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ON ORTHOGONALLY σ-COMPLETE LATTICE ORDERED GROUPS

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Abstract. In this paper we prove a theorem of Cantor-Bernstein type for orthogonally σ-complete lattice ordered groups.

Keywords: lattice ordered group, orthogonal σ-completeness, direct factor

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Sikorski [6] and Tarski [10] (cf. also Sikorski [7]) proved a theorem of Cantor-Bernstein type for σ-complete Boolean algebras.

In a modified (but equivalent) form this theorem can be expressed as follows.

(A) Let B_1 and B_2 be σ-complete Boolean algebras. Suppose that

- (i) there exists an element $b_1 \in B_1$ such that B_2 is isomorphic to the interval $[0, b_1]$ of B_1;
- (ii) there exists an element $b_2 \in B_2$ such that B_1 is isomorphic to the interval $[0, b_2]$ of B_2.

Then B_1 is isomorphic to B_2.

Let us remark that each interval $[0, b_1]$ in B_1 is isomorphic to a direct factor of the lattice B_1 (since B_1 is a bounded distributive lattice and b_1 has a complement in B_1); conversely, for each direct factor X of the lattice B_1 there exists $b_1 \in B_1$ such that X is isomorphic to $[0, b_1]$.

Hence (i) is equivalent with the condition

- (i_1) there exists a direct factor of B_1 which is isomorphic to B_2.

Let G be a lattice ordered group. A nonempty subset $A \subseteq G^+$ is said to be orthogonal (or disjoint) if $a_1 \wedge a_2 = 0$ whenever a_1 and a_2 are distinct elements of A. If each denumerable orthogonal subset of G has a supremum in G, then G will be called orthogonally σ-complete.

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In the present paper we prove the following result:

(B) Let G_1 and G_2 be orthogonally σ-complete lattice ordered groups. Suppose that

(i$_2$) there exists a direct factor of G_1 which is isomorph to G_2;

(ii$_2$) there exists a direct factor of G_2 which is isomorphic to G_1.

Then G_1 is isomorphic to G_2.

By an example we show that the condition of orthogonal σ-completeness cannot be omitted in (B).

Other results of Cantor-Bernstein type for lattice ordered groups and for MV-algebras were obtained in [2]–[5] and [9].

1. Direct factors of a lattice ordered group

Let I be a nonempty set and for each $i \in I$ let H_i be a lattice ordered group. The direct product of the indexed system $(H_i)_{i\in I}$ has the usual meaning; we denote it by

$$H = \prod_{i\in I} H_i.$$

If $I = \{1, 2, \ldots, n\}$, then we write also $H = H_1 \times H_2 \times \ldots \times H_n$. For $h \in H$ and $i \in I$, let h_i be the component of H in H_i.

Suppose that ψ is an isomorphism of a lattice ordered group G onto H. For each $i \in I$ we put

$$H_i^0 = \{g \in G\colon\ \psi(g)_j = 0 \quad \text{for each } \ j \in I \setminus \{i\}\}.$$

Further, for each $x \in G$ and each $i \in I$ we denote by $\varphi_i(x)$ the element of H_i^0 such that

$$\psi(\varphi_i(x))_i = \psi(x)_i.$$

Then H_i^0 is a convex ℓ-subgroup of G and the mapping

$$\varphi\colon\ G \to \prod_{i\in I} H_i^0$$

defined by

$$\varphi(x) = (\varphi_i(x))_{i\in I}$$

is an isomorphism of G onto $\prod\limits_{i\in I} H_i^0$; it is called an internal direct product decomposition of G. All ℓ-subgroups H_i^0 of G that can be constructed in this way are called internal direct factors of G.

Let $\mathcal{I}(G)$ be the system of all internal direct factors of G; this system is partially ordered by the set-theoretical inclusion. The following facts are well-known:

1.1. $\mathcal{I}(G)$ is a Boolean algebra with the greatest element G and the least element $\{0\}$.

