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MODULAR FUNCTIONS ON MULTILATTICES

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Abstract. We prove that every modular function on a multilattice L with values in a topological Abelian group generates a uniformity on L which makes the multilattice operations uniformly continuous with respect to the exponential uniformity on the power set of L.

Keywords: multilattices, modular functions

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Introduction

The foundations of the theory of multilattices were laid in the fifties by M. Benado in [11], motivated by numerous examples of posets which are multilattices but not lattices, and the research was carried on many papers (for example, [10], [16], [18], [19], [20] and many others). In particular, by [10], examples of multilattices are the intervals of modular interval spaces, which are a common generalization of L_1 type Banach spaces, hyperconvex metric spaces and modular lattices.

The aim of the present paper is to prove that modular functions on multilattices generate a topological structure analogously as modular functions on lattices. We recall that in [13] I. Fleischer and T. Traynor, extending a result of K. Birkhoff in [12] for increasing real-valued modular functions, proved that every modular function on a lattice L with values in a topological Abelian group generates a lattice uniformity on L, i.e. a uniformity which makes the lattice operations of L uniformly continuous.

This result allowed to use the theory of lattice uniformities developed in [22], [23], [27], [7], to extend to modular functions on lattices many results of classical measure theory, which have applications in particular in non-commutative measure theory and in fuzzy measure theory (see, for example, [1], [2], [4], [5], [6], [3], [8], [9], [13], [14], [15], [21], [23], [24], [25], [27]).

In [19], the results of [12] have been extended to modular functions on multilattices, proving that every increasing real-valued modular function on a multilattice L generates a pseudometric on L.

In the present paper, extending the results of [13], we prove that every modular function μ on a multilattice L with values in a topological Abelian group generates a multilattice uniformity $\mathcal{U}(\mu)$ on L, i.e. a uniformity which makes the multilattice operations of L uniformly continuous with respect to the exponential uniformity on the power set of L, and $\mathcal{U}(\mu)$ is the weakest multilattice uniformity which makes μ uniformly continuous (see Theorem 2.2.3). For increasing real-valued modular functions, $\mathcal{U}(\mu)$ coincides with the uniformity generated by the pseudometric in [19] and, if L is a lattice, coincides with the lattice uniformity of [13].

The paper is organized as follows: in Section 1, we study properties of multilattice uniformities and give a characterization of multilattice uniformities (Theorem 1.4) which allows to simplify the proof of the main result. In Section 2.1, we study properties of a set associated to a modular function, which are essential tools for the proof of the main result. Finally, in Section 2.2, we prove the main result.

Preliminaries

Let $(L, \leqslant)$ be a poset. For $a, b \in L$ denote by $U(a,b)$ and $L(a,b)$ the sets of all upper and lower bounds of the set $\{a,b\}$, respectively. Further, let $a \vee b$ be the set of all minimal elements of $U(a,b)$ and $a \wedge b$ the set of all maximal elements of $L(a,b)$.

L is said to be a *(directed) multilattice* if:

(1) For every $a, b \in L$, $U(a,b) \neq \emptyset$ and $L(a,b) \neq \emptyset$.

(2) For every $c \in U(a,b)$, there exists $d \in a \vee b$ with $d \leqslant c$.

(3) For every $c \in L(a,b)$, there exists $d \in a \wedge b$ with $d \geqslant c$.

If G is an Abelian group, a function $\mu\colon\ L \to G$ is called *modular* if, for every $a, b \in L$, $c \in a \wedge b$ and $d \in a \vee b$, $\mu(a) + \mu(b) = \mu(c) + \mu(d)$. Then, if μ is modular and $a, b \in L$, we have $\mu(r) = \mu(s)$ for every $r, s \in a \vee b$ and $\mu(t) = \mu(u)$ for every $t, u \in a \wedge b$.

A *congruence* on L is an equivalence relation θ such that $(a,b) \in \theta$ and $(c,d) \in \theta$ imply $(a \vee c, b \vee d) \in' \theta$ and $(a \wedge c, b \wedge d) \in' \theta$, where $(a \vee c, b \vee d) \in' \theta$ means that:

(1) For every $z \in a \vee c$, there exists $z' \in b \vee d$ with $(z', z) \in \theta$.

(2) For every $z' \in b \vee d$, there exists $z \in a \vee c$ with $(z, z') \in \theta$.

The meaning of $(a \wedge c, b \wedge d) \in' \theta$ is analogous.

The following result holds.

Theorem ([19], Th. 2.2). *Let L be a directed multilattice and θ reflexive binary relation on L. Then θ is a congruence relation iff the following conditions hold:*

(1) $(a,b) \in \theta$ *iff there exists* $c \in a \wedge b$ *and* $d \in a \vee b$ *such that* $(c,d) \in \theta$.

(2) $(a,b) \in \theta$, $(b,c) \in \theta$ *and* $a \leqslant b \leqslant c$ *imply* $(a,c) \in \theta$.

(3) $(a,b) \in \theta$ *and* $a \leqslant b$ *imply* $(a \vee c, b \vee c) \in' \theta$ *and* $(a \wedge c, b \wedge c) \in' \theta$.

Through the paper, L will denote *a directed multilattice* and G a *topological Abelian group*.

We set $\Delta = \{(a,b) \in L \times L\colon\ a = b\}$. If $a,b \in L$ and $a \leqslant b$, we set $[a,b] = \{c \in L\colon\ a \leqslant c \leqslant b\}$. We say that a subset A of L is *convex* if, for every $a,b \in A$ with $a \leqslant b$, $[a,b] \subseteq A$. A *filter* on L is a non-empty family $\mathcal{U}$ of non-empty subsets of L which is closed with respect to the intersections and contains the oversets of its elements.

