota)$ and $\operatorname{coker} \iota$ is torsion as an abelian group. Moreover, $\ker T_A(\iota) = 0$ as an epimorphic image of $\operatorname{Tor}^1_{E(A)}(H_A(G)/M, A)$ which vanishes by Lemma 2.1a. Clearly, this gives that $\theta$ is one-to-one if $G$ is $A$-solvable. Conversely, assume that $\theta$ is a monomorphism. For $x \in \operatorname{Im} T_A(\iota) \cap \ker \theta_G$, there is $y \in T_A(M)$ such that $x = T_A(\iota)(y)$. Then, $\theta(y) = \theta_G T_A(\iota)(y) = 0$ yields $x = 0$. Therefore, $\ker \theta_G$ is isomorphic to a subgroup of the torsion group $\operatorname{coker} T_A(\iota) \cong T_A(\operatorname{coker} \iota)$ which results in a contradiction unless $G$ is $A$-solvable since $\ker \theta_G$ is torsion-free divisible by Theorem 2.2. □

**Theorem 3.2.** *Let $A \in \mathcal{G}$. The following conditions are equivalent for a reduced abelian group $G$:*

a) *$G$ is a finitely $A$-generated $A$-solvable group.*

b) *$G$ is an $A$-solvable group in $\mathcal{G}$.*

c) *$G$ admits an almost $A$-balanced sequence $0 \to U \xrightarrow{\alpha} A^n \xrightarrow{\beta} G \to 0$ such that $U$ is $A$-generated.*

*Moreover, the sequence in c) can be chosen in such a way that*

$$T_A(H_A(G))/\operatorname{Im} H_A(\beta) = 0.$$

Proof. The equivalence of a) and b) is an immediate consequence of Theorem 2.2a. Moreover, a reduced finitely $A$-generated group is in $\mathcal{G}$ by Theorem 2.2b. By Lemma 3.1, the implication c) $\Rightarrow$ b) is true.

b) $\Rightarrow$ c): Since $G$ has finite torsion-free rank, we can choose a finite independent subset $X = \{x_1, \dots, x_n\}$ of $G$ such that $\langle X\rangle$ is free and $G/\langle X\rangle$ is torsion. Because $G \in \mathcal{G}$, we have $G_p = \langle x_{1p}, \dots, x_{np}\rangle$ for almost all primes $p$ of $\mathbb{Z}$. There are maps $\varphi_1, \dots, \varphi_m \in H_A(G)$ such that $X \subseteq \varphi_1(A) + \ldots + \varphi_m(A)$. We write $A = A_p \oplus A^p$ and $G = G_p \oplus G^p$ for each prime $p$ such that $A^p$ and $G^p$ are fully invariant. There are $a_{1p}, \dots, a_{mp} \in A_p$ and $b_{1p}, \dots, b_{mp} \in A^p$ such that $x_i = \sum_{j=1}^m \varphi_j(a_{jp} + b_{jp})$. Since $\operatorname{Hom}(A_p, G^p) = 0 = \operatorname{Hom}(A^p, G_p)$, we have $x_{ip} = \sum_{j=1}^m \varphi_j(a_{jp})$; and $G_p \subseteq \varphi_1(A) + \ldots + \varphi_m(A)$ for all but finitely many primes $p$. By adding finitely many maps to $\{\varphi_1, \dots, \varphi_m\}$ if necessary, we may assume that $tG \subseteq \varphi_1(A) + \ldots + \varphi_m(A)$ and $G/\big[\varphi_1(A) + \ldots + \varphi_m(A)\big]$ is torsion. Since $H_A(G)$ has finite torsion-free rank, no generality is lost if we assume that $M = \langle \varphi_1, \dots, \varphi_m\rangle$ has the additional property that $H_A(G)/M$ is torsion as an abelian group.

We define a map $\theta\colon T_A(M) \to G$ by $\theta(\alpha \otimes a) = \alpha(a)$, and observe $\theta = \theta_G T_A(\iota)$ where $\iota\colon M \to H_A(G)$ is the inclusion map. Since $\theta_G$ is an isomorphism and $\ker T_A(\iota) = 0$ as an image of $\operatorname{Tor}^1_{E(A)}\big(H_A(G)/M, A\big)$ which vanishes by Lemma 2.1, $\theta$ is one-to-one. Furthermore, $\operatorname{Im}\theta = \varphi_1(A) + \ldots + \varphi_m(A)$ yields that $\operatorname{coker} T_A(\iota) \cong G/\operatorname{Im}\theta \cong (G/tG)/(\operatorname{Im}\theta/tG)$ is torsion and divisible. In particular, $(G/\operatorname{Im}\theta)_p$ either vanishes or is unbounded. On the other hand, since $M$ and $H_A(G)$ are $E(A)$-modules, the same holds for $(\operatorname{coker}\iota)_p$. However, the $E(A)$-module-structure of the latter module is completely determined by its $E_p$-module-structure. Since $E_p$ is finite, $(\operatorname{coker}\iota)_p$ is bounded, and $\operatorname{coker} T_A(\iota) \cong T_A(\operatorname{coker}\iota)$ has bounded $p$-components. This is only possible if $\operatorname{coker} T_A(\iota) = 0$. Therefore, $T_A(\iota)$ is an isomorphism, and the same holds for $\theta$.

Choose a projective resolution $0 \to V \xrightarrow{\lambda} E(A)^m \xrightarrow{\pi} M \to 0$ of $M$. Since $U = \ker T_A(\pi) = \operatorname{Im} T_A(\lambda)$ is $A$-generated, it remains to show that $0 \to U \to T_A\big(E(A)^m\big) \xrightarrow{T_A(\pi)} T_A(M) \to 0$ is almost $A$-balanced. An application of the functor

$H_A$ yields the diagram

$$\begin{array}{ccccc} H_AT_A\big(E(A)^m\big) & \xrightarrow[H_AT_A(\pi)]{} & H_AT_A(M) & & \\ \wr\Big\uparrow \varphi_{E(A)^m} & & \Big\uparrow \varphi_M & & \\ E(A)^m & \xrightarrow[\pi]{} & M & \longrightarrow & 0. \end{array}$$

It gives $\mathrm{Im}\big(H_AT_A(\pi)\big) = \mathrm{Im}\,\varphi_M$, and the proof is complete once we have shown that $\varphi_M$ has a torsion cokernel. For this, we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_AT_A(M) & \xrightarrow[H_AT_A(\iota)]{} & H_AT_AH_A(G) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \Big\uparrow \varphi_M & & \wr\Big\uparrow \varphi_{H_A(G)} & & \Big\uparrow \varphi & & \\ 0 & \longrightarrow & M & \xrightarrow[\iota]{} & H_A(G) & \longrightarrow & \mathrm{coker}\,\iota & \longrightarrow & 0 \end{array}$$

whose first row is exact since $T_A(\iota)$ is an isomorphism. By the Snake-Lemma, $\mathrm{coker}\,\varphi_M \cong \ker\varphi = \mathrm{coker}\,\iota$. As we have shown before $\mathrm{coker}\,\iota$ is torsion as abelian group. □

We say that an $A$-generated group $G \in \mathcal{G}$ is *$\mathcal{G}_A$-presented* if there exists an almost $A$-balanced exact sequence $0 \to U \to A^n \to G \to 0$ in which $U$ is $A$-generated and in $\mathcal{G}$. By Lemma 3.1, any $\mathcal{G}_A$-presented group is $A$-solvable.

