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STRONG ASYMMETRIC DIGRAPHS WITH PRESCRIBED INTERIOR AND ANNULUS

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Abstract. The directed distance $d(u,v)$ from u to v in a strong digraph D is the length of a shortest $u-v$ path in D. The eccentricity $e(v)$ of a vertex v in D is the directed distance from v to a vertex furthest from v in D. The center and periphery of a strong digraph are two well known subdigraphs induced by those vertices of minimum and maximum eccentricities, respectively. We introduce the interior and annulus of a digraph which are two induced subdigraphs involving the remaining vertices. Several results concerning the interior and annulus of a digraph are presented.

1. Introduction

A digraph D is *strong* if for every two vertices u and v of D, there is both a $u-v$ (directed) path and a $v-u$ path in D. For vertices u and v in a strong digraph D, the *directed distance* $d(u,v)$ from u to v is the length of a shortest $u-v$ path in D. We say that a digraph D is *asymmetric* if whenever (u,v) is an arc of D, then (v,u) is not an arc of D.

For a vertex v in a strong digraph D, the *eccentricity* $e(v)$ of v is the directed distance from v to a vertex furthest from v in D. The *radius* rad D of D is the minimum eccentricity among the vertices of D; while its *diameter* diam D is the maximum eccentricity. The *center* $C(D)$ of D is the subdigraph induced by those vertices of D having minimum eccentricity; while the *periphery* $P(D)$ is the subdigraph induced by those vertices having maximum eccentricity. In Figure 1, we give an example of a strong asymmetric digraph D with its center $C(D)$ and periphery $P(D)$. In addition, the radius of D is 3, the diameter of D is 4, and the eccentricity of each vertex is indicated.

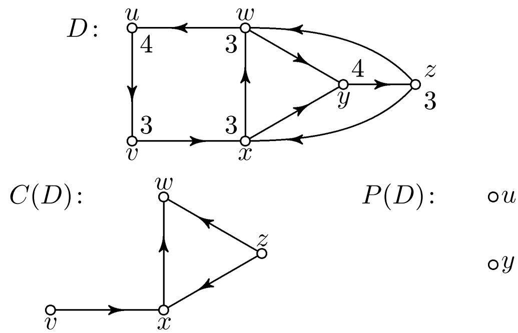

Fig. 1

If F and H are subdigraphs of a strong digraph D, then the standard *directed distance* $d(F,H)$ from F to H is defined by

$$d(F,H) = \min\{d(u,v) \mid u \in V(F), \quad v \in V(H)\}.$$

If D is the digraph given in Figure 1 with subdigraphs $F = \langle\{u,v\}\rangle$ and $H = \langle\{z\}\rangle$, then $d(F,H) = 3$, while $d(H,F) = 2$. Clearly, this distance is not a metric, but it is a generalization of the directed distance from one vertex to another. The first result shows that it is possible to prescribe both the center and periphery of a digraph at the same time.

Theorem 1. *For every two asymmetric digraphs D_1 and D_2, there exists a strong asymmetric digraph H such that $C(H) \cong D_1$ and $P(H) \cong D_2$.*

Proof. We define a strong asymmetric digraph H (see Figure 2) by

$$V(H) = V(D_1) \cup V(D_2) \cup \{z_i \mid 1 \leqslant i \leqslant 6\}$$

and

$$E(H) = E(D_1) \cup E(D_2) \cup \{(z_i, z_{i+1}) \mid 1 \leqslant i \leqslant 5\} \cup \{(x, z_1), (z_5, x) \mid x \in V(D_2)\} \\ \cup \{(x, z_1), (x, z_4), (z_5, x), (z_6, x) \mid x \in V(D_1)\}.$$

From the construction of H, we have
(i) $e(x) = 3$ for $x \in V(D_1)$,
(ii) $e(z_5) = e(z_6) = 4$,
(iii) $e(z_i) = 5$ for $1 \leqslant i \leqslant 4$, and
(iv) $e(x) = 6$ for $x \in V(D_2)$.
Thus, $C(H) \cong D_1$ and $P(H) \cong D_2$. □

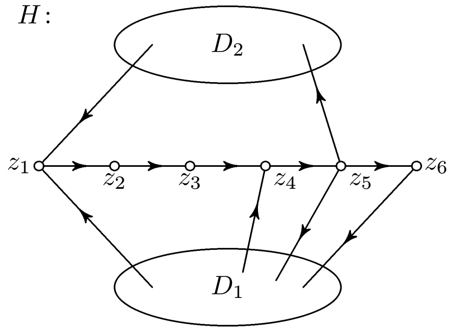

Fig. 2

The topological concepts of interior an annulus for a connected graph were studied in [1]. For a strong digraph D with rad $D < \operatorname{diam} D$, the *interior* $\operatorname{Int}(D)$ of D is defined by

$$\operatorname{Int}(D) = \langle \{v \in V(D) \mid e(v) < \operatorname{diam} D\} \rangle .$$

If rad $D = \operatorname{diam} D$, we define

$$\operatorname{Int}(D) = D.$$

The *annulus* $\operatorname{Ann}(D)$ of a strong digraph D is defined only when rad $D < \operatorname{diam} D - 1$ and is defined by

$$\operatorname{Ann}(D) = \langle \{v \in V(D) \mid \operatorname{rad} D < e(v) < \operatorname{diam} D\} \rangle .$$

Otherwise, we say that D has no annulus. A strong digraph D is shown in Figure 3 with its interior $\operatorname{Int}(D)$ and annulus $\operatorname{Ann}(D)$. The eccentricity of each vertex of D is also indicated.