1.2. If $G_1 \in \mathcal{I}(G)$, then $\mathcal{I}(G_1) \subseteq \mathcal{I}(G)$.

For $X \subseteq G$ we denote

$$X^\delta = \{g \in G\colon\ |g| \wedge |x| = 0 \quad \text{for each } x \in X\}.$$

Then we have

1.3. If $G_1 \in \mathcal{I}(G)$, then G_1^δ is the complement of G_1 in $\mathcal{I}(G)$.

The set X^δ is said to be a polar in G. For the basic properties of polars cf. Šik [8].

1.4. Let X be a convex ℓ-subgroup of G. Then the following conditions are equivalent:

(i) If $0 \leqslant g \in G$, then the set

$$X_g = \{x \in X\colon\ 0 \leqslant x \leqslant g\}$$

has a greatest element.

(ii) X is an internal direct factor of G.

Moreover, if (i) is valid, then the greatest element of the set X_g is the component of g in the internal direct factor X; this component will be denoted by $g(X)$. For $x \in X$ we have $x(X) = x$.

1.5. Lemma. *Let G be a lattice ordered group which is orthogonally σ-complete. Suppose that G_n $(n = 1, 2, 3, \ldots)$ are internal direct factors of G such that $G_{n(1)} \cap G_{n(2)} = \{0\}$ whenever $n(1)$ and $n(2)$ are distinct positive integers. Denote*

$$D = \left(\bigcup_{n=1}^{\infty} G_n\right)^\delta.$$

Then D is an internal direct factor of G.

P r o o f. In view of the definition, D is a convex ℓ-subgroup of G. Let $0 \leqslant g \in G$. Put $g_n = g(G_n)$ for each $n \in \mathbb{N}$. Then $0 \leqslant g_n \leqslant g$ and the system $(g_n)_{n\in\mathbb{N}}$ is orthogonal. Hence there exists $x \in G$ such that

$$x = \bigvee_{n=1}^{\infty} g_n.$$

Let $x_n = x(G_n)$. Since $x \geqslant g_n$, we have $x_n \geqslant g_n(G_n) = g_n$. Further,

$$x_n = x_n \wedge x = x_n \wedge \left(\bigvee_{m=1}^{\infty} g_m\right) = \bigvee_{m=1}^{\infty} (x_n \wedge g_m) = x_n \wedge g_n,$$

whence $x_n \leqslant g_n$. Summarizing, $x_n = g_n$.

We denote $y = g - x$. Hence $0 \leqslant y \leqslant g$. Let $n \in \mathbb{N}$ and $y_n = y(G_n)$. Thus $0 \leqslant y_n \leqslant y$. Since $g = y + x$, we obtain $g_n = y_n + x_n = y_n + g_n$ and therefore $y_n = 0$. In view of 1.4, this yields that y belongs to D. Also, $y \wedge x = 0$, whence $g = y \vee x$.

Let $z \in D$, $0 \leqslant z \leqslant g$. Then $z \wedge x = 0$, thus

$$z = z \wedge g = z \wedge (y \vee x) = z \wedge y.$$

Then $z \leqslant y$. By applying 1.4 we infer that $D \in \mathcal{I}(G)$. □

1.6. Lemma. *Let G, G_n $(n \in \mathbb{N})$ and D be as in* 1.5. *Then G is an internal direct product of its convex ℓ-subgroups D and G_n $(n \in \mathbb{N})$.*

Proof. In view of 1.5, the symbol $g(D)$ is defined for each $g \in G$. Consider the mapping ψ of G into

$$A = D \times \prod_{i \in I}^{\infty} G_n$$

such that, for each $g \in G$,

$$\psi(g)(D) = g(D), \quad \psi(g)(G_n) = g(G_n).$$

Then ψ is a homomorphism of G into A. Let $0 \neq g \in G$. Hence $|g| \neq 0$. Thus either $|g|_n = |g|(G_n) > 0$ for some $n \in \mathbb{N}$, or $|g| \in D$; in the latter case we have $|g|(D) = |g| > 0$. Therefore $\psi(|g|) > 0$. This yields that $\psi(g) \neq 0$ and hence ψ is an isomorphism of G into A.

