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OSCILLATION AND NONOSCILLATION OF HIGHER ORDER SELF-ADJOINT DIFFERENTIAL EQUATIONS

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Abstract. Oscillation and nonoscillation criteria for the higher order self-adjoint differential equation

$$(*)\qquad (-1)^n(t^\alpha y^{(n)})^{(n)} + q(t)y = 0$$

are established. In these criteria, equation $(*)$ is viewed as a perturbation of the conditionally oscillatory equation

$$(-1)^n(t^\alpha y^{(n)})^{(n)} - \frac{\mu_{n,\alpha}}{t^{2n-\alpha}}y = 0,$$

where $\mu_{n,\alpha}$ is the critical constant in conditional oscillation. Some open problems in the theory of conditionally oscillatory, even order, self-adjoint equations are also discussed.

Keywords: self-adjoint differential equation, oscillation and nonoscillation criteria, variational method, conditional oscillation

MSC 2000: 34C10

1. Introduction

The aim of this paper is to investigate oscillatory properties of the even order, self-adjoint, linear differential equation

$$(1)\qquad (-1)^n(t^\alpha y^{(n)})^{(n)} + q(t)y = 0,$$

where $\alpha \notin \{1, 3, \ldots, 2n-1\}$ and no sign restriction is imposed on the function q. The problem of oscillation/nonoscillation of higher order self-adjoint differential equations

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of the form

$$(-1)^n (r(t) y^{(n)})^{(n)} + q(t) y = 0 \tag{2}$$

was investigated in several recent papers [4], [6], [7], [8], [10], [12], [16], [17] and various conditions for oscillation/nonoscillation of this equation were established. In these papers equation (2) is mostly viewed as a "perturbation" of the nonoscillatory one-term equation

$$(-1)^n (r(t) y^{(n)})^{(n)} = 0 \tag{3}$$

and conditions on the "perturbation function" q are given which guarantee that (2) is oscillatory/nonoscillatory.

In our paper we employ a somewhat different approach. Motivated by the papers [8], [12], equation (1) is not investigated as a perturbation of the equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} = 0, \tag{4}$$

but as a perturbation of the Euler-type equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} - \frac{\mu_{n,\alpha}}{t^{2n-\alpha}} y = 0, \tag{5}$$

where $\mu_{n,\alpha}$ is the so-called *critical constant* in (5) (which will be specified later). We show that if the value $q(t) + \mu_{n,\alpha} t^{\alpha-2n}$ is sufficiently negative/not too negative, then the perturbed equation (1) remains nonoscillatory/becomes oscillatory. This method was used in [8], [12] to study the oscillatory properties of the equation

$$(-1)^n y^{(2n)} + q(t) y = 0$$

(which corresponds to the case $\alpha = 0$ in (1)), viewed as a perturbation of the Euler equation

$$(-1)^n y^{(2n)} - \frac{\mu_n}{t^{2n}} y = 0 \qquad \text{with } \mu_n := \frac{[(2n-1)!!]^2}{4^n},$$

so our results can be viewed as a direct extension of the criteria given there.

Similarly to the above mentioned papers, we use a variational technique which is based on the relationship between the nonoscillation of even order self-adjoint equations and the positivity of a certain associated quadratic functional. An important role is also played by the connection between the self-adjoint equations investigated and the linear Hamiltonian systems. Finally, note that the (non)oscillation criteria for the self-adjoint, even order, differential equations are closely related to the spectral properties of the associated differential operators and were investigated in several recent papers [1], [5], [8], [14], [16], [17].

The paper is organized as follows. In the next section we recall basic oscillatory properties of the self-adjoint, even order, differential equations, in particular, the relationship between the oscillatory properties of these equations and the positivity of a certain associated quadratic functional. The third section is devoted to nonoscillation criteria for (1) and the following section to their oscillation counterparts. Section 5 contains remarks and comments concerning the results of the paper. In the last section we collect some technical results used in the previous sections.

2. Auxiliary results

In this section we recall basic oscillatory properties of self-adjoint, even order, differential equations

$$L(y) := \sum_{k=0}^{n} (-1)^k (r_k(t) y^{(k)})^{(k)} = 0, \quad r_n(t) > 0. \tag{6}$$

Oscillatory properties of these equations can be investigated within the scope of the oscillation theory of linear Hamiltonian systems (further LHS)

$$x' = Ax + B(t)u, \quad u' = C(t)x - A^T u, \tag{7}$$

where A, B, C are $n \times n$ matrices with B, C symmetric. Indeed, if y is a solution of (6) and we set

$$x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u = \begin{pmatrix} (-1)^{n-1}(r_n y^{(n)})^{(n-1)} + \ldots + r_1 y' \\ \vdots \\ -(r_n y^{(n)})' + r_{n-1} y^{(n-1)} \\ r_n y^{(n)} \end{pmatrix},$$

then (x, u) solves (7) with A, B, C given by

$$\begin{gathered} B(t) = \operatorname{diag}\{0, \ldots, 0, r_n^{-1}(t)\}, \quad C(t) = \operatorname{diag}\{r_0(t), \ldots, r_{n-1}(t)\}, \\ A = A_{i,j} = \begin{cases} 1, & \text{if } j = i+1, \ i = 1, \ldots, n-1, \\ 0, & \text{elsewhere.} \end{cases} \end{gathered} \tag{8}$$

In this case we say that the solution (x, u) of (7) *is generated* by the solution y of (6). Moreover, if $y_1, \ldots, y_n$ are solutions of (6) and the columns of the matrix solution

(X,U) of (7) are generated by the solutions $y_1,\dots,y_n$, we say that the solution (X,U) is *generated by the solutions* $y_1,\dots,y_n$.

