Jinjin Li

A note on $g$-metrizable spaces

*Czechoslovak Mathematical Journal*, Vol. 53 (2003), No. 2, 491–495

Persistent URL: http://dml.cz/dmlcz/127816

**Czechoslovak Mathematical Journal, 53 (128) (2003), 491–495**

# A NOTE ON $g$-METRIZABLE SPACES

Jinjin Li, Shantou

(Received July 26, 2000)

*Abstract.* In this paper, the relationships between metric spaces and $g$-metrizable spaces are established in terms of certain quotient mappings, which is an answer to Alexandroff's problems.

*Keywords*: metric spaces, $g$-metrizable spaces, 1-sequence-covering mappings, $\sigma$-mappings, quotient mappings

*MSC 2000*: 54E99, 54C10, 54D55

## 1. Introduction

A central question of Alexandroff's problem is that the relationships between various topological spaces and metric spaces are established by means of various mappings [1]. The concept of a $g$-metrizable space was first introduced by F. Siwiec in [2], as a generalization of metric spaces. How to characterize a $g$-metrizable space by a nice image of a metric space? S. Lin introduced the concept of 1-sequence-covering mappings in order to give characterizations for spaces with a point-countable weak base [3]. Recently, S. Lin introduced the concept of $\sigma$-mappings [4], and showed that a space $X$ is a $\sigma$-space if and only if it is a $\sigma$-image of a metric space. This shows that 1-sequence-covering mappings and $\sigma$-mappings are very important in answering Alexandroff's problems. In this paper, the relationships between metric spaces and $g$-metrizable spaces are established by means of 1-sequence-covering and quotient $\sigma$-mappings, which is also an answer to Alexandroff's problems.

All spaces in this paper are assumed to be regular and $T_1$. Mappings are continuous and onto. $\mathbb{N}$ denotes the set of positive integers, $\omega = \{0\} \cup \mathbb{N}$.

This work is supported by the NSF of China (10271056).

We recall some definitions.

Let $X$ be a space, and $\mathscr{P}$ be a cover of $X$. $\mathscr{P}$ is a network for $X$ if, whenever $x \in U$ with $U$ open in $X$, then $x \in P \subset U$ for some $P \in \mathscr{P}$. A subfamily $\mathscr{P}'$ of $\mathscr{P}$ is a network at $x \in X$ if $x \in \bigcap \mathscr{P}'$ and whenever $x \in U$ with $U$ open in $X$, then $P \subset U$ for some $P \in \mathscr{P}'$. A space is a $\sigma$-space if it has a $\sigma$-locally finite network.

**Definition 1.1.** Let $f\colon X \to Y$ be a mapping.

(1) $f$ is called a $\sigma$-mapping [4] if $X$ has a base $\mathscr{B}$ such that $f(\mathscr{B}) = \{f(B)\colon B \in \mathscr{B}\}$ is $\sigma$-locally finite in $Y$.

(2) $f$ is a 1-sequence-covering mapping [3] if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to $y$ in $Y$ there is a sequence $\{x_n\}$ converging to $x$ in $X$ with $x_n \in f^{-1}(y_n)$ for all $n$.

(3) $f$ is sequentially quotient [5] if for each convergent sequence $L$ of $Y$, there is a convergent sequence $S$ of $X$ such that $f(S)$ is a subsequence of $L$.

(4) $f$ is quotient [6] if, whenever $f^{-1}(U)$ is open, then $U$ is open in $Y$.

(5) $f$ is pseudo-open [6] if, whenever $f^{-1}(y) \subset V$ with $V$ open in $X$, then $y \in \operatorname{int}(f(V))$.

Obviously, every 1-sequence-covering mapping is sequentially quotient.