**Proposition 3.3.** *For a group $A \in \mathcal{G}$, the class of $\mathcal{G}_A$-presented groups is closed with respect to direct summands.*

P r o o f. Let $G = B \oplus C$ be a $\mathcal{G}_A$-presented group. Choose an almost $A$-balanced exact sequence $0 \to U \xrightarrow{\alpha} A^n \xrightarrow{\beta} G \to 0$ with $U \in \mathcal{G}$, and let $\pi_B\colon G \to B$ be the projection along $C$. We consider the induced exact sequence $(\mathcal{E})$ $0 \to V \xrightarrow{\lambda} A^n \xrightarrow{\pi_B\beta} B \to 0$ in which $\lambda$ is the inclusion-map. If $\varphi \in H_A(B)$, then there is $\gamma \in H_A(G)$ with $\varphi = \pi_B\gamma = H_A(\pi_B)(\gamma)$. We can find a non-zero integer $m$ and a map $\delta \in H_A(A^n)$ with $m\gamma = H_A(\beta)(\delta)$. Therefore, $m\varphi = H_A(\pi_B)(m\gamma) = H_A(\pi_B\beta)(\delta)$, and $(\mathcal{E})$ is almost $A$-balanced. Furthermore, $B$ is $A$-solvable and in $\mathcal{G}$ as a direct summand of an $A$-solvable group in $\mathcal{G}$. If $M = \mathrm{Im}\,H_A(\pi_B\beta) \subseteq H_A(B)$, then the evaluation map $\theta\colon T_A(M) \to B$ satisfies $\theta = \theta_BT_A(\iota)$ where $\iota\colon M \to H_A(B)$ is the inclusion map. Since $\mathrm{coker}\,\iota$ is torsion, we have that $T_A(\iota)$ is a monomorphism, and the same holds for $\theta$. By [1, Lemma 2.1], $\mathrm{coker}\,\theta_V \cong \ker\theta = 0$. Therefore, $V$ is an $A$-generated subgroup of $A^n$, and it remains to show that $V \in \mathcal{G}$. Since $V \in \Gamma_\infty$ as in the proof of Theorem 2.2, it suffices to establish that $\mathrm{Hom}(V, tV)$ is torsion:

Observe that $C = \beta(V)$ and fits into the exact sequence $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta|_V} C \to 0$ in which $U$ and $C$ are $A$-generated groups in $\mathcal{G}$. By Theorem 2.2, $U$ and $C$ are finitely $A$-generated. In particular, $\mathrm{Hom}(U, tA)$ and $\mathrm{Hom}(C, tA)$ are torsion. Since $V \subseteq A^n$, we have $\mathrm{Hom}(V, tV) \subseteq \bigoplus_n \mathrm{Hom}(V, tA)$. But $\mathrm{Hom}(V, tA)$ fits into the exact sequence $0 \to \mathrm{Hom}(C, tA) \to \mathrm{Hom}(V, tA) \to \mathrm{Hom}(U, tA)$ in which the first and third Hom-group are torsion. $\square$

**Corollary 3.4.** *Let $A$ be in $\mathcal{G}$. The following conditions are equivalent for a reduced abelian group $G$:*

a) *$G$ is an $A$-solvable group in $\mathcal{G}$.*

b) *There exists an almost $A$-balanced exact sequence $0 \to T \to H \to G \to 0$ in which $H$ is $\mathcal{G}_A$-presented and $T$ is a reduced $A$-generated torsion group.*

Proof. b) $\Rightarrow$ a): Since $\mathcal{G}_A$-presented groups are finitely $A$-generated, $G$ is in $\mathcal{G}$ by Theorem 2.2b as a reduced finitely $A$-generated group. Because $T$ is $A$-solvable, $G$ is $A$-solvable by Theorem 3.2.

a) $\Rightarrow$ b): Since $G$ is $A$-solvable, we can find an almost $A$-balanced exact sequence $0 \to U \xrightarrow{\alpha} A^n \xrightarrow{\beta} G \to 0$ in which $U$ is $A$-generated and $\alpha$ is an inclusion map. By Theorem 2.2c, there exists an $A$-generated subgroup $V$ of $U$ with $V \in \mathcal{G}$ and $U = V + tU$. We consider the induced sequence $(\mathcal{E})$ $0 \to U/V \xrightarrow{\overline{\alpha}} A^n/V \xrightarrow{\overline{\beta}} G \to 0$. Observe that $U/V \cong tU/tV$ is a reduced $A$-generated torsion-group which is $A$-solvable.

Let $\pi_1\colon U \to U/V$ and $\pi_2\colon A^n \to A^n/V$ be the canonical projections. Since $\pi_2\alpha = \overline{\alpha}\pi_1$ and $\overline{\beta}\pi_2 = \beta$, we obtain the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_A(U) & \xrightarrow[H_A(\alpha)]{} & H_A(A^n) & \xrightarrow[H_A(\beta)]{} & H_A(G) \\
 & & \Big\downarrow{\scriptstyle H_A(\pi_1)} & & \Big\downarrow{\scriptstyle H_A(\pi_2)} & & \Big\downarrow{\scriptstyle 1_{H_A(G)}} \\
0 & \longrightarrow & H_A(U/V) & \xrightarrow[H_A(\overline{\alpha})]{} & H_A(A^n/V) & \xrightarrow[H_A(\overline{\beta})]{} & H_A(G).
\end{array}$$

Given $x \in H_A(G)$, there is a non-zero integer $m$ such that $mx = H_A(\beta)(y)$ for some $y \in H_A(A^n)$. Then, $mx = H_A(\overline{\beta})H_A(\pi_2)(y)$, and $(\mathcal{E})$ is almost $A$-balanced. Set $M = \mathrm{Im}\, H_A(\overline{\beta})$, and let $\iota\colon M \to H_A(G)$ be the inclusion map. Since $\mathrm{coker}\,\iota$ is torsion and the evaluation map $\theta\colon T_A(M) \to G$ satisfies $\theta = \theta_G T_A(\iota)$, the map $T_A(\iota)$ is a monomorphism, and the same holds for $\theta$. The map $\theta$ fits into the commutative diagram

$$\begin{array}{ccccccccc}
 & & T_A H_A(U/V) & \longrightarrow & T_A H_A(A^n/V) & \longrightarrow & T_A(M) & \longrightarrow & 0 \\
 & & {\wr}\Big\downarrow{\scriptstyle \theta_{U/V}} & & \Big\downarrow{\scriptstyle \theta_{A^n/V}} & & \Big\downarrow{\scriptstyle \theta} & & \\
0 & \longrightarrow & U/V & \longrightarrow & A^n/V & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

It follows that $\theta_{A^n/V}$ is an isomorphism. In particular, $A^n/V$ is reduced by Theorem 2.2a. An application of Theorem 2.2b gives $A^n/V \in \mathcal{G}$.