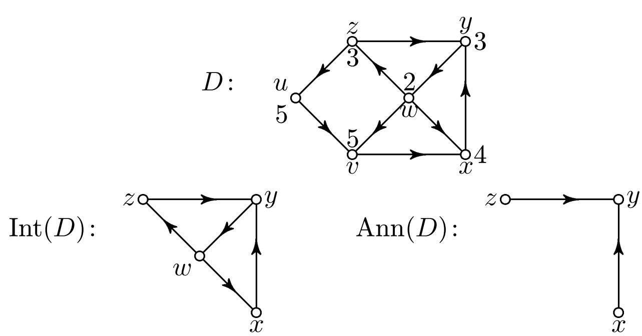

Fig. 3. The interior and annulus of a strong digraph

In this paper, we investigate strong asymmetric digraphs with a pair of prescribed induced subdigraphs, where the pair is chosen from the center, interior, annulus, and periphery.

From the definition of interior, it is clear that the center of a strong digraph H is an induced subdigraph of the interior of H. In addition, the interior of H is isomorphic to the center of H if and only if $\operatorname{rad} H \geqslant \operatorname{diam} H - 1$. For any subdigraph F of an asymmetric digraph D, our first result states precisely when D can be embedded in some strong asymmetric digraph H such that the interior and center of H are D and F, respectively.

Theorem 2. *Let D be an asymmetric digraph and let F be an induced subdigraph of D. Then there exists a strong asymmetric digraph H containing D as an induced subdigraph such that* $\operatorname{Int}(H) = D$ *and* $C(H) = F$ *if and only if $F = D$ or for each $y \in V(F)$, there exists $x \in V(D) - V(F)$ such that there is an $x - y$ path in D.*

P r o o f. Assume that $F \neq D$ and assume further that for each $y \in V(F)$, there is an $x - y$ path in D for some $x \in V(D) - V(F)$. Let $S = \{x \in V(D) - V(F) \mid d(\langle\{x\}\rangle, F) = 1\}$ and let $m = \max\limits_{y \in V(F)} d(\langle S\rangle, \langle\{y\}\rangle)$. We define a strong asymmetric digraph H (see Figure 4) by

$$V(H) = V(D) \cup \{v_i, w_i \mid 1 \leqslant i \leqslant m + 4\}$$

and

$$\begin{aligned} E(H) = E(D) &\cup \{(x, v_1), (v_2, x), (v_{m+4}, x), (x, w_1), (w_{m+4}, x) \mid x \in V(D) - V(F)\} \\ &\cup \{(y, v_2), (y, w_2) \mid y \in V(F)\} \cup \{(v_{m+4}, w_1), (w_{m+4}, v_1)\} \\ &\cup \{(v_i, v_{i+1}), (v_j, v_{m+4}), (w_i, w_{i+1}), (w_j, w_{m+4}) \mid 1 \leqslant i \leqslant m + 2, \\ &\quad 1 \leqslant j \leqslant m + 3\}. \end{aligned}$$

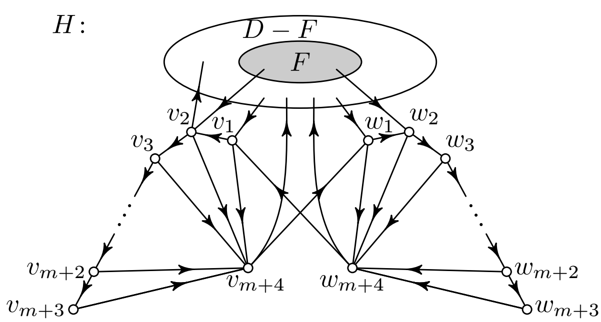

Fig. 4

For each $y \in V(F)$, observe that

(i) $d(y,x) \leqslant 2$ for $x \in V(D) - V(F)$;

(ii) $d(y,v_i) = d(y,w_i) = i-1$ for $2 \leqslant i \leqslant m+3$; and

(iii) $d(y,v_1) \leqslant 3$, $d(y,w_1) \leqslant 3$, and $d(y,v_{m+4}) = d(y,w_{m+4}) = 2$.

If $y, y_1 \in V(F)$, then $d(y,y_1) \leqslant d(y,x) + d(x,y_1) \leqslant 2+m$ for some $x \in S$. Since $d(y,v_{m+3}) = m+2$, we have $e(y) = m+2$ for each $y \in V(F)$.

For $x \in V(D) - V(F)$, if follows that

(i) $d(x,x_1) \leqslant 3$ for $x_1 \in V(D) - V(F)$;

(ii) $d(x,v_i) = d(x,w_i) = i$ for $1 \leqslant i \leqslant m+3$; and

(iii) $d(x,v_{m+4}) = d(x,w_{m+4}) = 2$.

For $y \in V(F)$, we have $d(x,y) \leqslant d(x,x_1) + d(x_1,y) \leqslant 3+m$ for some $x_1 \in S$. Thus, $e(x) = m+3$ for each $x \in V(D) - V(F)$.

We now show that $e(v_i) = e(w_i) = m+4$ for $1 \leqslant i \leqslant m+4$. By the construction of H, for $1 \leqslant i \leqslant m+3$, we have

(i) $d(v_i,x) \leqslant 2$ and $d(w_i,x) = 2$ for $x \in V(D) - V(F)$;

(ii) $d(v_i,y) \leqslant d(v_i,x_1) + d(x_1,y) \leqslant 2+m$ for $y \in V(F)$ and some $x_1 \in S$;

(iii) $d(w_i,y) \leqslant d(w_i,x_1) + d(x_1,y) \leqslant 2+m$ for $y \in V(F)$ and some $x_1 \in S$;

(iv) $d(v_i,w_j) = d(w_i,v_j) = j+1$ for $1 \leqslant j \leqslant m+3$;

(v) $d(v_i,w_{m+4}) = d(w_i,v_{m+4}) = 3$; and

(vi) $d(v_i,v_j) \leqslant m+4$ and $d(w_i,w_j) \leqslant m+4$ for $1 \leqslant j \leqslant m+4$.