Let us recall some basic definitions. Let $X$ be a space, and let $\mathscr{P}$ be a cover of $X$. A space $X$ is determined by $\mathscr{P}$ if $U \subset X$ is open (closed) in $X$ if and only if $U \cap P$ is open (closed) in $P$ for every $P \in \mathscr{P}$. A space $X$ is a $k$-space (a sequential space), if it is determined by the cover consisting of all compact (all compact metric) subsets of $X$. A space is a Fréchet if, whenever $x \in \overline{A}$, there is a sequence $\{a_n\colon n \in \mathbb{N}\}$ in $A$ with $a_n \to x$.

**Definition 1.2.** Let $X$ be a space, and let $\mathscr{P} = \bigcup\{\mathscr{P}_x\colon x \in X\}$ be a collection of subsets in $X$ satisfying the following conditions:

(a) $\mathscr{P}_x$ is a network at $x \in X$.

(b) For any $U, V \in \mathscr{P}_x$, there is $W \in \mathscr{P}_x$ such that $W \subset U \cap V$.

Then,

(1) $\mathscr{P}$ is called a weak base for $X$ [7] if for $G \subset X$, $x \in G$, there is $P \in \mathscr{P}_x$ with $x \in P \subset G$. Then $G$ is open in $X$ [13]. A space $X$ is weakly first countable if $X$ has a weak base $\mathscr{P} = \bigcup\{\mathscr{P}_x\colon x \in X\}$ such that each $\mathscr{P}_x$ is countable. A space $X$ is a $g$-metrizable space [2] if it has a $\sigma$-locally finite weak base.

(2) $\mathscr{P}$ is called a sequential neighborhood network for $X$ [3] if any $P \in \mathscr{P}_x$ is a sequential neighborhood of $x$ in $X$ (that is, if for each convergent sequence $x_n \to x$, there exists $m \in \mathbb{N}$ such that $\{x\} \cup \{x_n\colon n \geqslant m\} \subset P$). This $\mathscr{P}_x$ is called a sequential neighborhood network of $x$.

## 2. Results

**Lemma 2.1.** *The following conditions are equivalent for a space $X$:*

(1) *$X$ is a 1-sequence-covering and $\sigma$-image of a metric space.*

(2) *$X$ has a $\sigma$-locally finite sequential neighborhood network.*

P r o o f. (1) $\Rightarrow$ (2). Suppose that $f\colon M \to X$ is a 1-sequence-covering and $\sigma$-mapping, where $M$ is a metric space, then there is a base $\mathscr{B}$ for $M$ such that $f(\mathscr{B})$ is $\sigma$-locally finite in $X$. For each $x \in X$, there is $\beta_x \in f^{-1}(x)$ satisfying Definition 1.1 (2). Put $\mathscr{P}_x = \{f(B)\colon \beta_x \in B \in \mathscr{B}\}$, $\mathscr{P} = \bigcup\{\mathscr{P}_x\colon x \in X\}$, then it is easy to check that $\mathscr{P}$ is a $\sigma$-locally finite sequential neighborhood network for $X$.

(2) $\Rightarrow$ (1). Let $X$ have a $\sigma$-locally finite sequential neighborhood network $\mathscr{P} = \bigcup\{\mathscr{P}_n\colon n \in \mathbb{N}\}$, where $\mathscr{P}$ is closed under finite intersections, and each $\mathscr{P}_n$ is locally finite. We can assume that $X \in \mathscr{P}_n \subset \mathscr{P}_{n+1}$. Let $\mathscr{P}_n = \{P_\alpha\colon \alpha \in A_n\}$. For each $n \in \mathbb{N}$, $A_n$ is endowed with discrete topology. Put $M = \big\{\alpha = (\alpha_n) \in \prod\limits_{n\in\mathbb{N}} A_n\colon \{P_{\alpha_n}\colon n \in \mathbb{N}\}$ is a network at some point $x_\alpha$ in $X\big\}$, and equip with $M$ the subspace topology induced by the product topology of the product space $\prod\limits_{n\in\mathbb{N}} A_n$. Then $M$ is a metric space. The point $x_\alpha$ is unique in $X$ because $X$ is $T_2$. We define $f\colon M \to X$ by $f(\alpha) = x_\alpha$. Then

(a) $f$ is surjective. For each $x \in X$, there is a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $\alpha_{n_i} \in A_{n_i}$ and $\{P_{\alpha_{n_i}}\colon i \in \mathbb{N}\}$ is a network at $x$. For $n \in \mathbb{N} \setminus \{n_i\colon i \in \mathbb{N}\}$, take $\alpha_n \in A_n$ with $P_{\alpha_n} = X$. Let $\alpha = (\alpha_n)$. Then $\alpha \in M$ and $f(\alpha) = x$. Thus $f$ is surjective.