Let $a \in A$. Consider the system $(y^0, y^1, y^2, \ldots)$ where $y^0 = a^+(D)$, $y^n = a^+(G_n)$ for each $n \in \mathbb{N}$. Since G is orthogonally σ-complete, there exists $y \in G^+$ such that

$$y = \bigvee_{n=0}^{\infty} y_n.$$

It is easy to verify (by an analogous argument as in the proof of 1.5) that

$$y(D) = y^0, \quad y(G_n) = y^n \quad (n = 1, 2, \ldots).$$

Hence $\psi(y) = a^+$. Similarly we can verify that there exists $z \in G$ with $\psi(z) = a^-$. Then $\psi(y - z) = a$. Thus ψ is an epimorphism.

If $d \in D$, then $\psi(g)(D) = 0$; similarly, if $g \in G_n$, then $\psi(g)(G_n) = g_n$. This yields that ψ is an internal direct product decomposition of G. □

2. Proof of (B)

If G is a lattice ordered group, $A, B \in \mathcal{I}(G)$ and $A \subseteq B$, then we denote by $B \ominus A$ the relative complement of A in the interval $[\{0\}, B]$ of the Boolean algebra $\mathcal{I}(G)$. Hence we have

$$B = A \times (B \ominus A)$$

(meaning that B is an internal direct product of A and $B \ominus A$).

For proving (B) we apply Lemma 1.6 and use similar steps as in the well-known proof of the classical Cantor-Bernstein Theorem of the set theory.

2.1. Lemma. *Let G be a lattice ordered group which is orthogonally σ-complete. Assume that $A_1, A_2 \in \mathcal{I}(G)$, $A_1 \supseteq A_2$ and that A_2 is isomorphic to G. Then A_1 is isomorphic to G as well.*

Proof. Let φ be an isomorphism of G onto A_2. Put $A_3 = \varphi(A_1)$. Hence A_3 is an internal direct factor of A_2. Further, we denote $A_4 = \varphi(A_2)$. Thus A_4 is an internal direct factor of A_3. By further analogous steps we construct a sequence

$$G \supseteqq A_1 \supseteqq A_2 \supseteqq A_3 \supseteqq A_4 \supseteqq A_5 \supseteqq \dots$$

of internal direct factors of G (cf. 1.2).

Let the symbol $\simeq$ denote the relation of isomorphism between lattice ordered groups. From the construction of the elements of the sequence under consideration we get

$$G \simeq A_2, \quad A_1 \simeq A_3, \quad A_2 \simeq A_4, \quad A_3 \simeq A_5, \quad \dots$$

and, moreover,

$$G \ominus A_1 \simeq A_2 \ominus A_3, \quad A_1 \ominus A_2 \simeq A_3 \ominus A_4, \quad A_2 \ominus A_3 \simeq A_4 \ominus A_5, \quad \dots$$

Denote

$$G \ominus A_1 = G_1, \quad A_1 \ominus A_2 = G_2, \quad A_2 \ominus A_3 = G_3, \quad A_3 \ominus A_4 = G_4, \quad \dots$$

We obtain

$$G_n \simeq G_{n+2} \quad \text{for each } n \in \mathbb{N}. \tag{1}$$

Further, we have

$$G_{n(1)} \cap G_{n(2)} = \{0\}$$

whenever $n(1)$ and $n(2)$ are distinct positive integers. Let D be as in 1.5. According to 1.6 we obtain

α) G is an internal direct product of its ℓ-subgroups D and G_n $(n \in \mathbb{N})$.