Since $d(v_i,w_{m+3}) = d(w_i,v_{m+3}) = m+4$ for $1 \leqslant i \leqslant m+3$, it follows that $e(v_i) = e(w_i) = m+4$.

To complete this part of the proof, we need to show that $e(v_{m+4}) = e(w_{m+4}) = m+4$. Using Figure 4, we observe that

(i) $d(v_{m+4},x) = d(w_{m+4},x) = 1$ for $x \in V(D) - V(F)$;

(ii) $d(v_{m+4},y) \leqslant d(v_{m+4},x_1) + d(x_1,y) \leqslant 1+m$ for $y \in V(F)$ and some $x_1 \in S$;

(iii) $d(w_{m+4},y) \leqslant d(w_{m+4},x_1) + d(x_1,y) \leqslant 1+m$ for $y \in V(F)$ and some $x_1 \in S$;

(iv) $d(v_{m+4},w_j) = d(w_{m+4},v_j) = j$ for $1 \leqslant j \leqslant m+3$;

(v) $d(v_{m+4},w_{m+4}) = d(w_{m+4},v_{m+4}) = 2$; and

(vi) $d(v_{m+4},v_j) = d(w_{m+4},w_j) = j+1$ for $1 \leqslant j \leqslant m+3$.

Thus, $e(v_{m+4}) = e(w_{m+4}) = m+4$, and we conclude that $\mathrm{Int}(H) = D$ and $C(H) = F$.

Now suppose that $F = D$. We define a strong asymmetric digraph H (see Figure 5) by

$$V(H) = V(D) \cup \{v_i, w_i \mid 1 \leqslant i \leqslant 4\}$$

and

$$\begin{aligned}E(H) = E(D) &\cup \{(x, v_1), (x, w_1), (v_4, x), (w_4, x) \mid x \in V(D)\}\\ &\cup \{(v_i, v_4), (w_i, w_4) \mid 1 \leqslant i \leqslant 3\} \cup \{(v_4, w_1), (w_4, v_1)\}\\ &\cup \{(v_i, v_{i+1}), (w_i, w_{i+1}) \mid 1 \leqslant i \leqslant 2\}.\end{aligned}$$

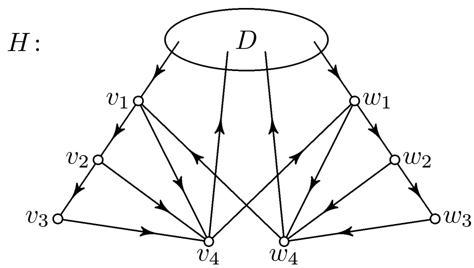

Fig. 5

For $x, x_1 \in V(D)$ and $1 \leqslant i \leqslant 4$, we have $d(x, x_1) \leqslant 3$ and $d(x, v_i) = d(x, w_i) \leqslant 3$. Since $d(x, v_3) = d(x, w_3) = 3$, it follows that $e(x) = 3$ for $x \in V(D)$. For $z \in V(H) - V(D)$ and $1 \leqslant i \leqslant 4$, we have from the construction of H that $d(v_i, z) \leqslant 4$ and $d(w_i, z) \leqslant 4$. It is clear that $d(v_i, x) \leqslant 2$ and $d(w_i, x) \leqslant 2$ for $x \in V(D)$ and $1 \leqslant i \leqslant 4$. Since

$$d(v_i, w_3) = d(v_4, v_3) = d(w_i, v_3) = d(w_4, w_3) = 4$$

for $1 \leqslant i \leqslant 3$, we conclude that $e(v_i) = e(w_i) = 4$ $(1 \leqslant i \leqslant 4)$. Thus, $\mathrm{Int}(H) = C(H) = D = F$.

We claim that these are precisely the conditions needed for the existence of a strong asymmetric digraph H with $\mathrm{Int}(H) = D$ and $C(H) = F$. That is, if $F \neq D$ and if there exists some $y \in V(F)$ such that for each $x \in V(D) - V(F)$, there is no $x - y$ path in D, then there does not exist a strong asymmetric digraph H with $\mathrm{Int}(H) = D$ and $C(H) = F$. Suppose, to the contrary, that there is some strong asymmetric digraph H with $\mathrm{Int}(H) = D$ and $C(H) = F$. Assume that the vertex y_1 of F has the property for each $x \in V(D) - V(F)$, there is no $x - y_1$ path in D. Since $F \neq D$, we have $P(H) = \langle V(H) - V(D) \rangle$. Observe that if $e_H(y) = m$ for $y \in V(F)$, then $e_H(z) \geqslant m + 2$ for $z \in V(P(H))$. For $y \in V(F)$ and $x \in V(H)$ with $(x, y) \in E(H)$, it follows that $e_H(x) \leqslant 1 + e_H(y) = 1 + m$. Thus, we must have $x \in V(D)$. But this says that for each $x \in V(H) - V(F)$, there is no $x - y_1$ path in H, which contradicts that H is strong. $\square$

Corollary 3. *Let D be an asymmetric digraph and let F be a proper induced subdigraph of D. Then there exists a strong asymmetric digraph H containing D as in induced subdigraph such that* $\mathrm{Int}(H) = D$ *and* $\mathrm{Ann}(H) = F$ *if and only if for each $y \in V(D) - V(F)$, there exists $x \in V(F)$ such that there is an $x - y$ path in D.*

The proof of Corollary 3 follows directly from Theorem 2, which states that there exists a strong asymmetric digraph H with $\mathrm{Int}(H) = D$ and $C(H) = D - V(F)$ if and only if $D - V(F) = D$ or for each $y \in V(D) - V(F)$, there exists an $x \in V(F)$ such that there is an $x - y$ path in D. Since $D \neq F$ and $V(F) \neq \emptyset$ in Corollary 3, we have $\mathrm{Ann}(H) = \mathrm{Int}(H) - V(C(H)) = F$.