We recall that, if $(L,\mathcal{U})$ is a uniform space, the *exponential uniformity* on the power set $P(L)$ of L is the uniformity which has as its base the family consisting of the sets

$$2^U = \{(A,B) \in P(L) \times P(L)\colon\ \forall x \in A, \exists y \in B\colon\ (x,y) \in U;\\ \forall y \in B, \exists x \in A\colon\ (x,y) \in U\},$$

where $U \in \mathcal{U}$. For $U,V \in \mathcal{U}$ and $x \in L$ we set $U^{-1} = \{(a,b) \in L \times L\colon\ (b,a) \in U\}$, $U \circ V = \{(a,b) \in L \times L\colon\ \exists c \in L\colon\ (a,c) \in U,\ (c,b) \in V\}$ and $U(x) = \{y \in L\colon\ (x,y) \in U\}$.

1. Multilattice uniformities

In this section we introduce and study multilattice uniformities, since in the next section we will see that every modular function generates a multilattice uniformity.

A uniformity $\mathcal{U}$ on L is called a *multilattice uniformity* if the maps

$$\vee\colon\ (a,b) \in L \times L \to a \vee b \in P(L),\quad \wedge\colon\ (a,b) \in L \times L \to a \wedge b \in P(L)$$

are uniformly continuous with respect to the product uniformity in $L \times L$ and the exponential uniformity in $P(L)$. Then $\mathcal{U}$ is a multilattice uniformity iff, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $(a,b) \in V$ and $(c,d) \in V$ imply $(a \vee c, b \vee d) \in 2^U$ and $(a \wedge c, b \wedge d) \in 2^U$.

Lemma 1.1. *Let $\mathcal{U}$ be a uniformity on L. Then $\mathcal{U}$ is a multilattice uniformity iff, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $(a,b) \in V$ and $c \in L$ imply $(a \vee c, b \vee c) \in 2^U$ and $(a \wedge c, b \wedge c) \in 2^U$.*

P r o o f. $\Rightarrow$ is trivial.

$\Leftarrow$ Let $U, V \in \mathcal{U}$ be such that $V \circ V \subseteq U$ and choose, corresponding to V, $V' \in \mathcal{U}$ as in the assumption. Let $(a,b) \in V'$, $(c,d) \in V'$ and $z \in a \vee c$. Then we can choose $z' \in b \vee c$ such that $(z, z') \in V$ and, corresponding to z', we can choose $z'' \in d \vee b$ such that $(z', z'') \in V$. Therefore $(z, z'') \in V \circ V \subseteq U$.

In a similar way we obtain the other conditions. $\square$

Proposition 1.2. *Let $\mathcal{U}$ be a multilattice uniformity. Then, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V \subseteq U$ and the following property: for every $(a,b) \in V$, there exist $c \in a \wedge b$ and $d \in a \vee b$ such that $[c,d] \times [c,d] \subseteq V$.*

P r o o f. Let $U \in \mathcal{U}$ and

$$V = \{(a,b) \in L \times L \colon \exists\, c \in a \wedge b,\ \ d \in a \vee b \colon\ [c,d] \times [c,d] \subseteq U\}.$$

Trivially $V \subseteq U$. Let $(a,b) \in V$, $c \in a \wedge b$ and $d \in a \vee b$ be such that $[c,d] \times [c,d] \subseteq U$. We prove that $[c,d] \times [c,d] \subseteq V$.

Let $x, y \in [c,d]$. Then we can choose $e \in x \wedge y$ such that $e \geqslant c$ and $f \in x \vee y$ such that $f \leqslant d$. Then $[e,f] \times [e,f] \subseteq [c,d] \times [c,d] \subseteq U$, hence $(x,y) \in V$.

It remains to prove that $V \in \mathcal{U}$. Choose a symmetric $W_0 \in \mathcal{U}$ such that $W_0 \circ W_0 \subseteq U$ and, for every $i \in \{1,2,3\}$, $W_i \in \mathcal{U}$ with the following property: $(a,b) \in W_i$ and $(c,d) \in W_i$ imply $(a \vee c, b \vee d) \in 2^{W_{i-1}}$ and $(a \wedge c, b \wedge d) \in 2^{W_{i-1}}$. We prove that $W_3 \subseteq V$. Let $(a,b) \in W_3$, $c \in a \wedge b$, $d \in a \vee b$ and $x, y \in [c,d]$. We have to prove that $(x,y) \in U$. By $(a,b) \in W_3$ and $(a,a) \in W_3$, we get $(a,d) \in W_2$ by the choice of W_3. Moreover, by $(a,d) \in W_2$, $(x,x) \in W_2$ and $x \wedge d = x$ and by the choice of W_2, we can choose $e \in x \wedge a$ such that $(e,x) \in W_1$. Finally, by $(a,b) \in W_3 \subseteq W_2$ and $(a,a) \in W_2$ we get $(a,c) \in W_1$. By $(e,x) \in W_1$, $(a,c) \in W_1$, $c \vee x = x$ and $e \vee a = a$ we get $(a,x) \in W_0$ by the choice of W_1. In a similar way we obtain that $(a,y) \in W_0$. Therefore $(x,y) \in W_0^{-1} \circ W_0 = W_0 \circ W_0 \subseteq U$. $\square$

Proposition 1.3. *Let $\mathcal{U}$ be a multilattice uniformity. Then:*

(1) *For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \subseteq U$ and, for every $x \in L$, $V(x)$ is convex.*

(2) *The topology generated by $\mathcal{U}$ is locally convex, i.e. every $x \in L$ has a base of convex neighbourhoods.*

P r o o f. (2) follows by (1).

(1) The proof is similar to the proof of 1.1.6 of [22] for lattice uniformities. We repeat the proof for completeness.