It remains to show that $0 \to V \to A^n \xrightarrow{\pi_2} A^n/V \to 0$ is almost $A$-balanced. Given $\varphi \in H_A(A^n/V)$, there is a non-zero integer $\ell$ such that $H_A(\overline{\beta})(\ell\varphi) = H_A(\beta)(\psi) = H_A(\overline{\beta})H_A(\pi_2)(\psi)$ for some $\psi \in H_A(A^n)$. Thus, $\ell\varphi - H_A(\pi_2)(\psi) \in H_A(\overline{\alpha})\big(H_A(U/V)\big)$ which is a torsion group by Lemma 2.1b. Hence, $k\ell\varphi \in \operatorname{Im} H_A(\pi_2)$ for some non-zero integer $k$. □

Looking at the almost $A$-balanced sequences of the form $. \xrightarrow{\beta} . \to 0$ constructed in Theorem 3.2 and Corollary 3.4, we see that $N = \operatorname{coker} H_A(\beta)$ always satisfies $T_A(N) = 0$. Since $N$ is torsion, we obtain the exact sequence $0 = \operatorname{Tor}^1_{E(A)}(N, A/tA) \to N \otimes_{E(A)} tA \to T_A(N) = 0$. It yields $N_p \otimes_{E_p} A_p \cong N_p \otimes_{E(A)} A_p = 0$ for all primes $p$. Since $A_p$ is homogeneous if and only if it is faithfully flat as an $E_p$-module [3], we have $N = 0$ if $A_p$ is homogeneous for all primes $p$. We thus have shown:

**Corollary 3.5.** *Let $A \in \mathcal{G}$ have homogeneous $p$-components for all primes $p$, and suppose that $G \in \mathcal{G}$ is $A$-solvable.*

a) *There exists an $A$-balanced exact sequence $0 \to U \to A^n \to G \to 0$ with $S_A(U) = U$. Moreover, $G$ is $\mathcal{G}_A$-presented if and only if the sequence can be chosen such that $U \in \mathcal{G}$.*

b) *There exists an $A$-balanced exact sequence $0 \to T \to H \to G \to 0$ in which $T$ is a torsion $A$-solvable group and $H$ is $\mathcal{G}_A$-presented.*

## 4. Direct sum decompositions of $\mathcal{G}_A$-presented groups

In the following, projection modulo the torsion subgroup of a given abelian group will be indicated by an overscore. This section investigates how the existence of non-trivial direct sum decompositions of the right $E(A)/tE(A)$-module $\overline{H_A(G)}$ is related to decompositions of the $A$-solvable group $G$. Observe that the $E(A)/tE(A)$-module structure of $\overline{M}$ coincides with its $E(A)$-module structure for any right $E(A)$-module $M$.

**Proposition 4.1.** *Let $A \in \mathcal{G}$, and $G$ be an $A$-solvable group in $\mathcal{G}$. If $\overline{H_A(G)}$ is an indecomposable $E(A)/tE(A)$-module, then $G$ is essentially indecomposable.*

P r o o f. There is nothing to show if $G$ is torsion. Hence, suppose that $G$ is an honest mixed group. If it is not essentially indecomposable, then there exists an idempotent $e \in E(G)$ such that $e, 1-e \notin tE(G)$. Since $G$ is $A$-solvable, we have $E(G) \cong \mathrm{End}_{E(A)}(H_A(G))$ as has been shown in [2, Theorem 4.4]. Let $f$ be the idempotent corresponding to $e$ under this isomorphism. Clearly, $f$ induces an idempotent endomorphism $\overline{f}\colon \overline{H_A(G)} \to \overline{H_A(G)}$ by $\overline{f}(\overline{x}) = \overline{f(x)}$. Since $\overline{H_A(G)}$ is indecomposable as an $E(A)/tE(A)$-module, its endomorphism ring has no non-trivial idempotents. Without loss of generality, we have $\overline{f} = 0$. Thus, $f\colon H_A(G) \to tH_A(G)$. However, $tH_A(G) = H_A(tG)$ by Lemma 2.1 since $G \in \mathcal{G}$. We view $T_A(f)$ as a map from $T_AH_A(G)$ into $T_AH_A(tG)$. Since $tG$ and $G$ are $A$-solvable, $\theta_{tG}T_A(f)\theta_G^{-1}$ is an element of the torsion group $\mathrm{Hom}(G, tG)$. We obtain that $mT_A(f) = 0$ for some non-zero integer $m$. Observe that $H_AT_A(f)\varphi_{H_A(G)} = \varphi_{H_A(tG)}f$ yields $\varphi_{H_A(tG)}mf = 0$. Since $tG$ is $A$-solvable by Lemma 2.1b, we have that $\varphi_{H_A(tG)}$ is an isomorphism. Hence, $mf = 0$, which is not possible by the choice of $f$. $\square$

We now show that the converse of this result is true if $G$ is $\mathcal{G}_A$-presented.

**Theorem 4.2.** *Let $A \in \mathcal{G}$. A $\mathcal{G}_A$-presented group $G$ is essentially indecomposable if and only if $\overline{H_A(G)}$ is an indecomposable $E(A)/tE(A)$-module.*

P r o o f. As before, it is enough to consider the case that $G$ is an honest mixed group. Assume that $G$ is essentially indecomposable. Let $0 \to U \xrightarrow{\alpha} A^n \xrightarrow{\beta} G \to 0$ be an almost $A$-balanced exact sequence where $U \in \mathcal{G}$ is $A$-generated. It induces the exact sequence $0 \to H_A(U) \xrightarrow{H_A(\alpha)} H_A(A^n) \xrightarrow{H_A(\beta)} M \to 0$ in which $M = \mathrm{Im}\, H_A(\beta)$ is a submodule of $H_A(G)$ with $H_A(G)/M$ torsion. Observe that $G \cong T_A(M)$ by [1, Lemma 2.1]. Moreover, $K = H_A(G)/\bigl[M + tH_A(G)\bigr]$ is torsion as an abelian group and fits into the exact sequence $0 \to M/tM \to H_A(G)/tH_A(G) \to K \to 0$ because of $(M + tH_A(G))/tH_A(G) \cong M/\bigl(M \cap tH_A(G)\bigr) = M/tM$. Since $M/tM$ and $H_A(G)/tH_A(G)$ are torsion-free divisible, the same has to hold for $K$. Therefore, $K = 0$, and $\overline{M} \cong \overline{H_A(G)}$. Once we have shown that

(I) if $\varphi\colon \overline{M} \to \overline{M}$ is an $E(A)/tE(A)$-morphism, then there is a map $\tau\colon M \to M$ such that $\varphi(\overline{x}) = \overline{\tau(x)}$ for all $x \in M$, and