Recall that two different points t_1, t_2 are said to be *conjugate* relative to system (7) if there exists a nontrivial solution (x,u) of this system such that $x(t_1)=0=x(t_2)$. Consequently, by the above mentioned relationship between (6) and (7), these points are conjugate relative to (6) if there exists a nontrivial solution y of this equation such that $y^{(i)}(t_1)=0=y^{(i)}(t_2)$, $i=0,1,\dots,n-1$. System (7) (and hence also equation (6)) is said to be *oscillatory* if for every $T\in\mathbb{R}$ there exists a pair of points $t_1,t_2\in[T,\infty)$ which are conjugate relative to (7) (relative to (6)), in the opposite case (7) (or (6)) is said to be *nonoscillatory.*

Using the relation between (6), (7) and the so-called Roundabout Theorem for linear Hamiltonian systems (see [18], [19]), one can easily prove the following variational lemma which plays a crucial role in our investigation of oscillatory properties of (1).

Lemma 2.1 ([15]). *Equation* (6) *is nonoscillatory if and only if there exists* $T\in\mathbb{R}$ *such that*

$$I(y;T,\infty):=\int_T^\infty\left[\sum_{k=0}^n r_k(t)(y^{(k)}(t))^2\right]\mathrm{d}t>0$$

for any nontrivial $y\in W^{n,2}(T,\infty)$ *with compact support in* (T,∞).

We also use the following Wirtinger-type inequality.

Lemma 2.2 ([16]). *Let* $y\in W^{1,2}(T,\infty)$ *have compact support in* (T,∞) *and let* M *be a positive differentiable function such that* $M'(t)\neq 0$ *for* $t\in[T,\infty)$. *Then*

$$\int_T^\infty|M'(t)|y^2\,\mathrm{d}t\leqslant 4\int_T^\infty\frac{M^2(t)}{|M'(t)|}y'^2\,\mathrm{d}t.$$

We say that $y_1,\dots,y_n,\tilde y_1,\dots,\tilde y_n$ form an *ordered* system of solutions of equation (6) (at ∞) if $y_i>0$, $\tilde y_i>0$, $i=1,\dots,n$ and

$$\frac{y_i}{y_{i+1}}\to 0,\quad \frac{\tilde y_i}{\tilde y_{i+1}}\to 0,\quad \frac{y_n}{\tilde y_1}\to 0,\quad i=1,\dots,n-1,$$

for $t\to\infty$.

In Section 4 devoted to oscillation criteria for (1) we use the following statement proved essentially (i.e. with a minor modification with respect to the presentation given here) in [8].

Theorem 2.1. *Let $y_1, \dots, y_n$, $\tilde y_1, \dots, \tilde y_n$, be an ordered system of solutions of* (3) *and let* (X, U), $(\tilde X, \tilde U)$ *be the solutions of matrix LHS* (7) *generated by* $y_1, \dots, y_n$ *and* $\tilde y_1, \dots, \tilde y_n$, *respectively. Further, let* $L = X^T \tilde U - U^T \tilde X$ *(this is a constant matrix as can be verified directly by differentiation). If there exists* $i \in \{1, \dots, n\}$ *such that*

$$\limsup_{t\to\infty} \frac{\tilde W(t)}{L_{i,l} \tilde W_{l,i}(t)} \int_t^\infty q(s) y_i^2(s)\,\mathrm{d}s < -1, \tag{9}$$

where $l = \min\{j \in \{1, \dots, n\}, L_{i,j} \neq 0\}$ *($L_{i,j}$ are entries of L) and*

$$\tilde W(t) = W(\tilde y_1, \dots, \tilde y_n), \quad \tilde W_{i,j}(t) = W(\tilde y_1, \dots, \tilde y_{i-1}, y_j, \tilde y_{i+1}, \dots, \tilde y_n),$$

then equation (2) *is oscillatory. Moreover, if* $q(t) \leqslant 0$ *eventually, then* $\limsup$ *in* (9) *can be replaced by* $\liminf$.

Theorem 2.2. *Let* $y_1, \dots, y_n, \tilde y_1, \dots, \tilde y_n$, (X, U), $(\tilde X, \tilde U)$ *and* L *be the same as in the previous theorem,*

$$W(t) := W(y_1, \dots, y_n), \quad W_{i,j}(t) = W(y_1, \dots, y_{i-1}, \tilde y_j, y_{i+1}, \dots, y_n).$$

If there exists $i \in \{1, \dots, n\}$ *such that*

$$\limsup_{t\to\infty} \frac{W(t)}{L_{l,i} W_{l,i}(t)} \int^t q(s) \tilde y_i^2(s)\,\mathrm{d}s < -1, \tag{10}$$

where $l = \min\{j \in \{1, \dots, n\},\ L_{j,i} \neq 0\}$, *then equation* (2) *is oscillatory. Moreover, if* $q(t) \leqslant 0$ *eventually, then* $\limsup$ *in* (10) *can be replaced by* $\liminf$.