We state the following two corollaries without proof.

Corollary 4. *Let D and F be asymmetric digraphs. Then there exists a strong asymmetric digraph H such that* $\mathrm{Int}(H) = D_1 \cong D$ *and* $C(H) = F_1 \cong F$ *if and only if $F \cong D$ or there is some induced subdigraph of $D_1 \cong D$, say $F_1 \cong F$, with the property that for each $y \in V(F_1)$, there exists $x \in V(D_1) - V(F_1)$ such that there is an $x - y$ path in D_1.*

Corollary 5. *Let D and F by asymmetric digraphs. Then there exists a strong asymmetric digraph H such that* $\mathrm{Int}(H) = D_1 \cong D$ *and* $\mathrm{Ann}(H) = F_1 \cong F$ *if and only if there is some induced subdigraph of $D_1 \cong D$, say $F_1 \cong F$, with the property that for each $y \in V(D_1) - V(F_1)$, there exists $x \in V(F_1)$ such that there is an $x - y$ path in D_1.*

3. Strong asymmetric digraphs with prescribed annulus and periphery

The next result shows that for any two asymmetric digraphs D and F, there is some strong asymmetric digraph H such that the annulus and periphery of H are D and F, respectively. Furthermore, the distance from the annulus to the periphery of H can be arbitrarily large.

Theorem 6. *Let D and F be asymmetric digraphs and let $n \geqslant 2$ be an integer. Then there exists a strong asymmetric digraph H such that $P(H) \cong F$ and* $\mathrm{Ann}(H) \cong D$ *with* $d(\mathrm{Ann}(H), P(H)) = n$. *In addition, if $p(D) \geqslant 2$, then there exists a strong asymmetric digraph H such that $P(H) \cong F$ and* $\mathrm{Ann}(H) \cong D$ *with* $d(\mathrm{Ann}(H), P(H)) = 1$.

Proof. Let $t = \max\{3, n\}$. For $n = 2$, we define a strong asymmetric digraph H (see Figure 6 (a)) by

$$V(H) = V(D) \cup V(F) \cup \{v_i \mid 1 \leqslant i \leqslant 3\}$$

and

$$E(H) = E(D) \cup E(F) \cup \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$$
$$\cup \{(y, x), (x, v_1), (v_2, x), (v_i, y) \mid x \in V(D), y \in V(F), 1 \leqslant i \leqslant 3\}.$$

For $n \geqslant 3$, we define a strong asymmetric digraph H (see Figure 6(b)) by

$$V(H) = V(D) \cup V(F) \cup \{v_i \mid 1 \leqslant i \leqslant t\}$$

and

$$E(H) = E(D) \cup E(F) \cup \{(v_i, v_{i+1}) \mid 1 \leqslant i \leqslant t-1\}$$
$$\cup \{(y, x), (x, v_1), (v_{t-1}, x) \mid x \in V(D), y \in V(F)\}$$
$$\cup \{(v_{t-1}, y), (v_t, y), (v_t, v_1) \mid y \in V(F)\}.$$

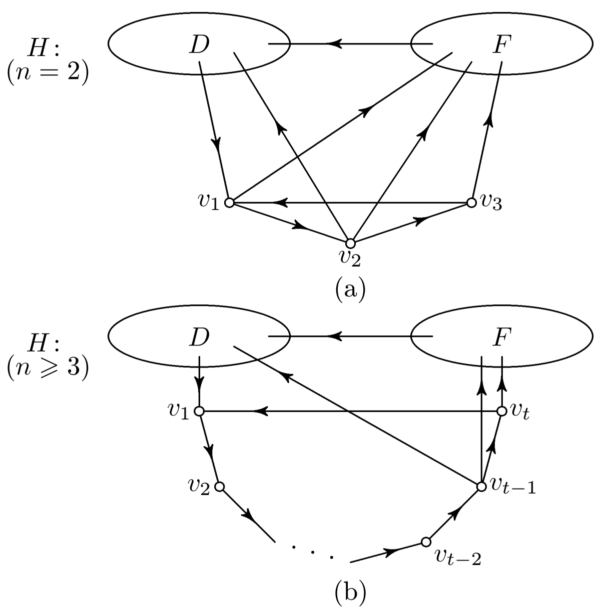

Fig. 6

For $x \in V(D)$, it follows from Figure 6 that

(i) $d(x, v_i) = i$ for $1 \leqslant i \leqslant t$;

(ii) $d(x, x_1) \leqslant d(x, v_{t-1}) + d(v_{t-1}, x_1) = (t-1) + 1 = t$ for $x_1 \in V(D)$;

(iii) $d(x, y) \leqslant d(x, v_{t-1}) + d(v_{t-1}, y) = (t-1) + 1 = t = n$ for $y \in V(F)$ and $n \geqslant 3$; and

(iv) $d(x,y)=2$ for $y\in V(F)$ and $n=2$.
Thus, $e(x)=t$ for $x\in V(D)$.

Observe that for $y\in V(F)$ and $x\in V(D)$, we have $e(y)\leqslant e(x)+1=t+1$ since $d(y,x)=1$. Because $d(y,v_t)=t+1$, it follows that $e(y)=t+1$ for each $y\in V(F)$.

It is clear from the construction of H that for $1\leqslant i\leqslant t$,

(i) $d(v_i,x)\leqslant t-1$ for $x\in V(D)$;
(ii) $d(v_i,y)\leqslant t-1$ for $y\in V(F)$; and
(iii) $d(v_i,v_j)\leqslant t-1$ for $1\leqslant j\leqslant t$.