(II) if $\tau \in \mathrm{End}_{E(A)}(M)$ satisfies $\overline{\tau(x)} = 0$ for all $x \in M$, then $\tau \in tE(A)$,

then the theorem is shown as follows:

Let $\pi\colon \overline{M} \to \overline{M}$ be an $E(A)$-morphism with $\pi^2 = \pi$. There is a map $\lambda\colon M \to M$ with $\overline{\lambda(x)} = \pi(\overline{x})$ for all $x \in M$ by (I). Since $\overline{\lambda^2(x)} = \pi\Bigl(\overline{\lambda(x)}\Bigr) = \pi^2(\overline{x}) = \pi(\overline{x})$, we have $k\lambda^2 = k\lambda$ for some non-zero integer $k$ using (II). Let $P$ be the set of primes dividing $k$, and write $A = A_1 \oplus A_2$ with $A_1 = \bigoplus_{q\in P} A_q$ and $\mathrm{Hom}(A_i, A_j) = 0$ if $i \neq j$. Then, $E(A) = E(A_1) \times E(A_2)$, and $M = M_1 \oplus M_2$ such that $M_iE(A_j) = 0$ for $i \neq j$. In particular, $M_1$ is bounded, and $\lambda(M_i) \subseteq M_i$ for $i = 1, 2$. Therefore, $\lambda|M_2$ is an idempotent of $\mathrm{End}_{E(A)}(M_2)$. Write $M_2 = \lambda(M_2) \oplus (1-\lambda)(M_2)$, and observe $G \cong T_A(M) = T_A\bigl(M_1 \oplus (1-\lambda)(M_2)\bigr) \oplus T_A\bigl(\lambda(M_2)\bigr)$. Since $G$ is essentially indecomposable, we have that one of the modules $T_A\bigl(\lambda(M_2)\bigr)$ or $T_A\bigl(M_1 \oplus (1-\lambda)(M_2)\bigr)$ is bounded.

In the first case, $H_AT_A\big(\lambda(M_2)\big)$ is bounded as an abelian group. The commutative diagram

$$\begin{tikzcd}
 & H_AT_A\big(\lambda(M_2)\big) \arrow[r] & H_AT_AH_A(G) \\
0 \arrow[r] & \lambda(M_2) \arrow[r] \arrow[u, "\varphi_{\lambda(M_2)}"'] & H_A(G) \arrow[u, "\wr", "\varphi_{H_A(G)}"']
\end{tikzcd}$$

yields that $\lambda(M_2)$ is isomorphic to a subgroup of the bounded group $H_AT_A\big(\lambda(M_2)\big)$. Consequently, $\lambda(M) = \lambda(M_1) \oplus \lambda(M_2)$ is bounded, and $\pi = \overline{\lambda} = 0$. On the other hand, if $T_A\big(M_1 \oplus (1-\lambda)(M_2)\big)$ is bounded, then the same argument as before gives that $M_1 \oplus (1-\lambda)(M_2)$ is bounded, from which we obtain that $0 = \overline{1-\lambda} = 1 - \overline{\lambda} = 1 - \pi$. In either case, $\overline{M}$ is indecomposable.

In order to verify the two statements, we first show that $\mathrm{Hom}(T_AH_A(U), T_A(tM))$ is torsion: For this, observe that $T_A(tM)$ is $A$-generated and has bounded $p$-components for each prime $p$ because $M_p$ is an $E_p$-module, and $E_p$ is finite. By Lemma 2.1b, $T_A(tM)$ is $A$-solvable, and $0 \to T_A(tM) \to T_A(M) \to T_A(M/tM) \to 0$ is exact. Moreover, $M/tM$ is torsion-free and divisible implies that $T_A(tM) = tT_A(M) \cong tG$. Since $G$ is $A$-solvable and in $\mathcal{G}$, we have that $tG$ is isomorphic to a subgroup of $tA^m$ for some $m < \omega$ by Lemma 2.1b. By Theorem 2.2a, $\ker\theta_U$ is torsion-free and divisible, and $T_AH_A(U) \cong U \oplus \bigoplus_k \mathbb{Q}$ for some $k < \omega$. Therefore, $\mathrm{Hom}\big(T_AH_A(U), T_A(tM)\big) \cong \mathrm{Hom}(U, tG)$ which is isomorphic to a subgroup of $\mathrm{Hom}(U, tA^m)$. The latter group is torsion, since $U \in \mathcal{G}$ implies that there is an epimorphism $A^s \to U \to 0$ for some $s < \omega$ which induces a monomorphism $0 \to \mathrm{Hom}(U, tA^m) \to \mathrm{Hom}(A^s, tA^m)$.

We view $\varphi$ as a map $M \to \overline{M}$, and can find a map $\psi\colon\ H_A(A^n) \to M$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & H_A(U) \arrow[r, "H_A(\alpha)"'] & H_A(A^n) \arrow[r, "H_A(\beta)"'] \arrow[d, "\psi"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & tM \arrow[r] & M \arrow[r, "\delta"'] & \overline{M} \arrow[r] & 0
\end{tikzcd}$$

commutative in which $\delta$ denotes the canonical projection. An easy diagram chase shows that $\psi H_A(\alpha) \in \mathrm{Hom}_{E(A)}\big(H_A(U), tM\big)$ and, hence, $T_A(\psi)T_AH_A(\alpha)\colon T_AH_A(U) \to T_A(tM)$. By the result of the previous paragraph, there is a non-zero integer $r$ such that $rT_A(\psi)T_AH_A(\alpha) = 0$. Because $r\varphi_{tM}\psi H_A(\alpha) = H_AT_A\big(r\psi H_A(\alpha)\big)\varphi_{H_A(U)} = 0$, we have $\varphi_{tM} r\psi H_A(\alpha) = 0$. But $\varphi_{tM}$ is one-to-one: To see this, observe that $tM \subseteq H_A(G)$ and that $\varphi_{H_A(G)}$ is an isomorphism such that $\varphi_{H_A(G)}\iota = H_AT_A(\iota)\varphi_{tM}$ where $\iota\colon\ tM \to H_A(G)$ is the inclusion map. Therefore, $r\psi H_A(\alpha) = 0$.

We can write $E(A) = R_1 \times R_2$ where $R_1$ is finite and multiplication by $r$ is an automorphism of $R_2$. Given an $E(A)$-module $N$, this ring-decomposition yields a corresponding decomposition $N = N_1 \oplus N_2$ such that $N_i R_j = 0$ for $i \neq j$. In particular, multiplication by $r$ is an automorphism of $N_2$. Therefore, $r\psi H_A(\alpha)\bigl(H_A(U)_2\bigr) = 0$ yields $\psi H_A(\alpha)(H_A(U)_2) = 0$.