P r o o f. Since this statement is not contained in any of the referred papers, we briefly sketch the proof. This proof is based on [5], Theorem 3.2 which claims that (2) is oscillatory provided there exists $c \in \mathbb{R}^n$ such that

$$\limsup_{t\to\infty} \frac{\int^t q(s)(c_1 \tilde y_1(s) + \ldots + c_n \tilde y_n(s))^2\,\mathrm{d}s}{c^T \left(\int_t^\infty \tilde X^{-1}(s) B(s) \tilde X^{T-1}(s)\,\mathrm{d}s\right)^{-1} c} < -1.$$

We take $c = e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$ (1 being the i-th entry) in the previous limit and show that

$$e_i^T \left(\int_t^\infty \tilde X^{-1}(s) B(s) \tilde X^{T-1}(s)\,\mathrm{d}s\right)^{-1} e_i \sim \frac{L_{l,i} W_{l,i}(t)}{W(t)} \quad \text{as } t \to \infty; \tag{11}$$

here $f(t) \sim g(t)$ for a pair of functions means $\lim\limits_{t\to\infty} f(t)/g(t) = 1$. Indeed, by a direct computation we have

$$(\tilde{X}^{-1}X)' = -\tilde{X}^{-1}B\tilde{X}^{T-1}L^T$$

and hence by Lemma 6.1

$$\left(\int_t^\infty \tilde{X}^{-1}B\tilde{X}^{T-1}\,\mathrm{d}s\right)^{-1}_{i,i} = (L^TX^{-1}\tilde{X})_{i,i} = \sum_{j=1}^n L_{j,i}(X^{-1}\tilde{X})_{j,i} = \sum_{j=1}^n L_{j,i}\frac{W_{j,i}}{W}.$$

Consequently, using Lemma 6.2 we see that (11) really holds. □

3. Nonoscillation criteria

Theorem 3.1. *Let $\alpha \notin \{1,3,\ldots,2n-1\}$, let $\nu_{n,\alpha}$ be the value of the greatest local minimum of the polynomial*

$$P_{n,\alpha}(\lambda) := \frac{(-1)^n\prod_{j=0}^{n-1}(\lambda-j)(\lambda-n+\alpha-j)-\mu_{n,\alpha}}{\left(\lambda-\frac{2n-1-\alpha}{2}\right)^2}, \tag{12}$$

where

$$\mu_{n,\alpha} = (-1)^n\prod_{j=0}^{n-1}(\lambda-j)(\lambda-n+\alpha-j) = |_{\lambda=\frac{2n-1-\alpha}{2}}, \tag{13}$$

and suppose that

$$\int^\infty \left[q(t)+\frac{\mu_{n,\alpha}}{t^{2n-\alpha}}\right]_- t^{2n-1-\alpha}\,\mathrm{d}t > -\infty, \tag{14}$$

where $[f(t)]_- := \min\{0, f(t)\}$ denotes the negative part of the function indicated.

If

$$\liminf_{t\to\infty} \lg t\int_t^\infty s^{2n-\alpha-1}\left[q(s)+\frac{\mu_{n,\alpha}}{s^{2n-\alpha}}\right]_-\,\mathrm{d}s > \frac{\nu_{n,\alpha}}{4}, \tag{15}$$

then equation (1) is nonoscillatory.

P r o o f. Denote

$$Q(t) := \left[q(t)+\frac{\mu_{n,\alpha}}{t^{2n-\alpha}}\right]_-.$$

According to (15) there exists $T \in \mathbb{R}$ such that

$$\lg t\int_t^\infty s^{2n-\alpha-1}Q(s)\,\mathrm{d}s > \frac{\nu_{n,\alpha}}{4}$$

for $t \geqslant T$. Then for any $y \in W^{n,2}(T,\infty)$ with $\operatorname{supp} y \subset (T,\infty)$ we have

$$\begin{aligned}\int_T^\infty Q(t)y^2(t)\,\mathrm{d}t &= 2\int_T^\infty Q(t)t^{2n-1-\alpha}\biggl[\int_T^t \frac{y(s)}{s^{\frac{2n-1-\alpha}{2}}}\biggl(\frac{y(s)}{s^{\frac{2n-1-\alpha}{2}}}\biggr)'\,\mathrm{d}s\biggr]\,\mathrm{d}t\\ &= 2\int_T^\infty \frac{y(t)}{t^{\frac{2n-1-\alpha}{2}}}\biggl(\frac{y(t)}{t^{\frac{2n-1-\alpha}{2}}}\biggr)'\frac{1}{\lg t}\lg t\biggl[\int_t^\infty s^{2n-1-\alpha}Q(s)\,\mathrm{d}s\biggr]\,\mathrm{d}t\\ &> \frac{\nu_{n,\alpha}}{2}\int_T^\infty\biggl|\frac{y(t)}{t^{\frac{2n-1-\alpha}{2}}}\biggr|\biggl|\biggl(\frac{y(t)}{t^{\frac{2n-1-\alpha}{2}}}\biggr)'\biggr|\frac{1}{\lg t}\,\mathrm{d}t\\ &\geqslant \nu_{n,\alpha}\int_T^\infty t\Bigl[\Bigl(t^{-\frac{2n-1-\alpha}{2}}y(t)\Bigr)'\Bigr]^2\,\mathrm{d}t\end{aligned}$$

by Lemma 2.2 with $M(t) = 1/\lg t$. Hence

$$\begin{aligned}&\int_T^\infty [t^\alpha(y^{(n)}(t))^2 + q(t)y^2(t)]\,\mathrm{d}t\\ &\qquad\geqslant \int_T^\infty\Bigl[t^\alpha(y^{(n)}(t))^2 - \frac{\mu_{n,\alpha}}{t^{2n-\alpha}}y^2(t) + Q(t)y^2(t)\Bigr]\,\mathrm{d}t\\ &\qquad> \int_T^\infty\Bigl[t^\alpha(y^{(n)}(t))^2 + \nu_{n,\alpha}t\Bigl[\Bigl(y(t)t^{-\frac{2n-1-\alpha}{2}}\Bigr)'\Bigr]^2 - \frac{\mu_{n,\alpha}}{t^{2n-\alpha}}y^2(t)\Bigr]\,\mathrm{d}t.\end{aligned}$$