For $A \subseteq L$, set $c(A) = \{x \in L \colon \exists\, a, b \in A \colon\ a \leqslant x \leqslant b\}$. It is easy to see that $c(A)$ is the smallest convex set which contains A. Let $U \in \mathcal{U}$. By (1.2), we can choose

$V_1 \in \mathcal{U}$ such that $V_1 \circ V_1 \subseteq U$ and, for every $(a,b) \in V_1$ with $a \leqslant b$, $[a,b] \times [a,b] \subseteq V_1$. Choose a symmetric $V_2 \in \mathcal{U}$ such that $V_2 \circ V_2 \subseteq V_1$ and set

$$V = \{(x,y) \in L \times L \colon\ y \in c(V_2(x))\}.$$

Then $V \in \mathcal{U}$ since $V_2 \subseteq V$ and, for every $x \in L$, $V(x)$ is convex since $V(x) = c(V_2(x))$. We prove that $V \subseteq U$. Let $(x,y) \in V$ and $a,b \in V_2(x)$ be such that $a \leqslant y \leqslant b$. Since $(x,a) \in V_2$, $(x,b) \in V_2$ and V_2 is symmetric, we get $(a,b) \in V_1$. By the choice of V_1, since $a,y \in [a,b]$, we get $(a,y) \in V_1$. Since $(x,a) \in V_2 \subseteq V_1$, we obtain $(x,y) \in V_1 \circ V_1 \subseteq U$. □

The following result gives a characterization of multilattice uniformities which allows to simplify the proof of the main result of the next section. It is similar to a characterization of lattice uniformities contained in a manuscript of Hans Weber.

Theorem 1.4. *Let $\mathcal{U}$ be a filter on $L \times L$. Then $\mathcal{U}$ is a multilattice uniformity iff the following conditions hold:*

(1) *For every $U \in \mathcal{U}$, $\Delta \subseteq U$.*

(2) *For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $(a,b) \in V$ implies that there exist $c \in a \wedge b$ and $d \in a \vee b$ with $(c,d) \in U$.*

(3) *For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $(c,d) \in V$, $c \in a \wedge b$ and $d \in a \vee b$ imply $(a,b) \in U$.*

(4) *For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $(a,b) \in V$, $(b,c) \in V$ and $a \leqslant b \leqslant c$ imply $(a,c) \in U$.*

(5) *For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $(a,b) \in V$, $a \leqslant b$ and $c \in L$ imply $(a \vee c, b \vee c) \in 2^U$ and $(a \wedge c, b \wedge c) \in 2^U$.*

Proof. $\Rightarrow$ If $\mathcal{U}$ is a multilattice uniformity, then (1), (4) and (5) hold by definition and (2), (3) follow by (1.2).

$\Leftarrow$ (i) We first prove that, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$.

Let $U \in \mathcal{U}$. By (3) we can choose $V \in \mathcal{U}$ such that $(c,d) \in V$, $c \in a \wedge b$ and $d \in a \vee b$ imply $(a,b) \in U$. Moreover, by (2) we can choose $V' \in \mathcal{U}$ such that $(a,b) \in V'$ implies that there exist $c \in a \wedge b$ and $d \in a \vee b$ such that $(c,d) \in V$. Let $(a,b) \in (V')^{-1}$. Then, since $(b,a) \in V'$, we can find $c \in a \wedge b$ and $d \in a \vee b$ such that $(c,d) \in V$. Therefore $(a,b) \in U$.

(ii) We prove that, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $(a,b) \in V$ and $a \leqslant b$ imply $[a,b] \times [a,b] \subseteq U$.

Let $U \in \mathcal{U}$. By (3), let $V_1 \in \mathcal{U}$ be such that $(c,d) \in V_1$, $c \in a \wedge b$ and $d \in a \vee b$ imply $(a,b) \in U$. By (5), let $V_2 \in \mathcal{U}$ be such that $(a,b) \in V_2$, $a \leqslant b$ and $c \in L$ imply $(a \wedge c, b \wedge c) \in 2^{V_1}$. Again by (5), let $V_3 \in \mathcal{U}$ be such that $(a,b) \in V_3$, $a \leqslant b$ and $c \in L$

imply $(a \vee c, b \vee c) \in 2^{V_2}$. Let $(x, y) \in V_3$ with $x \leqslant y$, and $a, b \in [x, y]$. We prove that $(a, b) \in U$. Since $x \leqslant a, b$ and $y \geqslant a, b$, we can choose $c \in a \wedge b$ and $d \in a \vee b$ such that $c \geqslant x$ and $d \leqslant y$. Hence $x \leqslant c \leqslant d \leqslant y$. Since $(x, y) \in V_3$, $c \vee x = c$ and $c \vee y = y$, by the choice of V_3 we get $(c, y) \in V_2$. Moreover, since $c \wedge d = c$ and $y \wedge d = d$, by the choice of V_2 we get $(c, d) \in V_1$. Then, by the choice of V_1, we get $(a, b) \in U$.

(iii) We prove that, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

Let $U \in \mathcal{U}$. By (ii), let $V_1 \in \mathcal{U}$ be such that $(a, b) \in V_1$ and $a \leqslant b$ imply $[a, b] \times [a, b] \subseteq U$. By (4), we can choose $V_2 \in \mathcal{U}$ such that $(a, b) \in V_2$, $(b, c) \in V_2$ and $(c, d) \in V_2$ with $a \leqslant b \leqslant c \leqslant d$, imply $(a, d) \in V_1$. By (5), we can choose $V_3 \in \mathcal{U}$ with $V_3 \subseteq V_2$ such that $(a, b) \in V_3$, $a \leqslant b$ and $c \in L$ imply $(a \vee c, b \vee c) \in 2^{V_2}$ and $(a \wedge c, b \wedge c) \in 2^{V_2}$. Finally, by (2) we can choose $V_4 \in \mathcal{U}$ such that $(a, b) \in V_4$ implies that there exist $c \in a \wedge b$ and $d \in a \vee b$ such that $(c, d) \in V_3$.

We prove that $V_4 \circ V_4 \subseteq U$. Let $(x, y) \in V_4$ and $(y, z) \in V_4$. By the choice of V_4 we can find $c \in x \wedge y$, $d \in x \vee y$, $e \in y \wedge z$ and $f \in y \vee z$ such that $(c, d) \in V_3$ and $(e, f) \in V_3 \subseteq V_2$. Since $(c, d) \in V_3$ with $c \leqslant d$ and $c \vee f = f$ by $f \geqslant y \geqslant c$, then by the choice of V_3 we can find $w_1 \in d \vee f$ such that $(f, w_1) \in V_2$. In a similar way, since $d \wedge e = e$ by $e \leqslant y \leqslant d$, we can find $w_2 \in c \wedge e$ such that $(w_2, e) \in V_2$.