We now define $\tau$: Write $x \in H_A(A^n)$ as $x = x_1 + x_2$ with $x_i \in H_A(A^n)_i$, and define a map $\nu\colon H_A(A^n) \to M$ by $\nu(x) = \psi(x_2)$. Since $H_A(\alpha)\bigl(H_A(U)_1\bigr) \subseteq H_A(A^n)_1 \subseteq \ker\nu$, we have $H_A(\alpha)\bigl(H_A(U)\bigr) \subseteq \ker\nu$, and $\nu$ induces a map $\tau\colon M \to M$ in the following way: For $x \in M$, choose $y \in H_A(A^n)$ with $H_A(\beta)(y) = x$, and define $\tau(x) = \nu(y)$. Write $x = x_1 + x_2$ and $y = y_1 + y_2$, and obtain $\delta\tau(x) = \delta\nu(y) = \delta\psi(y_2) = \varphi H_A(\beta)(y_2) = \varphi(x_2)$. Since $x_1 \in M_1 \subseteq tM$, we have $\varphi(x_1) = 0$, and $\tau$ is the desired map.

Moreover, if $\varrho\colon M \to M$ is a map with $\overline{\varrho(x)} = 0$ for all $x \in M$, then $\varrho(M) \subseteq tM$. However, this yields that $T_A(\varrho)$ is an element of $\mathrm{Hom}\bigl(T_A(M), tT_A(M)\bigr)$ which is isomorphic to the torsion group $\mathrm{Hom}(G, tG)$. Since $\varphi_{tM}\varrho = H_A T_A(\varrho)\varphi_M$ is torsion, we obtain as before that $\varrho$ has finite order. $\square$

**Corollary 4.3.** *Let $A \in \mathcal{G}$ and $G \in \mathcal{G}$ be an $A$-solvable group. If $0 \to T \to H \xrightarrow{\pi} G \to 0$ is an almost $A$-balanced exact sequence in which $T$ an $A$-solvable torsion group and $H$ is an essentially indecomposable $\mathcal{G}_A$-presented group, then $G$ is essentially indecomposable.*

Proof. Let $M = \operatorname{Im} H_A(\pi)$. Since $H_A(G)/M$ and $H_A(T)$ are torsion, we have $r_0(H_A(H)) = r_0(M) = r_0(H_A(G)) < \infty$. Thus, $\overline{H_A(H)} \cong \overline{M} \cong \overline{H_A(G)}$. By the last theorem, $\overline{H_A(H)}$ is indecomposable, and hence the same holds for $\overline{H_A(G)}$, from which it follows that $G$ is essentially indecomposable by Proposition 4.1. $\square$

**Corollary 4.4.** *The following conditions are equivalent for an abelian group $A \in \mathcal{G}$:*

a) *$A$ is essentially indecomposable.*
b) *$A$ is indecomposable in* Walk.
c) *$E(A)/tE(A)$ is local.*

Proof. To see that *a*) and *c*) are equivalent, observe that the fact that $A$ is $\mathcal{G}_A$-presented yields that $A$ is essentially indecomposable if and only if $E(A)/tE(A)$ is an indecomposable $E(A)/tE(A)$-module. However, an Artinian ring without non-trivial idempotents is local.

For the equivalence of *b*) and *c*), observe that $\mathrm{Hom}(A, tA) = tE(A)$. Hence, the Walk-endomorphism ring $E_W(A)$ of $A$ coincides with $E(A)/tE(A)$, and nothing is to prove. $\square$

Our first step toward showing that the class of $\mathcal{G}$-decomposable groups is closed with respect to direct summands is the verification of the fact that there is a Krull-Schmitt-Theorem for the groups in $\mathcal{G}$.

**Lemma 5.1.** *Let $A \in \mathcal{G}$. If $\{\overline{e_1}, \ldots, \overline{e_n}\}$ is a family of orthogonal idempotents of $E(A)/tE(A)$, then there are orthogonal idempotents $e_1, \ldots, e_n \in E(A)$ with $\overline{e_i} = e_i + tE(A)$ for $i = 1, \ldots, n$.*

Proof. Write $\overline{e}_1 = f_1 + tE(A)$ for some $f_1 \in E(A)$. Then, $f_1^2 - f_1 \in \bigoplus_{p \in P_1} E(A_p)$ for some finite subset $P_1$ of the set $P$ of all primes. As rings, $E(A) = \big(\times_{p \in P_1} E(A_p)\big) \times S$ for some subring $S$ of $E(A)$. There is a central idempotent $g_1 \in E(A)$ with $E(A)g_1 = \times_{p \in P_1} E(A_p)$. Since $E(A_p)$ is torsion, and $(1-g_1)(f_1^2 - f_1) = 0$, we have that $(1-g_1)f_1$ is an idempotent of $E(A)$ with $\overline{e}_1 = f_1 + tE(A) = (1-g_1)f_1 + tE(A)$. Hence, we can find a finite subset $P_1$ of $P$ and an idempotent $e_1$ of $E(A)$ with $\overline{e}_1 = e_1 + tE(A)$ and $e_1\big(\bigoplus_{p \in P_1} A_p\big) = 0$.

Assume that we have found finite subsets $P_1 \subseteq \ldots \subseteq P_n$ of $P$ and orthogonal idempotents $e_1, \ldots, e_n$ of $E(A)$ with $\overline{e}_i = e_i + tE(A)$ and $e_i\big(\bigoplus_{p \in P_i} A_p\big) = 0$. If $n < m$, then we choose $f_{n+1} \in E(A)$ with $\overline{e}_{n+1} = f_{n+1} + tE(A)$. As before, we can find a finite subset $Q_{n+1} \supseteq P_n$ of $P$ and a central idempotent $\tilde{g}_{n+1} \in E(A)$ such that $(1 - \tilde{g}_{n+1})\big(\bigoplus_{p \in Q_{n+1}} A_p\big) = 0$ and $h_{n+1} = (1 - \tilde{g}_{n+1})f_{n+1}$ is an idempotent of $E(A)$ with $\overline{e_{n+1}} = h_{n+1} + tE(A)$. Since $\overline{e}_i\overline{e}_{n+1} = \overline{e}_{n+1}\overline{e}_i = 0$, we can enlarge $Q_{n+1}$ to a finite subset $P_{n+1}$ of $P$ such that $e_i h_{n+1}, h_{n+1}e_i \in \bigoplus_{p \in P_{n+1}} E(A_p)$ for $i = 1, \ldots, n$. If we choose a central idempotent $g_{n+1}$ in $E(A)$ with $E(A)g_{n+1} = \times_{p \in P_{n+1}} E(A_p)$, then $e_{n+1} = (1 - g_{n+1})h_{n+1}$ is an idempotent of $E(A)$ with the desired properties. □

If $A$ is essentially indecomposable and $T$ is a bounded abelian group, then $A \oplus T$ is essentially indecomposable. To see this, write $A \oplus T = B \oplus C$, and let $P_1$ be the set of those primes $p$ for which $T[p] \neq 0$. Since $A \in \mathcal{G}$, we can write $A = D \oplus E$ with $E$ bounded and $\mathrm{Hom}(D, T \oplus E) = \mathrm{Hom}(E \oplus T, D) = 0$. We have $B = B_1 \oplus T_1$ and $C = C_1 \oplus T_2$ with $B_1, C_1 \subseteq D$ and $T_1, T_2 \subseteq E \oplus T$. Hence, $A = B_1 \oplus C_1 \oplus S$ for some bounded group $S$. Since $A$ is essentially indecomposable, $B_1$ or $C_1$ is bounded. This shows that $A \oplus T$ is essentially indecomposable.