To prove that (1) is nonoscillatory we need to show that the last integral is positive for any nontrivial $y \in W^{n,2}(T,\infty)$ with $\operatorname{supp} y \subset (T,\infty)$. The Euler-Lagrange equation corresponding to this integral is

$$(-1)^n(t^\alpha y^{(n)})^{(n)} - \frac{\nu_{n,\alpha}}{t^{\frac{2n-1-\alpha}{2}}}\biggl[t\biggl(\frac{y}{t^{\frac{2n-1-\alpha}{2}}}\biggr)'\biggr]' - \frac{\mu_{n,\alpha}}{t^{2n-\alpha}}y = 0 \tag{16}$$

and the characteristic equation of (16) is

$$\begin{aligned}(-1)^n\prod_{j=0}^{n-1}(\lambda-j)(\lambda+\alpha-n-j) - \nu_{n,\alpha}\Bigl(\lambda - \frac{2n-1-\alpha}{2}\Bigr)^2 - \mu_{n,\alpha}\\ = \Bigl(\lambda - \frac{2n-1-\alpha}{2}\Bigr)^2[P_{n,\alpha}(\lambda) - \nu_{n,\alpha}] = 0.\end{aligned}$$

Since $\nu_{n,\alpha}$ is the value of the greatest local minimum of the polynomial $P_{n,\alpha}$, the equation $P_{n,\alpha}(\lambda) - \nu_{n,\alpha} = 0$ has $2n-2$ real roots (counting multiplicity), which means that differential equation (16) possesses an ordered system of solutions. Hence any of its nontrivial solution has at most $2n-1$ zeros in the interval $[T,\infty)$ if T is sufficiently large, so this equation is disconjugate on this interval. Lemma 2.1 implies that

$$\int_T^\infty\Bigl[(t^\alpha y^{(n)})^2 + \nu_{n,\alpha}t\Bigl[\Bigl(yt^{-\frac{2n-1-\alpha}{2}}\Bigr)'\Bigr]^2 - \frac{\mu_{n,\alpha}}{t^{2n-\alpha}}y^2\Bigr]\,\mathrm{d}t > 0$$

for every nontrivial $y \in W^{n,2}(T,\infty)$ with $\operatorname{supp} y \subset (T,\infty)$ and this, again in view of Lemma 2.1, implies that (1) is nonoscillatory. $\square$

The next theorem deals with the case when the solution $y = t^{\frac{2n-1-\alpha}{2}}$ of (5) in the integral (14) is replaced by the solution $\tilde{y} = t^{\frac{2n-1-\alpha}{2}} \lg t$.

Theorem 3.2. *Let α, $\mu_{n,\alpha}$, $\nu_{n,\alpha}$ be the same as in the previous theorem. If*

$$\liminf_{t\to\infty} \frac{1}{\lg t} \int_a^t \Big[q(s) + \frac{\mu_{n,\alpha}}{s^{2n-\alpha}}\Big]_- s^{2n-1-\alpha} \lg^2 s \,\mathrm{d}s > \frac{\nu_{n,\alpha}}{4}, \tag{17}$$

then equation (1) *is nonoscillatory.*

Proof. Using the same method as in the previous proof (with $t^{\frac{2n-1-\alpha}{2}}$ replaced by $t^{\frac{2n-1-\alpha}{2}} \lg t$) we get

$$\int_T^\infty Q(t) y^2(t) \,\mathrm{d}t \geqslant \nu_{n,\alpha} \int_T^\infty t \lg^2 t \Big[\Big(\frac{y(t)}{t^{(2n-1-\alpha)/2} \lg t}\Big)'\Big]^2 \,\mathrm{d}t$$

and hence

$$\begin{aligned}\int_T^\infty & t^\alpha [(y^{(n)})^2 + q(t)y^2] \,\mathrm{d}t \\ &> \int_T^\infty \Big\{t^\alpha (y^{(n)})^2 - \frac{\mu_{n,\alpha}}{t^{2n-\alpha}} y^2 + \nu_{n,\alpha} t \lg^2 t \Big[\Big(\frac{y(t)}{t^{(2n-1-\alpha)/2} \lg t}\Big)'\Big]^2\Big\} \,\mathrm{d}t.\end{aligned}$$

Again, the equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} - \frac{\nu_{n,\alpha}}{t^{(2n-1-\alpha)/2} \lg t} \Big[t \lg^2 t \Big(\frac{y}{t^{(2n-1-\alpha)/2} \lg t}\Big)'\Big]' - \frac{\mu_{n,\alpha}}{t^{2n-\alpha}} y = 0$$

possesses an ordered system of solutions and this, by the same argument as above, implies that

$$\int_T^\infty [t^\alpha (y^{(n)})^2 + q(t) y^2] \,\mathrm{d}t > 0$$

for every $y \in W^{n,2}(T,\infty)$ with $\operatorname{supp} y \subset (T,\infty)$. This means, by Lemma 2.1, that (1) is nonoscillatory. $\square$

In this section we present oscillatory counterparts of the results given in the previous section.