By $(w_2, e) \in V_2$, $(e, f) \in V_2$ and $(f, w_1) \in V_2$ with $w_2 \leqslant e \leqslant f \leqslant w_1$ we get by the choice of V_2 that $(w_2, w_1) \in V_1$. Since $w_2 \leqslant w_1$, by the choise of V_1 we obtain $[w_2, w_1] \times [w_2, w_1] \subseteq U$. Now observe that $x, z \in [w_2, w_1]$, since $x \geqslant c \geqslant w_2$, $x \leqslant d \leqslant w_1$, $z \geqslant e \geqslant w_2$ and $z \leqslant f \leqslant w_1$. Then $(x, z) \in U$.

(iv) We prove that, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $(a, b) \in V$ and $c \in L$ imply $(a \vee c, b \vee c) \in 2^U$ and $(a \wedge c, b \wedge c) \in 2^U$.

Let $U \in \mathcal{U}$. By (iii), let $V_1 \in \mathcal{U}$ be symmetric and such that $V_1 \circ V_1 \circ V_1 \subseteq U$. By (5), choose $V_2 \in \mathcal{U}$ such that $(a, b) \in V_2$, $a \leqslant b$ and $c \in L$ imply $(a \vee c, b \vee c) \in 2^{V_1}$ and $(a \wedge c, b \wedge c) \in 2^{V_1}$. By (ii), let $V_3 \in \mathcal{U}$ be such that $(a, b) \in V_3$ and $a \leqslant b$ imply $[a, b] \times [a, b] \subseteq V_2$. Moreover, by (2), let $V_4 \in \mathcal{U}$ be such that $(a, b) \in V_4$ implies that there exist $c \in a \wedge b$ and $d \in a \vee b$ such that $(c, d) \in V_3$.

Let $(a, b) \in V_4$, $c \in L$ and $z \in a \vee c$. We prove that there exists $z' \in b \vee c$ such that $(z, z') \in U$. The other conditions can be proved in a similar way.

Since $(a, b) \in V_4$, we can find $r \in a \wedge b$ and $s \in a \vee b$ such that $(r, s) \in V_3$. Then $[r, s] \times [r, s] \subseteq V_2$. Since $r \leqslant a \leqslant s$ and $r \leqslant b \leqslant s$, we get $(r, a) \in V_2$ and $(b, s) \in V_2$. Since $(r, a) \in V_2$ with $r \leqslant a$, and $z \in a \vee c$, we can find $t \in r \vee c$ such that $(t, z) \in V_1$. Since $(r, s) \in [r, s] \times [r, s] \subseteq V_2$ and $t \in r \vee c$, we can find $u \in s \vee c$ such that $(t, u) \in V_1$. Finally, since $(b, s) \in V_2$ with $b \leqslant s$ and $u \in s \vee c$, we can find $z' \in b \vee c$ such that $(z', u) \in V_1$. Then $(z, z') \in V_1 \circ V_1 \circ V_1 \subseteq U$ by the symmetry of V_1.

By (i)–(iv), (1.1) and the assumptions, we conclude that $\mathcal{U}$ is a multilattice uniformity. $\square$

In this section, $\mu\colon L \to G$ will denote a *modular function*.

If $a, b \in L$ and $a \leqslant b$, we set

$$\mu(a, b) = \{\mu(d) - \mu(c)\colon\ a \leqslant c \leqslant d \leqslant b\}.$$

The aim of this section is to prove that μ generates a multilattice uniformity which has as its base the family consisting of the sets

$$\{(a, b) \in L \times L\colon\ \exists\, c \in a \wedge b,\ \ d \in a \vee b\colon\ \mu(c, d) \subseteq W\},$$

where W is a 0-neighbourhood in G (Theorem 2.2.3).

The essential steps to prove this result are contained in the following subsection.

2.1. We shall study the properties of the set $\mu(a, b)$.

Proposition 2.1.1. *Let $a, b \in L$, $c \in a \wedge b$ and $d \in a \vee b$. Then, for every $c' \in a \wedge b$ and $d' \in a \vee b$, we have $\mu(c, d) \subseteq \mu(c', d') + \mu(c', d')$.*

Proof. Let $e, f \in L$ be such that $c \leqslant e \leqslant f \leqslant d$.

(i) First suppose that $a \leqslant e \leqslant f \leqslant d$. Then $d \in e \vee b$ and $d \in f \vee b$. Since $c' \leqslant a \leqslant e$ and $c' \leqslant b$, we can find $t \in e \wedge b$ such that $t \geqslant c'$. Moreover, since $t \leqslant e \leqslant f$ and $t \leqslant b$, we can find $t' \in f \wedge b$ such that $t' \geqslant t$. Then, since μ is modular, we get

$$\mu(f) - \mu(e) = \mu(t') - \mu(t) \in \mu(c', d'),$$

since $c' \leqslant t \leqslant t' \leqslant b \leqslant d'$.

(ii) Now suppose $c \leqslant e \leqslant f \leqslant a$. Then $c \in e \wedge b$ and $c \in f \wedge b$. By $d' \geqslant a \geqslant f$ and $d' \geqslant b$, we can find $t \in f \vee b$ such that $t \leqslant d'$. Moreover, by $t \geqslant e, b$, we can find $t' \in e \vee b$ such that $t' \leqslant t$. Then we get

$$\mu(f) - \mu(e) = \mu(t) - \mu(t') \in \mu(c', d'),$$

since $c' \leqslant b \leqslant t' \leqslant t \leqslant d'$.