Since $d(v_i,v_{i-1})=d(v_1,x)=t-1$ for $2\leqslant i\leqslant t$ and $x\in V(D)$, it follows that $e(v_i)=t-1$ $(1\leqslant i\leqslant t)$. Thus, $P(H)\cong F$ and $\mathrm{Ann}(H)\cong D$. For $x\in V(D)$ and $y\in V(F)$, we have

$$d(x,y)=d(x,v_{n-1})+d(v_{n-1},y)=(n-1)+1=n.$$

Consequently, $d(\mathrm{Ann}(H),P(H))=n$.

Now suppose that $p(D)\geqslant 2$. We will show that there exists a strong asymmetric digraph H such that $P(H)\cong F$ and $\mathrm{Ann}(H)\cong D$ with $d(\mathrm{Ann}(H),P(H))=1$. For some $x_1\in V(D)$, let $S=V(D)-\{x_1\}$. Define a strong asymmetric digraph H (see Figure 7) by

$$V(H)=V(D)\cup V(F)\cup\{v_1,v_2,v_3\}$$

and

$$\begin{aligned}E(H)=E(D)\cup E(F)&\cup\{(y,x),(x_1,y)\mid x\in S,y\in V(F)\}\\&\cup\{(x,v_1),(v_2,x),(v_3,x),(v_i,y)\mid x\in V(D),y\in V(F),1\leqslant i\leqslant 3\}\\&\cup\{(v_1,v_2),(v_2,v_3),(v_3,v_1)\}.\end{aligned}$$

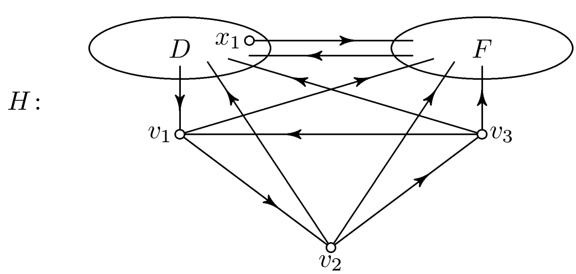

Fig. 7

It follows from the construction of H that for $x\in V(D)$.

(i) $d(x,v_i)=i$ for $1\leqslant i\leqslant 3$;
(ii) $d(x,y)\leqslant 2$ for $y\in V(F)$; and

(iii) $d(x, x_2) \leqslant 3$ for $x_2 \in V(D)$.
Thus, $e(x) = 3$ for $x \in V(D)$. Observe that for $y \in V(F)$ and $x \in S \subset V(D)$, we have $d(y, x) = 1$. This means that $e(y) \leqslant e(x) + 1 = 4$. Since $d(y, v_3) = 4$, it follows that $e(y) = 4$ for $y \in V(F)$. It is clear that $e(v_i) = 2$ for $1 \leqslant i \leqslant 3$. Thus, we conclude that $P(H) \cong F$ and $\mathrm{Ann}(H) \cong D$ with $d(\mathrm{Ann}(H), P(H)) = 1$. □

Corollary 7. *Let D and F be asymmetric digraphs. Then there exists a strong asymmetric digraph H such that $P(H) \cong F$ and $\mathrm{Ann}(H) \cong D$.*

From the previous theorem, if the annulus of a strong asymmetric digraph H is defined, then the distance from the annulus to the periphery of H may be arbitrarily large. On the other hand, our next result shows that the distance from the periphery to the annulus and the distance from the annulus to the center of H must be 1.

Theorem 8. *Let H be a strong asymmetric digraph containing an annulus. Then*

$$d(P(H), \mathrm{Ann}(H)) = d(\mathrm{Ann}(H), C(H)) = 1.$$

P r o o f. Assume that $\mathrm{rad}\, H = k$ and $\mathrm{diam}\, H = m$. Since H has an annulus, we have $m - 2 \geqslant k$. Observe that if $(y, z) \in E(H)$ for $y \in V(P(H))$ and $z \in V(H)$, then $e(z) \geqslant m - 1 > k$; that is, $z \in V(P(H)) \cup (\mathrm{Ann}(H))$. Thus, every arc that leaves $P(H)$ must be incident to a vertex of $\mathrm{Ann}(H)$. Since H is strong, we have $d(P(H), \mathrm{Ann}(H)) = 1$.

Now it follows from H being strong that there exists some $x \in V(H) - V(C(H))$ and $y \in V(C(H))$ such that $(x, y) \in E(H)$. But for each such x, we must have $e(x) \leqslant k + 1 < m$. Thus, $x \in V(\mathrm{Ann}(H))$ and we conclude that $d(\mathrm{Ann}(H), C(H)) = 1$. □

4. Strong asymmetric digraphs with prescribed center and annulus

The next result shows that for any two asymmetric digraphs D and F, there is some strong asymmetric digraph H such that the annulus and center of H are D and F, respectively. Furthermore, the distance from the annulus to the center of H can be arbitrarily large.

Theorem 9. *Let D and F be asymmetric digraphs and let n be a positive integer. Then there exists a strong asymmetric digraph H such that $\mathrm{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \mathrm{Ann}(H)) = n$ if and only if $n \geqslant 2$, $p(D) \geqslant 2$ or $q(F) \geqslant 1$.*

P r o o f. We consider four cases.