**Theorem 5.2.** *Let $A \in \mathcal{G}$.*

a) *There are essentially indecomposable subgroups $A_1, \ldots, A_n$ of $A$ with $A = A_1 \oplus \ldots \oplus A_n$.*

b) *If $A = A_1 \oplus \ldots \oplus A_n = B_1 \oplus \ldots \oplus B_m$ with $A_i$ and $B_j$ essentially indecomposable for all $i$ and $j$, then $n = m$ and, after reindexing, there are bounded groups*

*$C_1, \dots, C_n$ and $D_1, \dots, D_n$ with $A_i \oplus C_i = B_i \oplus D_i$. Moreover, if $1 \leqslant k \leqslant n$, then $A = B_1 \oplus \dots \oplus B_k \oplus A'_{l+1} \oplus \dots \oplus A'_n \oplus T$ where $A_j = A'_j \oplus S_j$ for bounded groups $T$, $S_1, \dots, S_n$.*

Proof. a) Since $E(A)/tE(A)$ is a finite dimensional $\mathbb{Q}$-algebra, we can find a finite set $\{\overline{e}_1, \dots, \overline{e}_n\}$ of orthogonal primitive idempotents of $E(A)/tE(A)$ with $1_A = \overline{e}_1 + \dots + \overline{e}_n$. By Lemma 5.1, each $\overline{e}_i$ is of the form $\overline{e}_i = e_i + tE(A)$ for orthogonal idempotents $e_1, \dots, e_n$ of $E(A)$. Setting $e = e_1 + \dots + e_n$ yields a decomposition $A = e_1(A) \oplus \dots \oplus e_n(A) \oplus (1-e)(A)$. We set $A_i = e_i(A)$ and $T = (1-e)(A)$. Since $(1-e) \in tE(A)$, we have $mT = 0$ for some non-zero integer $m$. Once we have shown that the $A'_i s$ are essentially indecomposable, $A_1 \oplus T$ is essentially indecomposable by the preceding remarks; and we have a decomposition of $A$ as in a). By Theorem 4.2, it is enough to show that $\overline{H_A(A_i)}$ is indecomposable:

Since $E(A) = H_A(A_1) \oplus \dots \oplus H_A(A_n) \oplus H_A(T)$, we have $\overline{E(A)} = \overline{H_A(A_1)} \oplus \dots \oplus \overline{H_A(A_n)}$. In particular, $\overline{H_A(A_i)} = e_iE(A)/tH_A(A_i) = e_iE(A)/\bigl[e_iE(A) \cap tE(A)\bigr] \cong \bigl[e_iE(A) + tE(A)\bigr]/tE(A) = \overline{e}_iE(A)$ yields that $\overline{H_A(A_i)}$ is an indecomposable $E(A)$-module.

b) Choose orthogonal idempotents $f_1, \dots f_m$ of $E(A)$ with $f_i(A) = B_i$ for $i = 1, \dots, m$. Since $\overline{H_A(B_i)}$ is an indecomposable $E(A)$-module for $i = 1, \dots, m$, the classical Krull-Schmitt-Theorem yields $n = m$ and $\overline{H_A(A_i)} \cong \overline{H_A(B_i)}$ for $i = 1, \dots, n$ after a possible reindexing.

Inverse $E(A)$-module isomorphisms $\overline{\sigma}\colon \overline{H_A(A_i)} \to \overline{H_A(B_i)}$ and $\overline{\tau}\colon \overline{H_A(B_i)} \to \overline{H_A(A_i)}$ can be lifted to a pair of maps $\sigma\colon H_A(A_i) \to H_A(B_i)$ and $\tau\colon H_A(B_i) \to H_A(A_i)$ such that $\overline{\sigma\tau} = 1_{\overline{H_A(B_i}}$ and $\overline{\tau\sigma} = 1_{\overline{H_A(A_i)}}$ using property (I) in the proof of Theorem 4.2. Furthermore, using property (II) in the same proof, we can find a non-zero integer $k$ with $k\sigma\tau = k1_{H_A(B_i)}$ and $k\tau\sigma = k1_{H_A(A_i)}$. By splitting off the $p$-components of $A_i$ and $B_i$ for those primes $p$ which divide $k$, we obtain decompositions $A_i = T_i \oplus C_i$ and $B_i = S_i \oplus D_i$ into direct sums of fully invariant subgroups such that multiplication by $k$ is an automorphism of $C_i$ and $D_i$. In particular, the restriction of $T_A(\sigma)$ to $T_AH_A(C_i)$ induces an isomorphism between $C_i$ and $D_i$. This shows $A_i \oplus T_i \cong B_i \oplus S_i$ as required.

Finally, $\overline{E(A)} = \overline{H_A(B_1)} \oplus \dots \oplus \overline{H_A(B_k)} \oplus \overline{H_A(A_{k+1})} \oplus \dots \oplus \overline{H_A(A_n)}$ by the classical Krull-Schmitt-Theorem. This gives $1_A - (f_1 + \dots + f_k) - (e_{k+1} + \dots + e_n) \in tE(A)$. By Lemma 5.1, we can extend $\{f_1, \dots, f_k\}$ to a set $\{f_1, \dots, f_k, g_{k+1}, \dots, g_n\}$ of orthogonal idempotents of $E(A)$ with $\overline{e}_j = g_j + tE(A)$. Moreover, the $g_j$'s can be chosen in such a way that $A_j = g_j(A) \oplus C_j$ for some bounded $C_j$. Setting $A'_j = g_j(A)$ and $f = f_1 + \dots f_k + g_{k+1} + \dots + g_n$ gives a decomposition $A = B_1 \oplus \dots \oplus B_k \oplus A'_{k+1} \oplus \dots \oplus A'_n \oplus (1-f)(A)$. Since $1 - f \in t\bigl(E(A)\bigr)$, we obtain that $(1-f)(A)$ is bounded, as desired. $\square$

Combining the last result with Proposition 3.3, we immediately obtain

**Corollary 5.3.** *Let $A \in \mathcal{G}$. Every $\mathcal{G}_A$-presented group is a direct sum of essentially indecomposable $\mathcal{G}_A$-presented groups.*

The proof of the next result is based on that of [4, Lemma 26.4], which is used in the proof of the Crawley-Jonnson-Warfield-Theorem, but several modifications to the arguments used in [4] are necessary due to the fact that [4, Lemma 26.4] deals with direct sums of modules whose endomorphism ring is local, while we consider direct sums of groups $A$ for which $\overline{E(A)}$, but not $E(A)$ is local.