(iii) Now we consider the general case. Since $c \leqslant e, a$, we can find $z \in e \wedge a$ such that $z \geqslant c$. Since $z \leqslant f, a$, we can find $z' \in f \wedge a$ such that $z' \geqslant z$. Moreover, since $d \geqslant f, a$, we can find $t \in f \vee a$ such that $t \leqslant d$. Finally, since $t \geqslant e, a$, we can find $t' \in e \vee a$ such that $t' \leqslant t$. Then

$$\mu(f) - \mu(e) = \mu(z') - \mu(z) + \mu(t) - \mu(t').$$

Since $c \leqslant z \leqslant z' \leqslant a$ and $a \leqslant t' \leqslant t \leqslant d$, we have $\mu(z') - \mu(z) \in \mu(c, a) \subseteq \mu(c', d')$ by (ii) and $\mu(t) - \mu(t') \in \mu(a, d) \subseteq \mu(c', d')$ by (i). Therefore $\mu(f) - \mu(e) \in \mu(c', d') + \mu(c', d')$. $\square$

Proposition 2.1.2. *Let $a, b \in L$, $c \in a \vee b$ and $d \in a \wedge b$. Then $\mu(a, c) = \mu(d, b)$.*

P r o o f. Let $a \leqslant e \leqslant f \leqslant c$. Then $c \in e \vee b$ and $c \in f \vee b$. Since $d \leqslant b$ and $d \leqslant a \leqslant e$, we can find $t \in e \wedge b$ such that $t \geqslant d$. Moreover, since $e \leqslant f$, we can find $t' \in f \wedge b$ such that $t' \geqslant t$. Then $d \leqslant t \leqslant t' \leqslant b$. Therefore

$$\mu(f) - \mu(e) = \mu(t') - \mu(t) \in \mu(d, b).$$

Now let $d \leqslant e \leqslant f \leqslant b$. Then $d \in a \wedge e$ and $d \in a \wedge f$. Since $c \geqslant a$ and $c \geqslant b \geqslant f$, we can choose $t \in a \vee f$ such that $t \leqslant c$. Moreover, since $e \leqslant f$, we can choose $t' \in a \vee e$ such that $t' \leqslant t$. Then $a \leqslant t' \leqslant t \leqslant c$. Therefore

$$\mu(f) - \mu(e) = \mu(t) - \mu(t') \in \mu(a, c).$$

□

Corollary 2.1.3.
(1) *If $a \leqslant b, c$, then $\mu(c, d) \subseteq \mu(a, b)$ for every $d \in b \vee c$.*
(2) *If $a \geqslant b, c$, then $\mu(d, c) \subseteq \mu(b, a)$ for every $d \in b \wedge c$.*

P r o o f. (1) Let $d \in b \vee c$. Since $a \leqslant b, c$, we can find $x \in b \wedge c$ such that $x \geqslant a$. By (2.1.2), $\mu(c, d) = \mu(x, b) \subseteq \mu(a, b)$, since $x, b \in [a, b]$.

(2) Let $d \in b \wedge c$. Since $a \geqslant b, c$, we can find $x \in b \vee c$ such that $x \leqslant a$. By (2.1.2), $\mu(d, c) = \mu(b, x) \subseteq \mu(b, a)$, since $x, b \in [b, a]$. □

Proposition 2.1.4. *If $a \leqslant b$ and $c, d \in [a, b]$, then there exist $z \in c \wedge d$ and $z' \in c \vee d$ such that $\mu(z, z') \subseteq \mu(a, b)$.*

P r o o f. Since $a \leqslant c, d$, we can find $z \in c \wedge d$ such that $z \geqslant a$. Since $b \geqslant c, d$, we can find $z' \in c \vee d$ such that $z' \leqslant b$. Hence, if $z \leqslant e \leqslant f \leqslant z'$, then $e, f \in [a, b]$. □

Proposition 2.1.5. *If $a \leqslant c \leqslant b$, then $\mu(a, b) \subseteq \mu(a, c) + \mu(c, b)$.*

P r o o f. Let $a \leqslant e \leqslant f \leqslant b$. Since $a \leqslant e, c$, we can find $t \in e \wedge c$ such that $t \geqslant a$. Since $t \leqslant f, c$, we can find $t' \in f \wedge c$ such that $t' \geqslant t$. Moreover, since $b \geqslant c, f$, we can choose $z \in f \vee c$ such that $z \leqslant b$ and, since $z \geqslant c, e$, we can choose $z' \in c \vee e$ such that $z' \leqslant z$. Then

$$\mu(f) - \mu(e) = \mu(t') - \mu(t) + \mu(z) - \mu(z').$$

Since $a \leqslant t \leqslant t' \leqslant c$ and $c \leqslant z' \leqslant z \leqslant b$, we have $\mu(t') - \mu(t) \in \mu(a, c)$ and $\mu(z) - \mu(z') \in \mu(c, b)$. □

Corollary 2.1.6. *If $a \leqslant b, d$, $c \leqslant b, d$, $z \in a \wedge c$ and $z' \in b \vee d$, then $\mu(z, z') \subseteq \mu(a, b) + \mu(a, b) + \mu(c, d)$.*

Proof. Since $a \leqslant b, d$, by (2.1.3) $\mu(d, z') \subseteq \mu(a, b)$. Since $b \geqslant a, c$, we can find $t \in a \vee c$ such that $t \leqslant b$. By (2.1.2), $\mu(z, c) = \mu(a, t) \subseteq \mu(a, b)$, since $a, t \in [a, b]$. Moreover, since $z \leqslant c \leqslant d \leqslant z'$, by (2.1.5) we get

$$\mu(z, z') \subseteq \mu(z, c) + \mu(c, d) + \mu(d, z') \subseteq \mu(a, b) + \mu(a, b) + \mu(c, d).$$

□

Proposition 2.1.7. *Let $a, b \in L$ with $a \leqslant b$, and $c \in L$. Then:*

(1) *For every $z \in b \vee c$ there exists $z' \in a \vee c$ such that $z' \leqslant z$ and $\mu(z', z) \subseteq \mu(a, b)$.*

(2) *For every $z \in a \vee c$ there exist $z' \in b \vee c$, $z_1 \in z \wedge z'$ and $z_2 \in z \vee z'$ such that $\mu(z_1, z_2) \subseteq \mu(a, b) + \mu(a, b)$.*

Proof. (1) Let $z \in b \vee c$. Since $z \geqslant b \geqslant a$ and $z \geqslant c$, we can find $z' \in a \vee c$ such that $z' \leqslant z$. Let $z' \leqslant e \leqslant f \leqslant z$. Since evidently $z \in b \vee z'$, by (2.1.3) (1) we have $\mu(z', z) \subseteq \mu(a, b)$.