(b) $f$ is continuous. For each $\alpha = (\alpha_n) \in M$ we have $f(\alpha) = x_\alpha \in X$. If $U$ is an open neighborhood of $x_\alpha$ in $X$, then there is $n \in \mathbb{N}$ such that $x_\alpha \in P_{\alpha_n} \subset U$ because $\{P_{\alpha_n}\colon n \in \mathbb{N}\}$ is a network at $x_\alpha$ in $X$. Put $W = \{\beta \in M\colon$ the $n$-th coordinate of $\beta$ is $\alpha_n\}$, then $W$ is an open neighborhood of $\alpha$ in $M$, and $f(W) \subset P_{\alpha_n} \subset U$. Hence $f$ is continuous.

(c) $f$ is a $\sigma$-mapping. For $n \in \mathbb{N}$, $\alpha_n \in A_n$, put $V(\alpha_1, \ldots, \alpha_n) = \{\alpha \in M\colon$ if $i \leqslant n$, then the $i$-th coordinate of $\alpha$ is $\alpha_i\}$, $\mathscr{B} = \{V(\alpha_1, \ldots, \alpha_n)\colon \alpha_i \in A_i,\ i \leqslant n$ and $n \in \mathbb{N}\}$. Then $\mathscr{B}$ is a base for $M$. It suffices to show that $f(V(\alpha_1, \ldots, \alpha_n)) = \bigcap\limits_{i\leqslant n} P_{\alpha_i}$. In fact, by the definition of $f$, $f(V(\alpha_1, \ldots, \alpha_n)) \subset P_{\alpha_i}$ for each $i \leqslant n$, and thus $f(V(\alpha_1, \ldots, \alpha_n)) \subset \bigcap\limits_{i\leqslant n} P_{\alpha_i}$. Conversely, since $f$ is surjective, for each $x \in \bigcap\limits_{i\leqslant n} P_{\alpha_i}$ there is $\beta = (\beta_j) \in M$ with $f(\beta) = x$. For $j \in \mathbb{N}$ we have $P_{\beta_j} \in \mathscr{P}_j \subset \mathscr{P}_{j+n}$, thus there is $\alpha_{j+n} \in A_{j+n}$ with $P_{\alpha_{j+n}} = P_{\beta_j}$. Put $\alpha = (\alpha_j)$. Then $\alpha \in V(\alpha_1, \ldots, \alpha_n)$ and $f(\alpha) = x$, and thus $f(V(\alpha_1, \ldots, \alpha_n)) = \bigcap\limits_{i\leqslant n} P_{\alpha_i}$. Hence $f$ is a $\sigma$-mapping.

(d) $f$ is 1-sequence-covering. For each $x \in X$, there is $\beta = (\alpha_i) \in M$ with $\beta \in f^{-1}(x)$. From the fact above, we have $f(V(\alpha_1, \ldots, \alpha_n)) = \bigcap_{i \leqslant n} P_{\alpha_i}$. For a convergent sequence $\{x_j\}$ of $X$ with $x_j \to x$, since $f(V(\alpha_1, \ldots, \alpha_n))$ is a sequential neighborhood of $x$ in $X$, there exists $i(n) \in \mathbb{N}$ such that if $i \geqslant i(n)$, then $x_i \in f(V(\alpha_1, \ldots, \alpha_n))$. Thus $f^{-1}(x_i) \cap V(\alpha_1, \ldots, \alpha_n) \neq \emptyset$. We may assume $1 < i(n) < i(n+1)$. For each $j \in \mathbb{N}$, if $j < i(1)$, we take $\beta_j \in f^{-1}(x_j)$; if $i(n) \leqslant j < i(n+1)$, we take $\beta_j \in f^{-1}(x_j) \cap V(\alpha_1, \ldots, \alpha_n)$ for $n \in \mathbb{N}$. Then it is easy to show that the sequence $\{\beta_j\}$ converges to $\beta$ in $M$. Hence $f$ is 1-sequence-covering.