Theorem 4.1. *Let* $\alpha \notin \{1, 3, \ldots, 2n-1\}$, *let* $P_{n,\alpha}$, $\mu_{n,\alpha}$ *be given by* (12), (13), *respectively,*

$$K_{n,\alpha} := P_{n,\alpha}\Big(\frac{2n-1-\alpha}{2}\Big) < 0$$

and suppose that the integral

$$\int^{\infty}\Big[q(t) + \frac{\mu_{n,\alpha}}{t^{2n-\alpha}}\Big]t^{2n-1-\alpha}\,\mathrm{d}t \tag{18}$$

is convergent. If

$$\limsup_{t\to\infty} \lg t \int_t^{\infty}\Big[q(s) + \frac{\mu_{n,\alpha}}{s^{2n-\alpha}}\Big]s^{2n-1-\alpha}\,\mathrm{d}s < K_{n,\alpha} \tag{19}$$

then equation (1) *is oscillatory. Moreover, if*

$$q(t) + \frac{\mu_{n,\alpha}}{t^{2n-\alpha}} \leqslant 0$$

eventually, then lim sup *in* (19) *can be replaced by* lim inf.

P r o o f. Our proof is based on a modified version of Theorem 2.1. A closer examination of the proof of this statement reveals that the result remains valid if the operator $(-1)^n(r(t)y^{(n)})^{(n)}$ is replaced by any self-adjoint operator $L(y)$ of the form (6) such that the equation $L(y) = 0$ possesses an ordered system of solutions. Obviously, equation (5) meets this assumption and $y_1 = t^{\alpha_1}, \ldots, y_{n-1} = t^{\alpha_{n-1}}$, $y_n = t^{\frac{2n-1-\alpha}{2}}$, $\tilde y_1 = t^{\frac{2n-1-\alpha}{2}} \lg t$, $\tilde y_2 = t^{2n-\alpha-1-\alpha_{n-1}}, \ldots, \tilde y_n = t^{2n-\alpha-1-\alpha_1}$ is an ordered system of solutions of this equation, where $\alpha_1, \ldots, \alpha_{n-1}$ are the first $n-1$ roots (ordered by size) of the polynomial

$$(-1)^n \prod_{j=0}^{n-1}(\lambda - j)(\lambda + \alpha - n - j) - \mu_{n,\alpha}.$$

Let (X, U), $(\tilde X, \tilde U)$ be the matrix solutions of the LHS associated with (5) generated by $y_1, \ldots, y_n$ and $\tilde y_1, \ldots, \tilde y_n$, respectively, and let $L := X^T\tilde U - U^T\tilde X$. Since the entry $L_{n,1}$ of this matrix appearing in the left lower corner is nonzero (see Lemma 6.5 given in the last section), we take $l = 1$ and $i = n$ in Theorem 2.1.

In the next part we compute the Wronskians

$$\tilde{W}(t) := W(\tilde{y}_1, \ldots, \tilde{y}_n), \quad \tilde{W}_{1,n}(t) := W(y_n, \tilde{y}_2, \ldots, \tilde{y}_n).$$

Denote

$$\lambda_k := \frac{2n - \alpha - 1 - 2\alpha_k}{2}, \qquad k = 1, \ldots, n-1.$$

Then using Lemma 6.3 (given again in the last section) and the Laplace rule for computing determinants we obtain

$$\begin{aligned}\tilde{W}(t) = t^{\frac{n(2n-1-\alpha)}{2}} \Big\{ &\lg t \Big(\prod_{i=1}^{n-1} \lambda_i\Big) W(t^{\lambda_{n-1}-1}, \ldots, t^{\lambda_1 - 1}) \\ &- t^{\lambda_{n-1}} \Big(\prod_{i=1}^{n-2} \lambda_i\Big) W(t^{-1}, t^{\lambda_{n-2}-1}, \ldots, t^{\lambda_1-1}) + \ldots \\ &+ (-1)^{n+1} t^{\lambda_1} \Big(\prod_{i=2}^{n-1} \lambda_i\Big) W(t^{-1}, t^{\lambda_{n-1}-1}, \ldots, t^{\lambda_2-1}) \Big\} \\ \sim t^{\frac{n(2n-1-\alpha)}{2}} &W(t^{\lambda_{n-1}-1}, \ldots, t^{\lambda_1-1}) \Big(\prod_{i=1}^{n-1} \lambda_i\Big) \lg t\end{aligned}$$

according to Lemma 6.2. Concerning the Wronskian $\tilde{W}_{1,n}$, we have (by Lemma 6.4)

$$\tilde{W}_{1,n}(t) = t^{\frac{n(2n-1-\alpha)}{2}} W(t^{\lambda_{n-1}-1}, \ldots, t^{\lambda_1-1}) \Big(\prod_{i=1}^{n-1} \lambda_i\Big).$$

Consequently,

$$\frac{\tilde{W}(t)}{L_{n,1} \tilde{W}_{1,n}(t)} \sim \frac{\lg t}{L_{n,1}}$$

and the theorem is proved since $L_{n,1} = -K_{n,\alpha}$ by Lemma 6.5. □

Using Theorem 2.2, Lemma 6.5 and a slight modification of the previous proof we get the following statement.

Theorem 4.2. *Let α, $\mu_{n,\alpha}$ and $K_{n,\alpha}$ be the same as in the previous theorem. If*

$$\limsup_{t\to\infty} \frac{1}{\lg t} \int^t \Big[q(s) + \frac{\mu_{n,\alpha}}{s^{2n-\alpha}}\Big] s^{2n-1-\alpha} \lg^2 s \, \mathrm{d}s < K_{n,\alpha}, \tag{20}$$

then equation (1) *is oscillatory. Moreover, if*

$$q(t) + \frac{\mu_{n,\alpha}}{t^{2n-\alpha}} \leqslant 0$$

eventually, then $\limsup$ *in* (20) *can be replaced by* $\liminf$.