Case 1. *Assume that* $n \geqslant 2$. For $n = 2$, we define a strong asymmetric digraph H (see Figure 8(a)) by

$$V(H) = V(D) \cup V(F) \cup \{u_i, v_i \mid 1 \leqslant i \leqslant 3\}$$

and

$$\begin{aligned} E(H) = E(D) \cup E(F) \cup \{(u_1, u_2), (u_2, u_3), (v_1, v_2), (v_2, v_3)\} \\ = \bigcup \{(u_i, x), (v_i, x), (x, y), (y, u_1), (y, v_1) \mid x \in V(D), y \in V(F), 1 \leqslant i \leqslant 3\}. \end{aligned}$$

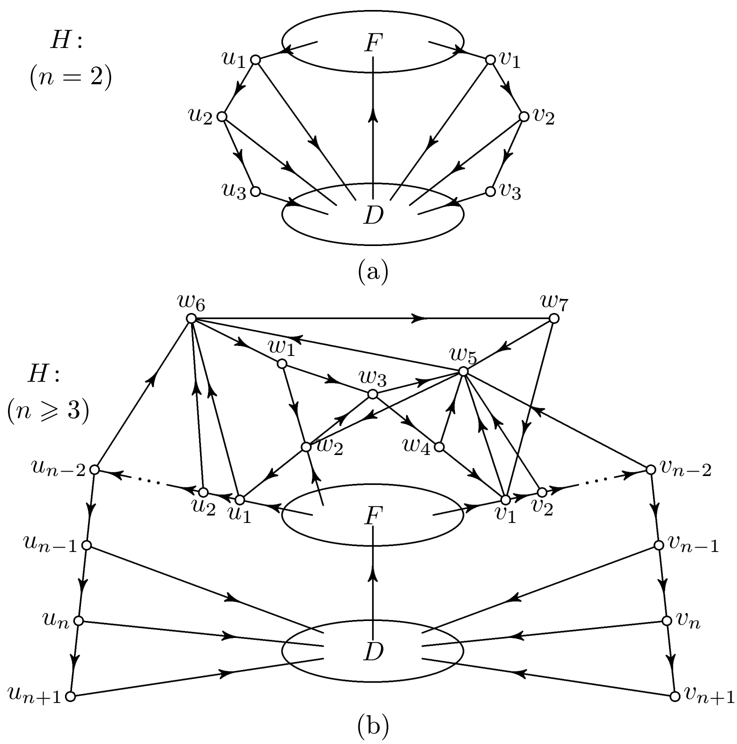

Fig. 8

For $n \geqslant 3$, define a strong asymmetric digraph H (see Figure 8(b)) by

$$V(H) = V(D) \cup V(F) \cup \{u_i, v_i \mid 1 \leqslant i \leqslant n+1\} \cup \{w_i \mid 1 \leqslant i \leqslant 7\}$$

and

$$
\begin{aligned}
E(H) = E(D) \cup E(F) \cup \{(u_i, u_{i+1}), (v_i, v_{i+1}) \mid 1 \leqslant i \leqslant n\} \\
\cup \{(u_i, x), (v_i, x), (x, y), (y, u_1), (y, v_1) \mid x \in V(D), y \in V(F), \\
n-1 \leqslant i \leqslant n+1\} \\
\cup \{(u_i, w_6), (v_i, w_5) \mid 1 \leqslant i \leqslant n-2\} \\
\cup \{(w_i, w_{i+1}) \mid 1 \leqslant i \leqslant 6\} \cup \{(y, w_2) \mid y \in V(F)\} \\
\cup \{(w_1, w_3), (w_2, u_1), (w_3, w_5), (w_4, v_1), (w_5, w_2), (w_6, w_1), (w_7, w_5), (w_7, v_1)\}.
\end{aligned}
$$

For $y \in V(F)$, observe that

(i) $d(y, u_i) = d(y, v_i) = i$ for $1 \leqslant i \leqslant n+1$;

(ii) $d(y, w_i) \leqslant 3$ for $1 \leqslant i \leqslant 7$, $n \geqslant 3$;

(iii) $d(y, x) = d(y, v_{n-1}) + d(v_{n-1}, x) = (n-1) + 1 = n$ for $x \in V(D)$; and

(iv) $d(y, y_1) \leqslant d(y, x) + d(x, y_1) = n+1$ for $y_1 \in V(F)$, $x \in V(D)$.

Since $d(y, v_{n+1}) = n+1$, we conclude that $e(y) = n+1$ for each $y \in V(F)$.

For each $x \in V(D)$ and $y \in V(F)$, we have $d(x, y) = 1$. Since $e(y) = n+1$ for $y \in V(F)$, it follows that $e(x) \leqslant n+2$ for $x \in V(D)$. But $d(x, u_{n+1}) = n+2$ implies that $e(x) = n+2$ for each $x \in V(D)$.

It is clear that $d(u_i, x) = d(v_i, x) = 1$ for $x \in V(D)$ and $n-1 \leqslant i \leqslant n+1$. Thus, in a fashion similar to above, we have $e(u_i) \leqslant n+3$ and $e(v_i) \leqslant n+3$ $(n-1 \leqslant i \leqslant n+1)$. Observing that $d(u_i, v_{n+1}) = d(v_i, u_{n+1}) = n+3$ for $n-1 \leqslant i \leqslant n+1$, we conclude that $e(u_i) = e(v_i) = n+3$ $(n-1 \leqslant i \leqslant n+1)$.

By the construction of H, it follows that $d(u_i, v_1) = d(v_i, u_1) = 3$ for $1 \leqslant i \leqslant n-2$. With this fact, it can be seen for $z \in V(H)$ that $d(u_i, z) \leqslant n+3$, $d(v_i, z) \leqslant n+3$, and $d(u_i, v_{n+1}) = d(v_i, u_{n+1}) = n+3$ $(1 \leqslant i \leqslant n-2)$. This means that $e(u_i) = e(v_i) = n+3$ for $1 \leqslant i \leqslant n-2$.