(2) Let $z \in a \vee c$ and $\overline{z} \in b \vee z$. Since $\overline{z} \geqslant b, c$, we can find $z' \in b \vee c$ such that $z' \leqslant \overline{z}$. Since $a \leqslant b, z$, we can find $p \in b \wedge z$ such that $p \geqslant a$. Since $p \leqslant z, z'$, we can find $q \in z \wedge z'$ such that $q \geqslant p$. Moreover, since $\overline{z} \in b \vee z$ and $z' \leqslant \overline{z}$, we have $\overline{z} \in z \vee z'$. We prove that $\mu(q, \overline{z}) \subseteq \mu(a, b) + \mu(a, b)$. Since $q \leqslant z \leqslant \overline{z}$, by (2.1.5) we obtain

$$\mu(q, \overline{z}) \subseteq \mu(q, z) + \mu(z, \overline{z}).$$

Let $u \in q \wedge c$. Since $z \in q \vee c$, using (2.1.2) and (2.1.3) we obtain

$$\mu(q, z) = \mu(u, c) \subseteq \mu(q, z') = \mu(z, \overline{z}).$$

Further, $\mu(z, \overline{z}) = \mu(p, b) \subseteq \mu(a, b)$, so that $\mu(q, \overline{z}) \subseteq \mu(a, b) + \mu(a, b)$. □

In a similar way we obtain the following dual statement of (2.1.7).

Proposition 2.1.8. *Let $a, b \in L$ with $a \leqslant b$, and $c \in L$. Then:*

(1) *For every $z \in a \wedge c$ there exists $z' \in b \wedge c$ such that $z' \geqslant z$ and $\mu(z, z') \subseteq \mu(a, b)$.*

(2) *For every $z \in b \wedge c$ there exist $z' \in a \wedge c$, $z_1 \in z \wedge z'$ and $z_2 \in z \vee z'$ such that $\mu(z_1, z_2) \subseteq \mu(a, b) + \mu(a, b)$.*

2.2. Now, using the results of Sections 1 and 2.1, we prove that μ generates a multilattice uniformity.

For every 0-neighbourhood W in G we set

$$U_W = \{(a,b) \in L \times L \colon \exists\, c \in a \wedge b,\ \ d \in a \vee b \colon \ \mu(c,d) \subseteq W\}$$

and denote by $\mathcal{U}(\mu)$ the family of all oversets of the sets U_W.

Lemma 2.2.1. *U_W has the following properties:*
(1) *If $a \leqslant b$, then $(a,b) \in U_W$ iff $\mu(a,b) \subseteq W$.*
(2) *$(a,b) \in U_W$ iff there exist $c \in a \wedge b$ and $d \in a \vee b$ such that $(c,d) \in U_W$.*
(3) *If $(a,b) \in U_W$ and $a \leqslant b$, then $[a,b] \times [a,b] \subseteq U_W$.*

Proof. (1) is trivial.

(2) follows by (1).

(3) Let $c, d \in [a,b]$. By (2.1.4), we can find $z \in c \wedge d$ and $z' \in c \vee d$ such that $\mu(z,z') \subseteq \mu(a,b) \subseteq W$. Then $(c,d) \in U_W$. □

Lemma 2.2.2. *For $a, b \in L$ with $a \leqslant b$, let $\mu^*(a,b) = \{\mu(d) - \mu(c) \colon \ c, d \in [a,b]\}$. Then $\mu(a,b) \subseteq \mu^*(a,b) \subseteq \mu(a,b) - \mu(a,b)$.*

Proof. The first inclusion is clear. Now let $c, d \in [a,b]$. Then $\mu(d) - \mu(c) = \mu(d) - \mu(a) - (\mu(c) - \mu(a)) \in \mu(a,b) - \mu(a,b)$. □

Theorem 2.2.3. *Let L be a directed multilattice, G a topological Abelian group and $\mu \colon L \to G$ a modular function. Then $\mathcal{U}(\mu)$ is the weakest multilattice uniformity which makes μ uniformly continuous. Further, $\mathcal{U}(\mu)$ has the following properties:*
(1) *For every $U \in \mathcal{U}(\mu)$ there exists $V \in \mathcal{U}(\mu)$ with $V \subseteq U$ such that $(a,b) \in V$, $c \in a \wedge b$ and $d \in a \vee b$ imply $[c,d] \times [c,d] \subseteq U$.*
(2) *For every $U \in \mathcal{U}(\mu)$ there exists $V \in \mathcal{U}(\mu)$ with $V \subseteq U$ such that $(a,b) \in V$, $a \leqslant b$, $c \geqslant a$, $e \leqslant b$, $d \in b \vee c$ and $f \in a \wedge e$ imply $(c,d) \in U$ and $(e,f) \in U$.*

Proof. (i) It is clear that $\mathcal{U}(\mu)$ is closed with respect to the intersections. To prove that $\mathcal{U}(\mu)$ is a multilattice uniformity, we prove that $\mathcal{U}(\mu)$ satisfies the following conditions of (1.4):
(a) For every $U \in \mathcal{U}(\mu)$, $\Delta \subseteq U$.
(b) For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $(a,b) \in V$ implies that there exists $c \in a \wedge b$ and $d \in a \vee b$ with $(c,d) \in U$.
(c) For every $U \in \mathcal{U}(\mu)$ there exists $V \in \mathcal{U}(\mu)$ such that $(c,d) \in V$, $c \in a \wedge b$ and $d \in a \vee b$ imply $(a,b) \in U$.
(d) For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $(a,b) \in V$, $(b,c) \in V$ and $a \leqslant b \leqslant c$ imply $(a,c) \in U$.
(e) For every $U \in \mathcal{U}(\mu)$, there exists $V \in \mathcal{U}(\mu)$ such that $(a,b) \in V$, $a \leqslant b$ and $c \in L$ imply $(a \vee c, b \vee c) \in 2^U$ and $(a \wedge c, b \wedge c) \in 2^U$.

(a) is trivial since, for every $a \in L$, $\mu(a,a) = \{0\}$.