Proof. Put $i=1$ in Theorem 2.2. This theorem and the fact that $L_{j,1}=0$, $j=1,\dots,n-1$, see Lemma 6.5, imply

$$\frac{W(t)}{L_{n,1}W_{n,1}(t)}\sim\frac{1}{L_{n,1}\lg t}.$$

□

5. Remarks and comments

(i) The oscillation and nonoscillation criteria proved in Sections 3, 4 are closely related to the concept of *conditionally oscillatory* equations. The equation

$$L(y)+q(t)y=0, \tag{21}$$

with $q(t)>0$ and a nonoscillatory self-adjoint $2n$-th order differential operator given by (6) is said to be *conditionally oscillatory* if there exists $\lambda_0<0$ such that (21), with $\lambda q(t)$ instead of $q(t)$, is oscillatory for $\lambda<\lambda_0$ and nonoscillatory for $\lambda>\lambda_0$. Theorems 3.1, 4.1 (and also Theorems 3.2, 4.2) show that the equation

$$(-1)^n(t^\alpha y^{(n)})^{(n)}-\left[\frac{\mu_{n,\alpha}}{t^{2n-\alpha}}-\frac{\lambda}{t^{2n-\alpha}\lg^2 t}\right]y=0 \tag{22}$$

is oscillatory for $\lambda<K_{n,\alpha}$ and nonoscillatory for $\lambda>\frac{\nu_{n,\alpha}}{4}$, where $\nu_{n,\alpha}$, $K_{n,\alpha}$ are given in Theorems 3.1, 4.1, respectively (observe that really $\nu_{n,\alpha}>4K_{n,\alpha}>0$); consequently, it is conditionally oscillatory. Conditionally oscillatory equations play an important role in the spectral theory of singular differential operators as is shown later.

(ii) Theorems 3.1, 3.2 can be formulated in a slightly more general form than that considered in Section 3. These theorems are a special case of the following statement which can be proved using the same ideas as for Theorems 3.1, 3.2.

Theorem 5.1. *Suppose that the $2n$-order equation $L(y)=0$ (L is the same as above) possesses an ordered system of solutions $y_1,\dots,y_{2n}$. Further suppose that there exist $i\in\{1,\dots,2n\}$, a positive differentiable function M satisfying $M'(t)\neq 0$ for large t and a constant ν such that the equation*

$$L(y)+\frac{\nu}{y_i}\left(\frac{M^2(t)}{|M'(t)|}\left(\frac{y}{y_i}\right)'\right)'=0$$

is nonoscillatory. Then equation (21) *is also nonoscillatory provided one of the following conditions holds:*

(i) $i \in \{1, \dots, n\}$ *and*

$$\liminf_{t\to\infty} \frac{1}{M(t)} \int_t^\infty q(s) y_i^2(s)\,\mathrm{d}s > \nu,$$

(ii) $i \in \{n+1, \dots, 2n\}$ *and*

$$\liminf_{t\to\infty} \frac{1}{M(t)} \int^t q(s) y_i^2\,\mathrm{d}s > \nu.$$

In Theorems 3.1, 3.2 we took $y_i = y_n = t^{\frac{2n-1-\alpha}{2}}$, $M(t) = \frac{1}{\lg t}$ or, respectively, $y_i = y_{n+1} = t^{\frac{2n-1-\alpha}{2}} \lg t$, $M(t) = \lg t$. Observe also that if we took another solution than y_n, y_{n+1}, then we would get results in a certain sense worse than in Theorems 3.1, 3.2. Indeed, these theorems are in view of the oscillation criteria given in Section 4 "optimal". The application of Theorem 2.1 with (4) replaced by (5) and e.g. $i = 1$ gives a sufficient condition for oscillation

$$\limsup_{t\to\infty} t^{2n-1-\alpha-2\alpha_1} \int_t^\infty \left[q(s) + \frac{\mu_{n,\alpha}}{s^{2n-\alpha}}\right] s^{2\alpha_1} < \overline{K}_{n,\alpha}$$

(the negative constant $\overline{K}_{n,\alpha}$ can be computed explicitly, but it is not important at this moment) and this condition, in contrast to that given in Theorem 4.1, does not apply to the equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} - \left[\frac{\mu_{n,\alpha}}{t^{2n-\alpha}} - \frac{\lambda}{t^{2n-\alpha}\lg^2 t}\right] y = 0$$

which is oscillatory if $\lambda < -K_{n,\alpha}$ by Theorem 4.1. A similar situation we have in the case of the nonoscillation criteria presented in Section 3. The reason for this phenomenon is, roughly speaking, that the "gap" between the solutions $y_n = t^{\frac{2n-1-\alpha}{2}}$ and $y_{n+1} = t^{\frac{2n-1-\alpha}{2}} \lg t$ is less than the "gap" between other consecutive solutions of the ordered system of solutions of (5).

(iii) In Theorem 4.1 we have supposed that the improper integral in (18) is convergent. If this integral diverges to ∞, then equation (21) is oscillatory by a higher order modification of the classical Leighton-Wintner oscillation criterion, see [6], which reads as follows.