**Lemma 5.4.** *Let $G = B \oplus C = N \oplus H$ be a $\mathcal{G}$-decomposable group. If $N \in \mathcal{G}$ is not torsion, then $G = N \oplus T \oplus B' \oplus C'$ for subgroups $B' \subseteq B$ and $C' \subseteq C$ and a sigma-cyclic torsion group $T$ such that $T_p \neq 0$ for only finitely many primes $p$.*

P r o o f. As in the proof of [4, Lemma 26.4], it suffices to consider the case that $G = N \oplus H \oplus H' = H \oplus K \oplus L$. We show that $G = N \oplus H \oplus T \oplus K' \oplus L'$ for some $K' \subseteq K$, $L' \subseteq L$, and a sigma-cyclic torsion group $T$ with $T_p \neq 0$ for only finitely many primes $p$.

Once this has been shown, we use Theorem 5.2 to write $N = \bigoplus_{i=1}^{k} N_i$ where $N_1, \dots, N_k$ are essentially indecomposable groups which are not torsion, and prove the lemma by induction on $k$. The case $k = 1$ is trivial since it corresponds to $H = 0$. Setting $H = N_1 \oplus \ldots \oplus N_k$, we can find $T_1$, $K_1 \subseteq B$, and $L_1 \subseteq C$ as desired with $G = H \oplus T_1 \oplus K_1 \oplus L_1$. By the result in the first paragraph, there are $K_2 \subseteq K_1$, $L_2 \subseteq L_1 \oplus T_1$, and a suitable torsion group $T_2$ with $G = N_{k+1} \oplus H \oplus T_2 \oplus K_2 \oplus L_2$. Let $p_1, \dots, p_t$ be the primes for which $T_1[p_i] \neq 0$. Since $G$ is $\mathcal{G}$-decomposable, we can write $G = V \oplus U$ such that $V = G_{p_1} \oplus \ldots \oplus G_{p_t}$ is a direct sum of cyclics and $U$ is a fully invariant subgroup of $G$ for which $U[p_i] = 0$ and $U = p_i U$ for $i = 1, \dots, t$. Therefore, $L_1 \oplus T_1 = T_1 \oplus (L_1 \cap V) \oplus (L_1 \cap U) = L_2 \oplus S$ for some subgroup $S \subseteq L_1 \oplus T_1$. Observe that $L_2 \cap V, S \cap V \subseteq T_1 \oplus (L_1 \cap V)$. Since $L_1 \cap U$ is a direct summand of $U$, it is fully invariant in $L_1 \oplus T_1$. Therefore, $T_1 \oplus (L_1 \cap V) = (L_2 \cap V) \oplus (S \cap V)$ and $(L_1 \cap U) = (L_2 \cap U) \oplus (S \cap U)$. We set $T = (L_2 \cap V) \oplus T_2$ and obtain $G = N \oplus T \oplus K_2 \oplus (L_2 \cap U)$ with $L_2 \cap U \subseteq L_1 \subseteq C$ as desired.

Following [4], we choose idempotents $e, e', f \in E(G)$ such that $ee' = e'e = 0$, $K = e(G)$, $L = e'(G)$, $H = (1 - e - e')(G)$, $N = f(G)$, and $H \oplus H' = (1 - f)(G)$. Because of $(1 - e - e')(G) \subseteq (1 - f)(G)$, we have $f = fef + fe'f$. Since $f$ is the identity in $fE(G)f \cong E(N)$, it is impossible that $\overline{fef}$ and $\overline{fe'f}$ are both in the Jacobson-radical of $\overline{fE(G)f}$. Observe that $\overline{E(N)}$ is non-zero since $N$ is not torsion. But $\overline{E(N)} \cong \overline{fE(G)f}$ is a local ring, and one of these two elements is a unit in $\overline{fE(G)f}$. Without loss of generality, this is the case for $\overline{fef}$. There is $r \in E(G)$

such that $s = frf$ satisfies $\overline{s}\,\overline{fef} = \overline{fef}\,\overline{s} = \overline{f}$. We can find $t_1, t_2 \in t(fE(G)f)$ with $sfef = f + t_1$ and $fefs = f + t_2$. Choose a non-zero integer $m$ with $mt_1 = mt_2 = 0$, and let $p_1, \ldots, p_n$ be the primes dividing $m$. As before, $T = G_{p_1} \oplus \ldots \oplus G_{p_n}$ is sigma-cyclic, and $G = T \oplus U$ for some fully invariant subgroup $U$ of $G$ for which multiplication by $p_i$ is an automorphism. In contrast to [4], $ese$ need not be an idempotent of $E(G)$. Nevertheless, we are able to show $U = (ese)(U) \oplus (1 - ese)(U)$:

Since $s = fs = fs$, we have $(ese)^2 = efsfefsfe = efs(f + t_1)fe = ese + efst_1fe$ with $mefst_1fe = 0$. Hence, $(ese)^2 = ese + t'$ for some $t' \in tE(G)$ with $t'(G) \subseteq T$. Since $U$ is fully invariant in $G$, it remains to show $ese(U) \cap (1 - ese)(U) = 0$. If $w = ese(u) = (1 - ese)(v)$ for some $u, v \in U$, then $ese(w) = [ese - (ese)^2](u) = (-t')(u) \in T \cap U$. Thus, $0 = (ese)^2(v) = ese(v) + t'(v) = ese(v) = w$.

Moreover, $(1 - e)(G) = L \oplus H$, and hence $(1 - e)(U) = (1 - e)(G) \cap U = (L \cap U) \oplus (H \cap U)$ is contained in $(1 - ese)(U)$. Since $U = (1 - e)(U) \oplus (K \cap U)$ by the full invariance of $U$ in $G$, we have $(1 - ese)(U) = (L \cap U) \oplus (H \cap U) \oplus (K \cap (1 - ese)(U))$. On the other hand, $ese(U) \subseteq K \cap U$. Hence, $K \cap U = ese(U) \oplus (K \cap (1 - ese)(U))$. We set $K' = K \cap (1 - ese)(U)$ and obtain $U = ese(U) \oplus K' \oplus (L \cap U) \oplus (H \cap U)$. We show that $N \cap U$ can replace $ese(U)$ in this decomposition. For this, define $\varphi\colon U \to ese(U)$ by $\varphi(u) = ese(u)$. Since $K' \oplus (L \cap U) \oplus (H \cap U) = \ker \varphi$, it suffices to show that $\varphi|(N \cap U)$ is an isomorphism. For $u \in U$, we have $fse(u) \in N \cap U$ and $ese(u) = (ese)^2(u) - t'(u) = ese(se)(u) = ese(fse)(u) \in \varphi(N \cap U)$. Furthermore, if $ese(x) = 0$ for some $x \in N \cap U$, then $x = f(x)$ yields $0 = esef(x) = efsfef(x) = ef(f + t_2)(x) = ef(x)$ since $meft_2 = 0$ yields $eft_2(x) \in T \cap U$. Thus, $0 = sfef(x) = (f + t_1)(x) = f(x) = x$.