(b) follows by (2.2.1) (2).

(c) Let $U \in \mathcal{U}(\mu)$ and let W be a 0-neighbourhood in G such that $U_W \subseteq U$. By (2.2.1) (3), (c) is satisfied with $V = U_W$.

(d) Choose U and V as in the proof of (c) and let W' be a 0-neighbourhood in G such that $W' + W' \subseteq W$. By (2.1.5), if $a \leqslant b \leqslant c$, then $\mu(a,c) \subseteq \mu(a,b) + \mu(b,c)$. Therefore (d) is satisfied with $V = U_{W'}$.

In a similar way we obtain (e) by (2.1.7) and (2.1.8).

By (1.4), $\mathcal{U}(\mu)$ is a multilattice uniformity.

(ii) To prove (1), let $U \in \mathcal{U}(\mu)$ and let W be a 0-neighbourhood in G such that $U_W \subseteq U$. Let W' be a 0-neighbourhood in G such that $W' + W' \subseteq W$. If $(a,b) \in U_{W'}$, by (2.2.1) (2) we can choose $c \in a \wedge b$ and $d \in a \vee b$ such that $(c,d) \in U_{W'}$. Let $r \in a \wedge b$ and $s \in a \vee b$. By (2.1.1), $\mu(r,s) \subseteq \mu(c,d) + \mu(c,d) \subseteq W' + W' \subseteq W$, from which $(r,s) \in U_W$. Since $r \leqslant s$, by (2.2.1) (3) we get $[r,s] \times [r,s] \subseteq U_W \subseteq U$.

In a similar way we obtain (2) by (2.1.3).

(iii) Now we prove that μ is uniformly continuous with respect to $\mathcal{U}(\mu)$.

Let W, W' be 0-neighbourhoods in G such that $W' - W' \subseteq W$. Let $(a,b) \in U_{W'}$, $c \in a \wedge b$ and $d \in a \vee b$ be such that $\mu(c,d) \subseteq W'$. Since $a, b \in [c,d]$, hence by (2.2.2) $\mu(a) - \mu(b) \in \mu^*(c,d) \subseteq W' - W' \subseteq W$.

(iv) Now let $\mathcal{U}$ be a multilattice uniformity which makes μ uniformly continuous. We prove that $\mathcal{U}(\mu) \leqslant \mathcal{U}$.

Let W be a 0-neighbourhood in G. Since μ is $\mathcal{U}$-uniformly continuous, we can choose $V \in \mathcal{U}$ such that

$$(*) \qquad (a,b) \in V \Rightarrow \mu(a) - \mu(b) \in W.$$

Since $\mathcal{U}$ is a multilattice uniformity, by (1.2) we can choose $V' \in \mathcal{U}$ such that $(a,b) \in V'$ implies that there exist $c \in a \wedge b$ and $d \in a \vee b$ with $[c,d] \times [c,d] \subseteq V$. We prove that $V' \subseteq U_W$.

Let $(a,b) \in V'$ and let $c \in a \wedge b$, $d \in a \vee b$ be such that $[c,d] \times [c,d] \subseteq V$. If $e, f \in [c,d]$ and $e \leqslant f$, then $(e,f) \in V$. By $(*)$, we get $\mu(f) - \mu(e) \in W$. Then $\mu(c,d) \subseteq W$, from which $(a,b) \in U_W$. □

Corollary 2.2.4. *Another base of $\mathcal{U}(\mu)$ is the family consisting of the sets*

$$U'_W = \{(a,b) \in L \times L \colon \ \mu(c,d) \subseteq W \ \forall\, c \in a \wedge b, \ \forall\, d \in a \vee b\},$$

where W is a 0-neighbourhood in G.

Proof. Let W be a 0-neighbourhood in G. It is clear that $U'_W \subseteq U_W$. Moreover, by (1) of (2.2.3), we can choose $V \in \mathcal{U}(\mu)$ such that $(a,b) \in V$, $c \in a \wedge b$ and $d \in a \vee b$

imply $(c,d) \in U_W$. Choose a 0-neighbourhood W' in G such that $U_{W'} \subseteq V$. Then $U_{W'} \subseteq U'_W$. $\square$

Proposition 2.2.5. *Let $\tau(\mu)$ be the topology generated by $\mathcal{U}(\mu)$. Then $\tau(\mu)$ has the following properties:*

(1) *Every $a \in L$ has a base of convex neighbourhoods in $\tau(\mu)$.*

(2) *For every $a \in L$ and every neighbourhood U_0 of a in $\tau(\mu)$, there exists a neighbourhood V_0 of a in $\tau(\mu)$ with $V_0 \subseteq U_0$ such that $b \in V_0$ implies $[c,d] \subseteq U_0$ for every $c \in a \wedge b$ and $d \in a \vee b$.*

Proof. (1) follows by (1.3) and (2.2.3).

(2) Let $a \in L$ and $U \in \mathcal{U}(\mu)$. By (2) of (2.2.3), we can choose $V \in \mathcal{U}(\mu)$ such that $(x,y) \in V$ implies $[c,d] \times [c,d] \subseteq U$ for every $c \in x \wedge y$ and every $d \in x \vee y$. Then $V(a) \subseteq U(a)$. Moreover, if $b \in V(a)$, $c \in a \wedge b$ and $d \in a \vee b$, then $(a,x) \in U$ for every $x \in [c,d]$, since $a \in [c,d]$. Then $[c,d] \subseteq U(a)$. $\square$

Using (2.2.5), with the same proof as in 3.2 of [24] we get the following result.

Corollary 2.2.6. *The topology $\tau(\mu)$ generated by $\mathcal{U}(\mu)$ is the weakest topology with the properties* (1) *and* (2) *of* (2.2.5) *which makes μ continuous.*

Now we prove that μ generates a congruence relation. We set

$$N(\mu) = \{(a,b) \in L \times L\colon\ \exists\, c \in a \wedge b,\ \ d \in a \vee b\colon\ \mu \text{ is constant on } [c,d]\}.$$

By (2.1.1), it is easy to see that $(a,b) \in N(\mu)$ iff μ is constant on $[c,d]$ for every $c \in a \wedge b$ and every $d \in a \vee b$. Moreover, if the topology of G is Hausdorff, by (2.2.4) we get $N(\mu) = \bigcap\{U\colon\ U \in \mathcal{U}(\mu)\}$.