For $X \subseteq A_1$ we put

$$X^{\delta_1} = \{g \in A_1 \colon |g| \wedge |x| = 0 \quad \text{for each } x \in X\}.$$

It is easy to verify that

$$\Bigl(\bigcup_{n=2}^{\infty} G_n\Bigr)^{\delta_1} = D,$$
$$G_n \subseteq A_1 \quad \text{for } n = 2, 3, \ldots$$

Thus by applying 1.6 again we infer

β) A_1 is an internal direct product of its convex ℓ-subgroups D and $G_2, G_3, G_4, \ldots$.

Now, from α), β) and (1) we get the relation $G \simeq A_1$. □

P r o o f of (B). Let G_1 and G_2 be orthogonally σ-complete lattice ordered groups. Suppose that the conditions (i_2) and (ii_2) are satisfied. Hence there exists $B^0 \in \mathcal{I}(G_2)$ such that $G_1 \simeq B^0$. Further, there exists $A^0 \in \mathcal{I}(G_1)$ such that $A^0 \simeq G_2$. Hence there is an isomorphism φ of B onto A^0. Put $\varphi(B^0) = A^1$. Then $B^0 \simeq A^1$ and thus $G_1 \simeq A^1$. Clearly $A^1 \in \mathcal{I}(G_1)$ and $A^1 \subseteq A^0$. Hence in view of 2.1, $G_1 \simeq A^0$. Therefore $G_1 \simeq G_2$. □

By means of simple examples we can show that if G is a lattice ordered group which is orthogonally σ-complete, then

(i) G need not be σ-complete,

(ii) G need not be orthogonally complete.

We conclude this section by remarking that the notion of orthogonal σ-completeness can be applied also for MV-algebras. By using the same steps as above we can verify that (B) remains valid if instead of lattice ordered groups G_1 and G_2 we consider MV-algebras M_1 and M_2.

3. An example

In this section we show that the assumption of orthogonal σ-completeness cannot be omitted in (B).

Let G be a lattice ordered group. An element u of G is called a strong unit if for each $g \in G$ there exists a positive integer n such that $g \leqslant nu$. An element s of G is said to be singular if $0 \leqslant s$ and the interval $[0, s]$ of G is a Boolean algebra.

The direct product decomposition of a lattice L is defined in the usual way. Suppose that L has a least element 0. Then we can define internal direct factors and internal direct product decompositions of L analogously as we did for the case of lattice ordered groups in Section 1 above; we omit the obvious details.

In what follows all direct product decompositions of G and of L are supposed to be internal. The set of all internal direct factors of L will be denoted by $\mathcal{I}(L)$.

3.1. *Let G be a lattice ordered group with a strong unit u. Let L be the interval $[0,u]$ of G. Assume that X belongs to $\mathcal{I}(L)$ and that A^0 is the convex ℓ-subgroup of G which is generated by the set X. Then $A^0 \in \mathcal{I}(G)$.*

Proof. Suppose that $L = X \times Y$. Then there exists a greatest element x^0 in X and a greatest element y^0 in Y; moreover, $u = x^0 \vee y^0$ and $x^0 \wedge y^0 = 0$. From the last relation we obtain $u = x^0 + y^0$. Then A^0 is the convex ℓ-subgroup of G which is generated by x^0. Let B^0 be the convex ℓ-subgroup of G which is generated by y^0. Then $A^0 \cap B^0 = \{0\}$, whence $A^0 + B^0 = B^0 + A^0$.

Let $g \in G$. There exists a positive integer n such that

$$g^+ \leqslant nu = nx^0 + ny^0 = nx^0 \wedge ny^0.$$

Hence

$$(*) \quad g^+ = g^+ \wedge (nx^0 \vee ny^0) = (g^+ \wedge nx^0) \vee (g^+ \wedge ny^0) = (g^+ \wedge nx^0) + (g^+ \wedge ny^0).$$

For the element g^- we obtain an analogous relation. Therefore $G = A^0 + B^0$. Thus the group G is a direct product of its subgroups A^0 and B^0.