Therefore, $U = (N \cap U) \oplus K' \oplus (L \cap U) \oplus (H \cap U)$ and $T = (N \cap T) \oplus (H \cap T) \oplus (H' \cap T)$. Consequently, $G = N \oplus H \oplus (H' \cap V) \oplus K' \oplus L'$ with $L' = L \cap U$. Observe that $H' \cap T$ is sigma-cyclic with only finitely many non-zero $p$-primary components. $\square$

**Lemma 5.5.** *Let $G = N \oplus H = B \oplus C$ be $\mathcal{G}$-decomposable where $N \in \mathcal{G}$ or $N$ is finite. If $X$ is a finite subset of $N \cap B$, then $B = B_0 \oplus B^0$ and $C = C_0 \oplus C^0$ such that $X \subseteq B_0$ and $N \oplus D \cong B_0 \oplus C_0 \oplus E$ for finite groups $D$ and $E$.*

Proof. We first consider the case that $N$ is finite. Since $G$ is $\mathcal{G}$-decomposable, we can write $G = T \oplus U$ where $U$ and $T$ are fully invariant and $T$ is a sigma-cyclic group containing $N$ with $T_p \neq 0$ for only finitely many primes $p$. We have $T = (T \cap B) \oplus (T \cap C)$. There are finite direct summands $B_0$ of $T \cap B$ and $C_0$ of $T \cap C$ with $N \subseteq B_0 \oplus C_0$.

In the case that $N \in \mathcal{G}$, we use Lemma 5.4 to write $G = N \oplus T \oplus B'' \oplus C''$ where $B'' \subseteq B$, $C'' \subseteq C$ and $T$ is a direct sum of cyclics with $T_p \neq 0$ for only finitely many primes $p$. Set $B_* = B \cap (N \oplus T \oplus C'')$ and $C_* = C \cap (N \oplus T \oplus B'')$. We have $X \subseteq B_*$, $B = B_* \oplus B''$, and $C = C_* \oplus C''$. Moreover, $B_* \oplus C_* \cong N \oplus T$. Let $p_1, \ldots, p_n$ be the primes with $T_p \neq 0$, and set $V = G_{p_1} \oplus \ldots \oplus G_{p_n}$. Write $G = V \oplus U$ for some fully invariant subgroup $U$ of $G$. Since $T \subseteq V$, we have $(N \cap V) \oplus T \cong (B_* \cap V) \oplus (C_* \cap V)$ and $N \cap U \cong (B_* \cap U) \oplus (C_* \cap U)$. Observe that $N \cap V$ is finite since it is a torsion direct summand of a group in $\mathcal{G}$. Moreover, $B_* \cap V$ is a direct sum of cyclics, and we can write $B_* \cap V = W \oplus E$ where $W$ is finite and $B_0 = B_* \oplus W$ contains $X$. We set $C_0 = C_* \cap U$, $B' = E \oplus B''$, and $C' = (C_* \cap V) \oplus C''$. Since $(N \cap V) \oplus B_0 \oplus C_0 = (B_* \cap U) \oplus (C_* \cap U) \oplus W \oplus (N \cap V) \cong (N \cap U) \oplus (N \cap V) \oplus W = N \oplus W$, we have obtained the desired decomposition of $G$. □

We are now able to show that the class of $\mathcal{G}$-decomposable groups is closed with respect to direct summands. By Theorem 5.2, every $\mathcal{G}$-decomposable group $G$ has the form $G = \bigoplus_{i \in I} G_i$ such that $G_i$ is either a cyclic $p$-group or an essentially indecomposable honest mixed group.

**Theorem 5.6.** *Let $G = B \oplus C = \bigoplus_{i \in I} G_i$ be $\mathcal{G}$-decomposable where each $G_i$ is either an essentially indecomposable group in $\mathcal{G}$ or a cyclic $p$-group. Then, $B \cong \bigoplus_{j \in J} H_j$ where, for each $j \in J$, we can find $i \in I$ such that $H_j$ is a direct summand of $G_i$.*

P r o o f. By Kaplansky's Theorem, we may assume that $G$ is countable. We write $B = \{b_n \mid n < \omega\}$ with $b_0 = 0$. We construct an ascending chain $0 = B_0 \subseteq \ldots \subseteq B_n \subseteq \ldots$ of direct summands of $B$ such that $B_n \in \mathcal{G}$ and contains $b_n$ for all $n < \omega$.

We write $B = B_n \oplus D_n$ and write $b_{n+1} = x + y$ with $x \in B_n$ and $y \in D_n$. We can find a finite subset $I_n$ of $I$ such that $y \in N = \bigoplus_{i \in I_n} G_i$. Observe that $N \in \mathcal{G}$. We apply Lemma 5.5 to the decomposition $G = D_n \oplus [B_n \oplus C]$ to obtain $D_n = K_n \oplus K'$ and $B_n \oplus C = L_n \oplus L'$ such that $y \in K_n$ and $K_n \oplus L_n \oplus S_n \cong N \oplus T_n$ for some finite groups $S_n$ and $T_n$. We set $B_{n+1} = B_n \oplus K_n$. Since $B = \bigcup_{n<\omega} B_n$, we have $B \cong \bigoplus_{n<\omega} K_n$. Every $p$-primary cyclic direct summand of $K_n$ is isomorphic a direct summand of $\bigoplus_{i \in I_n} (G_i)_p$ as desired. If $K_n$ has an essentially indecomposable direct summand $W$, then there is $i \in I_n$ such that $W \oplus U_i \cong G_i \oplus V_i$ for some finite groups $U_i$ and $V_i$. But then, $W = W' \oplus W''$ where $W'$ is isomorphic to a direct summand of $G_i$ and $W''$ is finite. Hence, $K_n = \left(\bigoplus_{i \in I_n} H_i\right) \oplus T_n$ where each $H_i$ is isomorphic to a direct summand of $G_i$ and $T$ is finite. This proves the theorem. □

**Corollary 5.7.** *Let $A \in \mathcal{G}$ and $G = B \oplus C = \bigoplus_{i \in I} G_i$.*

a) *If each $G_i$ is a reduced $A$-generated group in $\mathcal{G}$, then $B \cong T \oplus \bigoplus_{j \in J} B_j$, where each $B_j$ is a reduced $A$-generated group in $\mathcal{G}$, and $T$ is an $A$-solvable torsion group.*
b) *If each $G_i$ is $\mathcal{G}_A$-presented, then $B$ is a direct sum of an $A$-solvable torsion group and a direct sum of $\mathcal{G}_A$-presented groups.*

c) *Every $A$-projective group $P$ is of the form $P = \bigoplus_I P_i$ where each $P_i$ is isomorphic to a cyclic or essentially indecomposable direct summand of $A$.*

Proof. By Theorem 4.6, we have $B \cong \bigoplus_J B_j$ where each $B_j$ is either torsion or isomorphic to a direct summand of $G_i$. Let $J_1 = \{j \in J \mid B_j \text{ is torsion}\}$. Since $\bigoplus_{J_1} B_j$ is an $A$-generated reduced torsion group, it is $A$-solvable by Lemma 2.1. This proves the corollary. □

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