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ON HARPERS' RESULT CONCERNING THE BANDWIDTHS OF GRAPHS

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Abstract. In this paper, we improve the result by Harper on the lower bound of the bandwidth of connected graphs. In addition, we prove that considerating the interior boundary and the exterior boundary when estimating the bandwidth of connected graphs gives the same results.

Keywords: graphs, bandwidth, interior boundary, exterior boundary

MSC 2000: 05C78

1. Introduction

A classical optimization problem is to label the vertices of a graph with distinct integers so that the maximum difference between labels on adjacent vertices is minimized. In other words, the bandwidth problem for a graph $G=(V,E)$ is to label its n vertices v_i with distinct integers $f(v_i)$ so that the quantity $\max\{|f(v_i)-f(v_j)|\colon v_iv_j\in E\}$ is minimized. The corresponding problem for its adjacent matrix $M=(m_{ij})$ is to find a permutation of vertices such that the quantity $\max\{|i-j|\colon m_{ij}\neq 0\}$ is minimized. These considerations arise from computations with sparse symmetric matrices, where operations run faster when the matrix is permuted so that all nonzero entries lie near the diagonal.

The matrix bandwidth problem seems to originate in the 1950s when structural engineers first analyzed steel frameworks by computer manipulation of their structural matrices, while the bandwidth problem for graphs originated independently by L. H. Harper and A. W. Hales at the Jet Propulsion Laboratory in Pasadena in 1962. It minimized the maximum absolute error and the average absolute error in a 6-bit picture code represented by edge differences in a hypercube whose vertices were words of the code. Afterwards, R. R. Korfhage [9] began to work on the graph bandwidth problems and F. Harary [6] announced the problem at a conference in Prague. Early results on the bandwidth are surveyed in [2] and [4].

Since the mid-sixties, there has been strong interest in the bandwidth problem for graphs, with a steady growth of the theory developing alongside with the continuing search for a better bandwidth minimization algorithm. See, for example, [3], [5], [8] and [11] for details.

2. Basic terminology

Given a graph $G=(V,E)$, the bandwidth of G is defined by

$$B(G)=\min_{f}\max_{uv\in E}|f(u)-f(v)|$$

where $f\colon\ V\to\{1,2,\ldots,|V|\}$ is a bijection, which is called a labeling of G. A labeling attaining this minimum value is called an optimal labeling.

For a subset $S\subset V$, the interior and exterior boundaries of S are defined respectively as

$$\partial(S)=\{u\in S\colon\ \exists v\in V\setminus S \text{ such that } uv\in E\}$$

and

$$N(S)=\{u\in V\setminus S\colon\ \exists v\in S \text{ such that } uv\in E\}.$$

Suppose a labeling f is given. Let $u_i=f^{-1}(i)$, $1\leqslant i\leqslant n$, and define

$$S_k(f)=\{u_1,u_2,\ldots,u_k\}=f^{-1}(\{1,2,\ldots,k\}).$$

3. Results and theorems

Harper [7] proved the following theorem:

Theorem A. *For any connected graph G of order n,*

$$B(G)\geqslant\max_{1\leqslant k\leqslant n}\min\{|\partial(S)|\colon\ S\subset V(G) \text{ and } |S|=k\}.$$

We improve the above result to the following theorem:

Theorem 1. *For any connected graph G of order n,*

$$B(G) \geqslant \min_f \max_{1\leqslant k\leqslant n} |\partial(S_k(f))|$$

where the minimum is taken over all labelings.

Consideration the exterior boundary of a graph, we have the following theorem:

Theorem 2. *Suppose $G = (V, E)$ is a connected graph of order n. Then*

$$\min_f \max_{1\leqslant k\leqslant n} |N(S_k(f))| = \min_f \max_{1\leqslant k\leqslant n} |\partial(S_k(f))|.$$

Based on Theorem 1 and Theorem 2, we have the following corollary:

Corollary 1. *For any connected graph G of order n,*

$$B(G) \geqslant \min_f \max_{1\leqslant k\leqslant n} |N(S_k(f))|$$

where the minimum is taken over all labelings.

4. Proofs of the theorems

Proof of Theorem 1. Given a labeling f and a fixed integer $k < n$. Denote

$$S_k(f) = \{v \in V\colon\ f(v) \leqslant k\}$$

and choose a vertex $x \in \partial(S_k(f))$ such that

$$f(x) = \min\{f(v)\colon\ v \in \partial(S_k(f))\}.$$

Then

$$1 \leqslant f(x) \leqslant k - |\partial(S_k(f))| + 1.$$

The inequality $f(y) \geqslant k+1$ holds for any $y \in N(S_k(f))$. Therefore for $k < n$ and arbitrary $y \in N(S_k(f))$ we obtain

$$\begin{aligned} B(G, f) = {} & \max\{|f(v_i) - f(v_j)|\colon\ (v_i, v_j) \in E\} \geqslant |f(y) - f(x)| \\ & \geqslant |k + 1 - (k - |\partial(S_k(f))| + 1)| \geqslant |\partial(S_k(f))|. \end{aligned}$$

For $k = n$, $|\partial(S_k(f))| = 0$. Hence $B(G, f) \geqslant \max\limits_{1\leqslant k\leqslant n} |\partial(S_k(f))|$. In conclusion,

$$B(G) \geqslant \min_f \max_{1\leqslant k\leqslant n} |\partial(S_k(f))|.$$

□

P r o o f of Theorem 2. Arbitrarily choose a labeling f of G and consider the sets $S_k^*(f)$ which are defined by

$$S_k^*(f) = V \setminus S_k(f) \quad \text{for } k = 1, \ldots, n.$$

It is easy to check that

$$\partial(S_k(f)) = N(S_k^*(f)) \quad \text{for } k = 1, \ldots, n.$$

We define a labeling g on G by

$$S_k(g) = S_{n-k}^*(f) \quad \text{for } k = 1, \ldots, n-1$$

and

$$S_n(g) = V.$$

Since g depends uniquely on f and the labeling runs over all possible choices, we have

$$\max_{1 \leqslant k \leqslant n} |\partial(S_k(f))| = \max_{1 \leqslant k \leqslant n} |N(S_k^*(f))| = \max_{1 \leqslant k \leqslant n} |N(S_k(g))|$$

and hence

$$\min_f \max_{1 \leqslant k \leqslant n} |N(S_k(f))| = \min_f \max_{1 \leqslant k \leqslant n} |\partial(S_k(f))|.$$

The proof is completed. □

Remark. Corollary 1 follows directly from Theorem 1 and Theorem 2.

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