Proposition 2.2.7. *$N(\mu)$ is a congruence relation.*

Proof. It is clear that $N(\mu)$ is reflexive and symmetric.

We prove that $N(\mu)$ verifies the conditions of Theorem 2.2 of [19] cited in the Preliminaries.

The equivalence $(a,b) \in N(\mu)$ iff there exists $c \in a \wedge b$ and $d \in a \vee b$ such that $(c,d) \in N(\mu)$ is trivial.

The condition that $(a,b) \in N(\mu)$, $(b,c) \in N(\mu)$ and $a \leqslant b \leqslant c$ imply $(a,c) \in N(\mu)$ follows by (2.1.5).

The condition that $(a,b) \in N(\mu)$ and $a \leqslant b$ imply $(a \vee c, b \vee c) \in' N(\mu)$ and $(a \wedge c, b \wedge c) \in' N(\mu)$ follows by (2.1.7) and (2.1.8). $\square$

Remark. In [19] it has been proved that, if μ is an increasing real-valued modular function on a multilattice, the function defined by

$$d(a,b) = \mu(d) - \mu(c), \quad a,b \in L, \quad c \in a \wedge b, \quad d \in a \vee b,$$

is a pseudometric. Hence, in this case, $\mathcal{U}(\mu)$ coincides with the uniformity generated by d.

If L is a lattice and μ is a G-valued modular function, in [13] it has been proved that μ generates a lattice uniformity $\mathcal{U}_\mu$ which has as its base the family consisting of the sets

$$\{(a,b) \in L \times L\colon \ \mu(d) - \mu(c) \in W \ \forall\, c,d \in [a \wedge b, a \vee b], \ \ c \leqslant d\},$$

where W is a 0-neighbourhood in G. Then, if L is a lattice, $\mathcal{U}(\mu) = \mathcal{U}_\mu$.

References

[1] *A. Avallone*: Liapunov theorem for modular functions. Internat. J. Theoret. Phys. *34* (1995), 1197–1204.

[2] *A. Avallone*: Nonatomic vector-valued modular functions. Annal. Soc. Math. Polon. Series I: Comment. Math. *XXXIX* (1999), 37–50.

[3] *A. Avallone, G. Barbieri and R. Cilia*: Control and separating points of modular functions. Math. Slovaca *43* (1999).

[4] *A. Avallone and A. De Simone*: Extensions of modular functions on orthomodular lattices. Italian J. Pure Appl. Math. To appear.

[5] *A. Avallone and M. A. Lepellere*: Modular functions: Uniform boundedness and compactness. Rend. Circ. Mat. Palermo XLVII (1998), 221–264.

[6] *A. Avallone and J. Hamhalter*: Extension theorems (vector measures on quantum logics). Czechoslovak Math. J. *46 (121)* (1996), 179–192.

[7] *A. Avallone and H. Weber*: Lattice uniformities generated by filters. J. Math. Anal. Appl. *209* (1997), 507–528.

[8] *G. Barbieri and H. Weber*: A topological approach to the study of fuzzy measures. Funct. Anal. Econom. Theory. Springer, 1998, pp. 17–46.

[9] *G. Barbieri, M. A. Lepellere and H. Weber*: The Hahn decomposition theorem and applications. Fuzzy Sets Systems *118* (2001), 519–528.

[10] *H. J. Bandelt, M. Van de Vel and E. Verheul*: Modular interval spaces. Math. Nachr. *163* (1993), 177–201.

[11] *M. Benado*: Les ensembles partiellement ordonnes et le theoreme de raffinement de Schrelier II. Czechoslovak Math. J. *5* (1955), 308–344.

[12] *G. Birkhoff*: Lattice Theory, Third edition. AMS Providence, R.I., 1967.

[13] *I. Fleischer and T. Traynor*: Equivalence of group-valued measure on an abstract lattice. Bull. Acad. Pol. Sci. *28* (1980), 549–556.

[14] *I. Fleischer and T. Traynor*: Group-valued modular functions. Algebra Universalis *14* (1982), 287–291.

[15] *M. G. Graziano*: Uniformities of Fréchet-Nikodým type on Vitali spaces. Semigroup Forum *61* (2000), 91–115.

[16] *D. J. Hensen*: An axiomatic characterization of multilattices. Discrete Math. *33* (1981), 99–101.
[17] *J. Jakubík*: Sur les axiomes des multistructures. Czechoslovak Math. J. *6* (1956), 426–430.
[18] *J. Jakubík and M. Kolibiar*: Isometries of multilattice groups. Czechoslovak Math. J. *33* (1983), 602–612.
[19] *J. Lihová*: Valuations and distance function on directed multilattices. Math. Slovaca *46* (1996), 143–155.
[20] *J. Lihová and K. Repasky*: Congruence relations on and varieties of directed multilattices. Math. Slovaca *38* (1988), 105–122.
[21] *T. Traynor*: Modular functions and their Fréchet-Nikodým topologies. Lectures Notes in Math. *1089* (1984), 171–180.
[22] *H. Weber*: Uniform Lattices I: A generalization of topological Riesz space and topological Boolean rings; Uniform lattices II. Ann. Mat. Pura Appl. *160* (1991), 347–370; and *165* (1993), 133–158.
[23] *H. Weber*: Lattice uniformities and modular functions on orthomodular lattices. Order *12* (1995), 295–305.
[24] *H. Weber*: On modular functions. Funct. Approx. *XXIV* (1996), 35–52.
[25] *H. Weber*: Valuations on complemented lattices. Internat. J. Theoret. Phys. *34* (1995), 1799–1806.
[26] *H. Weber*: Complemented uniform lattices. Topology Appl. *105* (2000), 47–64.
[27] *H. Weber*: Two extension theorems. Modular functions on complemented lattices. Preprint.

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