Let $g, g' \in G$. There are uniquely determined elements $g_1, g_1' \in A^0$ and $g_2, g_2' \in B^0$ such that $g = g_1 + g_2$, $g' = g_1' + g_2'$. If $g_1 \leqslant g_1'$ and $g_2 \leqslant g_2'$, then $g \leqslant g'$. Conversely, suppose that $g \leqslant g'$. Denote $g'' = g' - g$. In view of $(*)$ there is a positive integer n such that

$$g'' = (g'' \wedge nx^0) + (g'' \wedge ny^0).$$

Then clearly $g'' \wedge nx_0 \in (A^0)^+$ and $(g'' \wedge ny^0) \in (B^0)^+$. Thus $g_1' - g_1 = g'' \wedge nx^0 \geqslant 0$, yielding that $g_1' \geqslant g_1$. Analogously, $g_2' \geqslant g_2$. Summarizing, we obtain for the lattice ordered group G the relation $G = A^0 \times B^0$. □

Let B be a Boolean algebra. We denote by $C(B)$ the system of all elementary Carathéodory functions on B (cf. [1]). Further, let $C_0(B)$ be the convex ℓ-subgroup of $C(B)$ which is generated by the greatest element of B. From the construction of $C_0(B)$ we conclude:

3.2. Lemma. *Let B_1 and B_2 be Boolean algebras. Then*

(i) *the lattice ordered groups* $C_0(B_1)$ *and* $C_0(B_2)$ *are isomorphic if and only if* B_1 *and* B_2 *are isomorphic;*

(ii) *for* $i \in \{1, 2\}$, B_i *is the set of all singular elements of* $C_0(B_i)$.

According to [7] (pp. 90 and 193) the hypothesis of σ-completeness is essential in (A).

This yields that there exist non-isomorphic Boolean algebras B_1 and B_2 which satisfy the conditions (i) and (ii) from (A).

Let us construct lattice ordered groups $G_1 = C_0(B_1)$ and $G_2 = C_0(B_2)$. Then in view of 3.2, G_1 is not isomorphic to G_2.

Let b_1 and b_2 be as in (A). Thus the interval $[0, b_1]$ of B_1 is isomorphic to B_2. Hence according to 3.2, $C_0([0, b_1])$ is isomorphic to $C_0(B_2)$. Further, $[0, b_0]$ is a direct factor of B_1, thus in view of 3.1, $C_0([0, b_1])$ is a direct factor of G_1. Therefore the condition (i_2) is valid for the lattice ordered groups B_1 and B_2. Similarly, by using the element b_2 we obtain that the condition (ii_2) is valid for B_1 and B_2.

Consequently, without the assumption of orthogonal σ-completeness the assertion of (B) need not hold.

References

[1] *J. Jakubík*: Cardinal properties of lattice ordered groups. Fund. Math. *74* (1972), 85–98.

[2] *J. Jakubík*: Cantor-Bernstein theorem for lattice ordered groups. Czechoslovak Math. J. *22(97)* (1972), 159–175.

[3] *J. Jakubík*: On complete lattice ordered groups with strong units. Czechoslovak Math. J. *46(121)* (1996), 221–230.

[4] *J. Jakubík*: Cantor-Bernstein theorem for MV-algebras. Czechoslovak Math. J. *49(124)* (1999), 517–526.

[5] *J. Jakubík*: Convex isomorphisms of archimedean lattice ordered groups. Mathware Soft Comput. *5* (1998), 49–56.

[6] *R. Sikorski*: A generalization of theorem of Banach and Cantor-Bernstein. Coll. Mat. *1* (1948), 140–144.

[7] *R. Sikorski*: Boolean algebras. Second edition, Springer Verlag, Berlin, 1964.

[8] *F. Šik*: To the theory of lattice ordered groups. Czechoslovak Math. J. *6(81)* (1956), 1–25. (In Russian.)

[9] *A. De Simone, D. Mundici and M. Navara*: A Cantor-Bernstein theorem for σ-complete MV-algebras. (Preprint).

[10] *A. Tarski*: Cardinal Algebras. Oxford University Press, New York, London, 1949.

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