Theorem 5.2. *Suppose that the* $2n$*-order operator* L *of the form* (6) *possesses an ordered system of solutions* $y_1, \dots, y_n, y_{n+1}, \dots, y_{2n}$. *If*

$$\int^\infty q(t) y_i^2(t)\,\mathrm{d}t = -\infty \quad \text{for some } i \in \{1, \dots, n\},$$

then equation (21) *is oscillatory.*

In our setting

$$L(y) = (-1)^n (t^\alpha y^{(n)})^{(n)} - \frac{\mu_{n,\alpha}}{t^{2n-\alpha}} y \quad \text{and} \quad y_i = t^{\frac{2n-1-\alpha}{2}}.$$

(iv) In Sections 3, 4 we have considered the case $\alpha \notin \{1, 3, \dots, 2n-1\}$. If α is in the "critical set" $\{1, 3, \dots, 2n-1\}$, then $\mu_{n,\alpha} = 0$ and the equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} + \frac{\lambda}{t^{2n-\alpha}} y = 0 \tag{23}$$

is oscillatory for any $\lambda < 0$. Indeed, let $y_1, \dots, y_n, \tilde{y}_1, \dots, \tilde{y}_n$ be an ordered system of solutions of the one term equation (4). One can verify by a direct computation that this ordered system can be chosen in such a way that $y_n = t^{\frac{2n-1-\alpha}{2}}$ (see also [10]) and by the previous Theorem 5.2 (23) it is oscillatory, since

$$\int^\infty \frac{\lambda}{t^{2n-\alpha}} y_n^2(t)\,\mathrm{d}t = \int^\infty \frac{\lambda}{t}\,\mathrm{d}t = -\infty.$$

Therefore, equations (4) and (5) coincide and the approach where (1) is considered as a perturbation of (4) has been already applied in [8], [13], [14] and other papers. On the other hand, as pointed out in [8], the "right term" which is to be added to (4) is $\mu t^{\alpha-2n} \lg^{-2} t$ since the equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} + \frac{\lambda}{t^{2n-\alpha} \lg^2 t} y = 0 \tag{24}$$

is conditionally oscillatory. However, we generally do not know the value of the constant of conditional oscilation λ_0 (i.e. the constant such that (24) is oscillatory for $\lambda < \lambda_0$ and nonoscillatory for $\lambda > \lambda_0$), so we cannot proceed in the same way as for $\alpha \notin \{1, 3, \dots, 2n-1\}$.

(v) A slight modification of the proofs of Theorem 4.1 in [12] and Theorem 5.2 in [11] gives the following criterion of conjugacy of (1). Recall that equation (6) is said to be *conjugate* in an interval (a, b) if there exists a nontrivial solution of this equation having at least two conjugate points in (a, b).

Theorem 5.3. *Suppose that*

$$\limsup_{t_1 \downarrow 0, t_2 \uparrow \infty} \int_{t_1}^{t_2} t^{\frac{2n-1-\alpha}{2}} \left[q(t) + \frac{\mu_{n,\alpha}}{t^{2n-\alpha}} \right] \mathrm{d}t \leqslant 0$$

and

$$q(t) + \frac{\mu_{n,\alpha}}{t^{2n-\alpha}} \not\equiv 0, \quad t \in (0, \infty).$$

Then (1) *is conjugate in the interval* $(0, \infty)$.

(vi) The oscillation theory of self-adjoint equations (6) is closely related to the spectral theory of singular differential operators. In particular, the spectrum of the operator

$$y \longmapsto w^{-1}(t)L(y)$$

(with a positive weight function w and L given by (6)) in the weighted Hilbert space $L^2_w(T,\infty)$ is discrete and bounded below (the so-called property BD) if and only if the equation $L(y)=\lambda w(t)y$ is nonoscillatory for every $\lambda\in\mathbb{R}$, see [15]. Consequently, by Theorems 3.1, 4.1 the operator

$$y \longmapsto \frac{1}{w(t)}\Big[(-1)^n(t^\alpha y^{(n)})^{(n)} - \frac{\mu_{n,\alpha}}{t^{2n-\alpha}}y\Big]$$

has property BD if and only if

$$\lim_{t\to\infty} \lg t \int_t^\infty w(s)s^{2n-\alpha-1}\,\mathrm{d}s = 0.$$

(vii) In all oscillation and nonoscillation criteria for (1), where this equation is viewed as a perturbation of (5), we have multiplied the function q (under the integral sign) by the square of a solution of this equation, compare (15), (17), (19), (20) and also other criteria along this line. In [7] we used a slightly different approach. Equation (2) is viewed there as a perturbation of the one term equation (4), but in oscillation criteria the function q is multiplied by the square of a general function and the only restriction on this function is that it can be, in a certain sense, "inserted" into the ordered system of solutions of (4). Of course, this method can be used also in the general setting when dealing with the general equation $L(y)+q(t)y=0$ viewed as a perturbation of (6).

6. Technical lemmata

We close the paper with technical results needed in the previous sections. Proofs of these statements can be found in [3].

Lemma 6.1. *Let $y_1,\dots,y_n,\tilde y_1,\dots,\tilde y_n\in C^{n-1}$ be a system of linearly independent functions and let X, $\tilde X$ be the Wronski matrices of $y_1,\dots,y_n$ and $\tilde y_1,\dots,\tilde y_n$, respectively. Then*

$$[\tilde X^{-1}X]_{i,j} = \frac{W(\tilde y_1,\dots,\tilde y_{i-1},y_j,\tilde y_{i+1},\dots,\tilde y_n)}{W(\tilde y_1,\dots,\tilde y_n)}.$$

Lemma 6.2. *Let $y_1, \dots y_m \in C^{m-1}$ be an ordered system of functions (at ∞). Then*

$$\lim_{t\to\infty} \frac{W(y_{i_1}, \dots, y_{i_k})}{W(y_{j_1}, \dots, y_{j_k})} = 0, \quad k = 1, \dots, m,$$

whenever $i_1 \leqslant j_1, \dots, i_k \leqslant j_k$ and at least one of the inequalities is strict.