To show that $e(w_i) = n+3$ $(1 \leqslant i \leqslant 7)$, we make the observation that for $i \in \{1, 2, 5\}$ and $j \in \{3, 4, 6, 7\}$, we have $d(w_i, v_1) = d(w_j, u_1) = 3$, $d(w_i, u_1) \leqslant 2$, and $d(w_j, v_1) \leqslant 2$. Also, $d(w_i, w_j) \leqslant 4$ for $1 \leqslant i \neq j \leqslant 7$. From this, it follows that $e(w_i) \leqslant n+3$ $(1 \leqslant i \leqslant 7)$. Since $d(w_i, v_{n+1}) = d(w_j, u_{n+1}) = n+3$ $(i \in \{1, 2, 5\}$ and $j \in \{3, 4, 6, 7\})$, we conclude that $e(w_i) = n+3$ $(1 \leqslant i \leqslant 7)$. Thus, $\mathrm{Ann}(H) \cong D$ and $C(H) \cong F$. From the construction of H, it is clear that $d(C(H), \mathrm{Ann}(H)) = n$.

Case 2. *Assume that* $n = 1$, $p(D) \geqslant 2$, *and* $p(F) \geqslant 2$. For some $x_1 \in V(D)$ and some $y_1 \in V(F)$, let $S = V(D) - \{x_1\}$ and $T = V(F) - \{y_1\}$. We define a strong asymmetric digraph H (see Figure 9) by

$$V(H) = V(D) \cup V(F) \cup \{u_i, v_i \mid 1 \leqslant i \leqslant 3\}$$

and

$$E(H) = E(D) \cup E(F) \cup \{(u_i, u_{i+1}), (v_i, v_{i+1}) \mid 1 \leqslant i \leqslant 2\}$$
$$\cup \{(u_i, x), (v_i, x), (y, u_1), (y, v_1) \mid x \in V(D), y \in V(F), 1 \leqslant i \leqslant 3\}$$
$$\cup \{(y_1, x_1)\} \cup \{(x_1, y) \mid y \in T\} \cup \{(x, y) \mid x \in S, y \in V(F)\}.$$

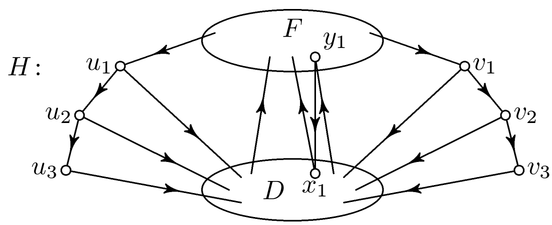

Fig. 9

For $y \in V(F)$, it is clear from the construction of H that $e(y) = 3$. Using a technique similar to the preceding case, we can show that $e(x) = 4$ and $e(u_i) = e(v_i) = 5$ for $x \in V(D)$ and $1 \leqslant i \leqslant 3$. Thus, $\mathrm{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \mathrm{Ann}(H)) = 1$.

Case 3. *Assume that* $n = 1$, $p(D) \geqslant 2$, *and* $p(F) = 1$. Let $V(F) = \{y\}$ and $S = V(D) - \{x_1\}$ for some $x_1 \in V(D)$. We define a strong asymmetric digraph H (see Figure 10) by

$$V(H) = V(D) \cup V(F) \cup \{u_i, v_i \mid 1 \leqslant i \leqslant 3\}$$

and

$$E(H) = E(D) \cup \{(y, x_1), (y, u_1), (y, v_1), (y, u_2), (y, v_2), (x_1, u_1), (x_1, v_1)\}$$
$$\cup \{(x, y), (u_1, x), (v_1, x) \mid x \in S\} \cup \{(u_i, u_{i+1}), (v_i, v_{i+1}) \mid 1 \leqslant i \leqslant 2\}$$
$$\cup \{(u_i, x), (v_i, x) \mid x \in V(D), 2 \leqslant i \leqslant 3\}.$$

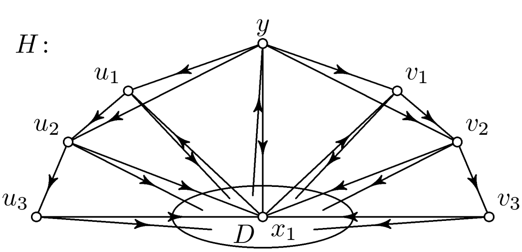

Fig. 10

It is clear from the construction of H that $e(y) = 2$. For $x \in S$, we have $d(x, y) = 1$ and, thus, $e(x) \leqslant 3$. Since $d(x, v_3) = 3$, it follows that $e(x) = 3$ for each $x \in S$. Observe that for $x \in S$, we have $d(x_1, x) \leqslant 2$, $d(x_1, y) \leqslant 3$, and $d(x_1, u_i) = d(x_1, v_i) = i$ $(1 \leqslant i \leqslant 3)$. Thus, $e(x_1) = 3$. Clearly, $d(u_i, x) = d(v_i, x) = 1$ for $x \in S$ and $1 \leqslant i \leqslant 3$. Consequently, $e(u_i) \leqslant 4$ and $e(v_i) \leqslant 4$ $(1 \leqslant i \leqslant 3)$. Since $d(u_i, v_3) = d(v_i, u_3) = 4$ $(1 \leqslant i \leqslant 3)$, it follows that $e(u_i) = e(v_i) = 4$. Therefore, $\mathrm{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \mathrm{Ann}(H)) = 1$.

Case 4. *Assume that* $n = 1$, $p(D) = 1$, *and* $q(F) \geqslant 1$. Let $y_1 \in V(F)$ such that the indegree of y_1 in F is at least 1. Suppose that $V(D) = \{x\}$ and $S = V(F) - \{y_1\}$. We define a strong asymmetric digraph H (see Figure 11) by

$$V(H) = V(D) \cup V(F) \cup \{u_i, v_i \mid 1 \leqslant i \leqslant 4\}$$

and

$$\begin{aligned} E(H) = E(F) &\cup \{(u_i, x), (v_i, x), (y, u_1), (y, v_1) \mid y \in V(F), 1 \leqslant i \leqslant 4\} \\ &\cup \{(x, y) \mid y \in S\} \cup \{(y_1, x)\}. \end{aligned}$$

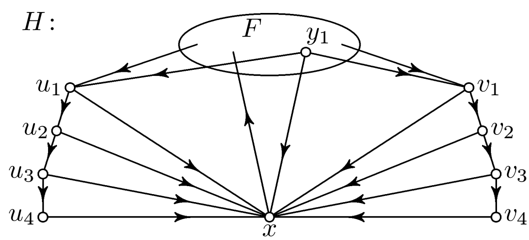

Fig. 11

Using the methods described in Cases 1 and 2, we find that $e(y) = 4$, $e(x) = 5$, and $e(u_i) = e(v_i) = 6$ for $y \in V(F)$ and $1 \leqslant i \leqslant 4$. Thus, $\mathrm{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \mathrm{Ann}(H)) = 1$.