From (a)–(d) above, $X$ is a 1-sequence-covering and $\sigma$-image of a metric space. □

By virtue of Definition 1.2 it is easy to check the following lemma (or see [3]).

**Lemma 2.2.** *Assume $\mathscr{P}$ is a cover of $X$.*

(1) *If $\mathscr{P}$ is a weak base for $X$, then $\mathscr{P}$ is a sequential neighborhood network for $X$.*
(2) *If $\mathscr{P}$ is a sequential neighborhood network of a sequential space, then $\mathscr{P}$ is a weak base for $X$.*

**Lemma 2.3** [2]. *Every $g$-first countable space is a sequential space.*

**Lemma 2.3** [5]. *Assume $f\colon X \to Y$ is a sequentially quotient mapping. If $Y$ is a sequential space, then f is a quotient mapping.*

**Theorem 2.4.** *The following conditions are equivalent for a space $X$.*

(1) *$X$ is a $g$-metrizable space.*
(2) *$X$ is a 1-sequence-covering and quotient $\sigma$-image of a metric space.*

Proof. (1) ⇒ (2). Suppose $X$ is a $g$-metrizable space, then $X$ has a $\sigma$-locally finite weak base. By Lemma 2.2 and Lemma 2.3, $X$ is a sequential space, and $X$ has a $\sigma$-locally finite sequential neighborhood network. By Lemma 2.1, $X$ is a 1-sequence-covering and $\sigma$-image of a metric space. In view of Lemma 2.3, the 1-sequence-covering mapping is a quotient mapping.

(2) ⇒ (1). Suppose $X$ is a 1-sequence-covering and quotient $\sigma$-image of a metric space. Then $X$ is a sequential space because $f$ is a quotient mapping. By Lemma 2.1, $X$ has a $\sigma$-locally finite sequential neighborhood network $\mathscr{P}$. In view of Lemma 2.2, $\mathscr{P}$ is $\sigma$-locally finite weak base for $X$. Hence $X$ is a $g$-metrizable space. □

By Theorem 14 in [8], we have

**Corollary 2.5.** *Every 1-sequence-covering and pseudo-open $\sigma$-mapping preserves metrizability.*

*References*

[1] *P. Alexandroff*: On some results concerning topological spaces and their continuous mappings. In: Proc. Symp. Gen. Top. (Prague, 1961). 1961, pp. 41–54.
[2] *F. Siwiec*: On defining a space by a weak base. Pacific J. Math. *52* (1974), 233–245.
[3] *Shou Lin*: On sequence-covering $s$-mappings. Adv. Math. (China) *25* (1996), 548–551.
[4] *Shou Lin*: $\sigma$-mappings and Alexandroff's problems. To appear.
[5] *J. R. Boone and F. Siwiec*: Sequentially quotient mappings. Czechoslovak Math. J. *26* (1976), 174–182.
[6] *R. Engelking*: General Topology. Polish Scientific Publishers, Warszawa, 1977.
[7] *A. V. Arhangel'skii*: Mappings and spaces. Russian Math. Surveys *21* (1966), 115–162.
[8] *Y. Tanaka*: $\sigma$-hereditarily closure-preserving $k$-networks and $g$-metrizability. Proc. Amer. Math. Soc. *112* (1991), 283–290.



*Author's address*: Department of Mathematics, Shantou University, Shantou 515063, Guangzhou, P. R. China; currently: Department of Mathematics, Zhangzhou Teachers College, Zhangzhou 363000, Fujian, P. R. China, e-mail: jinjinli@fjzs.edu.cn.