Lemma 6.3. *Let $y_1, \dots, y_m \in C^{m-1}$, $r \in C^{m-1}$ and $r \neq 0$. Then*

$$W(ry_1, \dots, ry_m) = r^m W(y_1, \dots, y_m).$$

Particularly, if $y_1 \neq 0$ we have

$$W(y_1, \dots, y_m) = y_1^m W((y_2/y_1)', \dots, (y_m/y_1)').$$

Lemma 6.4. *Let $y_1 = t^{\alpha_1}, \dots, y_n = t^{\alpha_n}$, $\alpha_i \in \mathbb{R}$, $i = 1, \dots, n$. Then*

$$W(y_1, \dots, y_n) = \prod_{1\leqslant i<j}^{n} (\alpha_j - \alpha_i) t^{\sum\limits_{k=1}^{n} \alpha_k - \frac{n(n-1)}{2}}.$$

Lemma 6.5. *Let $y_1 = t^{\alpha_1}, \dots, y_{n-1} = t^{\alpha_{n-1}}$, $y_n = t^{\frac{2n-1-\alpha}{2}}$, $\tilde{y}_1 = t^{\frac{2n-1-\alpha}{2}} \lg t$, $\tilde{y}_2 = t^{2n-1-\alpha-\alpha_{n-1}}$, $\dots$, $\tilde{y}_n = t^{2n-1-\alpha-\alpha_1}$ be an ordered system of solutions of* (5) *and let (X, U), $(\tilde{X}, \tilde{U})$ be the matrix solutions of the associated LHS* (7) *generated by $y_1, \dots, y_n$ and $\tilde{y}_1, \dots, \tilde{y}_n$, respectively. Further, let $L := X^T\tilde{U} - U^T\tilde{X}$ and let $L_{i,j}$, $i, j = 1, \dots, n$, be the entries of this (constant) matrix. Then $L_{j,1} = 0$ for $j = 1, \dots, n-1$ and*

$$L_{n,1} = -P_{n,\alpha}\Big(\frac{2n-1-\alpha}{2}\Big),$$

where the polynomial $P_{n,\alpha}$ is given by (12).

P r o o f. Let (x_j, u_j), $j = 1, \dots, n-1$, $(\tilde{x}, \tilde{u})$ be vector solutions of (7) with A given by (8) and

$$B(t) = \operatorname{diag}\{0, \dots, 0, t^{-\alpha}\}, \quad C(t) = \operatorname{diag}\Big\{\frac{\mu_{n,\alpha}}{t^{2n-\alpha}}, 0 \dots, 0\Big\},$$

generated by y_j, $j=1,\dots,n-1$, and $\tilde y_1$, respectively. Then $L_{j,1}=[x_j^T\tilde u-u_j^T\tilde x]$ are real constants. Further, let $\lambda\in\mathbb{R}$, $y(t)=t^\lambda$,

$$x(t)=\begin{pmatrix} y\\ y'\\ \vdots\\ y^{(n-1)}\end{pmatrix}=\begin{pmatrix} t^\lambda\\ \lambda t^{\lambda-1}\\ \vdots\\ \prod_{i=0}^{n-2}(\lambda-i)t^{\lambda-n+1}\end{pmatrix},$$

$$u(t)=\begin{pmatrix}(-1)^{n-1}(t^\alpha y^{(n)})^{(n-1)}\\ \vdots\\ -(t^\alpha y^{(n)})'\\ t^\alpha y^{(n)}\end{pmatrix}=\begin{pmatrix}\frac{(-1)^{n-1}}{\lambda+\alpha-2n+1}\prod_{i=0}^{n-1}(\lambda-i)(\lambda+\alpha-n-i)t^{\lambda+\alpha-2n+1}\\ \vdots\\ -(\lambda+\alpha-n)\prod_{i=0}^{n-1}(\lambda-i)t^{\lambda-n+\alpha-1}\\ \prod_{i=0}^{n-1}(\lambda-i)t^{\lambda-n+\alpha}\end{pmatrix}.$$

By a direct computation one can verify that

$$F(t):=x^T(t)\tilde u(t)-u^T(t)\tilde x(t)=t^{\lambda-\frac{2n-1-\alpha}{2}}[S(\lambda)\lg t+R(\lambda)],$$

where S, R are certain polynomials of degrees $2n-1$ and $2n-2$, respectively. Now F is a constant function if λ is a root of the polynomial

$$(-1)^n\prod_{i=0}^{n-1}(\lambda-i)(\lambda-n+\alpha-i)-\mu_{n,\alpha}=\Big(\lambda-\frac{2n-1-\alpha}{2}\Big)^2P_{n,\alpha}(\lambda),$$

which means that

$$F'(t)=t^{\lambda-\frac{2n+1-\alpha}{2}}\Big\{\Big(\lambda-\frac{2n-1-\alpha}{2}\Big)[S(\lambda)\lg t+R(\lambda)]+S(\lambda)\Big\}=0$$

for these λ. This implies that $R(\lambda)=0$ if $\lambda=\alpha_i$ and $\lambda=2n-1-\alpha-\alpha_i$, $i=1,\dots,n-1$. Hence, since the leading coefficients of $P(\lambda)$ and $R(\lambda)$ are $(-1)^n$ and $(-1)^{n-1}$, respectively, we have $R(\lambda)=-P_{n,\alpha}(\lambda)$. By a similar argument

$$S(\lambda)=\Big(\lambda-\frac{2n-1-\alpha}{2}\Big)P_{n,\alpha}(\lambda).$$

Consequently,

$$L_{j,1}=t^{\alpha_i-\frac{2n-1-\alpha}{2}}P_{n,\alpha}(\alpha_i)\Big\{\Big(\alpha_i-\frac{2n-1-\alpha}{2}\Big)\lg t-1\Big\}=0,\quad i=1,\dots,n-1,$$

and

$$L_{n,1}=-P_{n,\alpha}\Big(\lambda-\frac{2n-1-\alpha}{2}\Big).$$

$\square$

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