Conversely, assume that $n = 1$, $p(D) = 1$ and $q(F) = 0$. Suppose, to the contrary, that there is some strong asymmetric digraph H with the property that $\mathrm{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \mathrm{Ann}(H)) = 1$. Then there is some vertex y of $C(H)$ such that $(y, x) \in E(H)$ for some $x \in V(\mathrm{Ann}(H))$. Since H is an asymmetric digraph with $p(\mathrm{Ann}(H)) = 1$ and $q(C(H)) = 0$, we have $(z, y) \notin E(H)$ for each $z \in [V(C(H)) \cup V(\mathrm{Ann}(H))] - \{y\}$. If there is some vertex v of H such that v is adjacent to a vertex in $C(H)$, then $e(v) \leqslant \operatorname{rad} H + 1$. Thus, v must be a vertex in $C(H)$ or $\mathrm{Ann}(H)$. This means that y has indegree 0, which contradicts the fact that

H is strong. Therefore, it follows, for asymmetric digraphs D and F with $p(D) = 1$ and $q(F) = 0$, that there does not exist a strong asymmetric digraph H such that $\operatorname{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \operatorname{Ann}(H)) = 1$. □

Corollary 10. *Let D and F be asymmetric digraphs. Then there exists a strong asymmetric digraph H such that $\operatorname{Ann}(H) \cong D$ and $C(H) \cong F$.*

5. Strong asymmetric digraphs with prescribed interior and periphery

We now present a sufficient condition for two asymmetric digraphs D and F to be isomorphic to the periphery and interior, respectively, of some strong asymmetric digraph.

Theorem 11. *Let D and F be asymmetric digraphs. If F is nontrivial and strong, then there exists a strong asymmetric digraph H such that $P(H) \cong D$ and $\operatorname{Int}(H) \cong F$.*

Proof. Assume that $\operatorname{diam} F = m \geqslant 2$, and let $x, y \in V(F)$ such that $d(x, y) = m$. We define a strong asymmetric digraph H (see Figure 12) by

$$V(H) = V(D) \cup V(F)$$

and

$$E(H) = E(D) \cup E(F) \cup \{(z, x) \mid z \in V(D)\} \cup \{(v, z) \mid v \in V(F) - \{x\}, z \in V(D)\}.$$

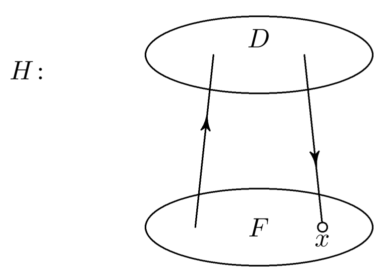

Fig. 12

For $z \in V(D)$ and $v \in V(F)$, it is clear, from the construction of H, that $e(z) = m + 1$ and $e(v) \leqslant m$. Thus, $P(H) \cong D$ and $\operatorname{Int}(H) \cong F$. □

If D and F are asymmetric digraphs such that F is not strong, then even when D is strong, we can draw no conclusion about the existence of a strong asymmetric digraph H such that $P(H) \cong D$ and $\mathrm{Int}(H) \cong F$. For example, define D by $V(D) = \{x_1, x_2, x_3\}$ and $E(D) = \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$, and define F by $V(F) = \{y_1, y_2\}$ and $E(F) = \{(y_1, y_2)\}$. We claim that no strong asymmetric digraph H exists with $P(H) \cong D$ and $\mathrm{Int}(H) \cong F$. Suppose, to the contrary, that there is some strong asymmetric digraph H with this property. It is clear that $V(H) = V(D) \cup V(F)$. Since H is strong and the indegree of vertex y_1 in F is 0, there must be some arc from D to y_1. Assume that $(x_1, y_1) \in E(H)$. From this, it follows that $e(x_1) = 2$. Since H is asymmetric, we have $(y_2, y_1) \notin E(H)$. Thus, $e(y_2) \geqslant 2$, which contradicts the fact that $\mathrm{Int}(H) \cong F$.

On the other hand, if we define D by $V(D) = \{x_1, x_2, x_3, x_4\}$ and $E(D) = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1)\}$, and define F by $V(F) = \{y_1, y_2\}$ and $E(F) = \{(y_1, y_2)\}$, then there exists a strong asymmetric digraph H such that $P(H) \cong D$ and $\mathrm{Int}(H) \cong F$. We define H (see Figure 13) by

$$V(H) = V(D) \cup V(F)$$

and

$$E(H) = E(D) \cup E(F) \cup \{(x_1, y_1), (x_3, y_1)\} \cup \{(y_2, x) \mid x \in V(D)\}.$$

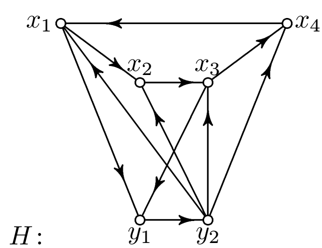

Fig. 13

It is clear from the construction of H that $e(x) = 3$ and $e(y) = 2$ for $x \in V(D)$ and $y \in V(F)$. Thus, $P(H) \cong D$ and $\mathrm{Int}(H) \cong F$.

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