Czechoslovak Mathematical Journal

Ladislav Nebeský

On properties of a graph that depend on its distance function

Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 2, 445–456

Persistent URL: `http://dml.cz/dmlcz/127902`

ON PROPERTIES OF A GRAPH THAT DEPEND ON ITS DISTANCE FUNCTION

Ladislav Nebeský, Praha

(Received October 2, 2001)

Abstract. If G is a connected graph with distance function d, then by a *step* in G is meant an ordered triple (u, x, v) of vertices of G such that $d(u, x) = 1$ and $d(u, v) = d(x, v) + 1$. A characterization of the set of all steps in a connected graph was published by the present author in 1997. In Section 1 of this paper, a new and shorter proof of that characterization is presented. A stronger result for a certain type of connected graphs is proved in Section 2.

Keywords: connected graphs, distance, steps, geodetically smooth graphs

MSC 2000: 05C12, 05C75

0. Introduction

The letters f–n will be reserved for denoting integers. All graphs considered here are finite undirected (without loops or multiple edges). For the terminology of graph theory, see the book [2]. Let G be a connected graph, and let d be its distance function. If $u, v \in V(G)$, α is an $u - v$ path of G and the length of α equals to $d(u, v)$, then we say that α is a geodesic in G.

By a *signpost system* we will mean an ordered pair (W, T), where W is a finite nonempty set, $T \subseteq W \times W \times W$, and the following axioms hold:

(Ax. 1) if $(u, x, v) \in T$, then $(x, u, u) \in T$ for all $u, v, x \in W$;
(Ax. 2) if $(u, x, v) \in T$, then $(x, u, v) \notin T$ for all $u, v, x \in W$;
(Ax. 3) if $u \neq v$, then there exists $z \in W$ such that $(u, z, v) \in T$ for all $u, v \in T$.

The notion of a signpost system was introduced in [5], but in a slightly different sense: (W, T) is a signpost system in our terminology if and only if T is a signpost system on W in the terminology of [5].

Research supported by Grant Agency of the Czech Republic, grant No. 401/01/0218.

Let (W, T) be a signpost system. By (Ax. 1),

$$(u, v, v) \in T \text{ if and only if } (v, u, u) \in T$$

for all $u, v \in T$. Combining (Ax. 1) and (Ax. 2), we see that

$$\text{(1)} \qquad \text{if } (u, x, v) \in T, \text{ then } v \neq u \neq x \text{ for all } u, v, x \in T.$$

By the *underlying graph* of (W, T) we mean the graph G defined as follows: $V(G) = W$ and

$$u \text{ and } v \text{ are adjacent in } G \text{ if and only if } (u, v, v) \in T$$

for all $u, v \in W$.

Let G be a connected graph, and let d denote its distance function. Following [6], by a *step* in G we mean an ordered triple (u, x, v), where $u, v, x \in V(G)$, $d(u, x) = 1$ and $d(u, v) = d(x, v) + 1$. If S denotes the set of all steps in G, then the ordered pair $(V(G), S)$ is called the *step system of* G. The step system of G is a useful instrument for studying those properties of G that depend on its distance function.

The next proposition will be used in Section 2.

Proposition 1. *Let G be a connected graph. Then G is biparite if and only if*

$$\textit{exactly one of the ordered triples } (u, x, v) \textit{ and } (x, u, v) \textit{ is a step in } G$$

for all $u, v, x \in V(G)$ such that $ux \in E(G)$.

Proof. Let d denote the distance function of G. Consider arbitrary $u, v, x \in V(G)$ such that $ux \in E(G)$. Then $d(u, v) < d(x, v) + 1$ or $d(x, v) < d(u, v) + 1$. Therefore, at most one of the ordered triples (u, x, v) and (x, u, v) is a step in G.

It is easy to see that G contains an odd cycle if and only if there exist $u_0, v_0, x_0 \in V(G)$ such that $u_0x_0 \in E(G)$ and neither (u_0, x_0, v_0) nor (x_0, u_0, v_0) is a step in G. □

Moreover, by a *step system* we mean an ordered pair (W, T) such that (W, T) is the step system of a connected graph. It is easy to see that every step system is a signpost system.

Proposition 2. *Let (W,T) be the step system of a connected graph G_1, and let G_2 be the underlying graph of (W,T). Then $G_1 = G_2$.*

P r o o f is obvious. □

Step systems were characterized by the present author in [6]. Using the notion of a signpost system, the characterization proved there can be described as follows:

Let (W,T) be a signpost system. Then (W,T) is a step system if and only if

(2) the underlying graph of (W,T) is connected

and a set **Ax** of axioms hold; **Ax** is a finite nonempty set of axioms that—similarly as (Ax. 1), (Ax. 2) and (Ax. 3)—can be reformulated in a language of the first-order logic.

Similar characterization of step systems was proved in [8] but the set of axioms was weaker there. Moreover, the correspondence between the characterization of a step system of a connected graph G and a characterization of the set of all geodesics of G was studied in [7].

The proofs of characterizations of step systems presented in [6] and [8] are long and complicated. In Section 1 of the present paper, a new, shorter and simpler proof of a characterization of step system will be given. The set of axioms used here will be weaker than in [8] (and thus than in [6]).

In the characterizations of step systems given in [6], [8] and here, the assumption that W be finite is combined with the assumption (2) and a finite set of axioms that can be reformulated in a language of the first-order logic. As was shown in [6] and [8], the assumption (2) cannot be omitted. The question whether the assumption (2) could be replaced by a finite nonempty set of axioms formulated in a language of the first-order logic is still open.

On the other hand, there exists a "rich" class **C** of connected graphs such that a theorem of the following structure can be proved:

Let (W,T) be a sigpost system. Then (W,T) is the step system of a graph belonging to the class **C** if and only if the set $\mathbf{Ax_C}$ of axioms holds; $\mathbf{Ax_C}$ is a finite set of axioms formulated in a language of the first-order logic.

As was shown by the present author in [9], the class of all trees has this property. As follows from the result of Mulder and Nebeský [5], the class of median graphs and the class of modular graphs have this property, too. (Note that every tree is a median graph, and every median graph is a modular one). In Section 2 of the present paper we will prove a theorem on this type for another class of graphs, a class involving median graphs, cycles and complete graphs.

1. A characterization of step systems

Let (W, T) be a step system. It is not difficult to show that it satisfies the following axioms:

(Ax. 4) if $(u, x, v), (v, y, x) \in T$, then $(v, y, u) \in T$ for all $u, v, x, y \in W$;
(Ax. 5) if $(u, x, v), (v, y, x) \in T$, then $(u, x, y) \in T$ for all $u, v, x, y \in W$;
(Ax. 6) if $(u, v, x), (v, u, y), (x, y, u) \in T$, then $(y, x, v) \in T$ for all $u, v, x, y \in W$;
(Ax. 7) if $(u, x, v), (v, y, y) \in T$, then $(v, y, u) \in T$ or $(y, v, x) \in T$ or $(u, x, y) \in T$ for all $u, v, x, y \in W$.

Verifications of (Ax. 4)–(Ax. 6) and of a stronger version of (Ax. 7) can be found in [6] on pages 154–155.

Remark 1. Let (W, T) be a signpost system satisfying (Ax. 6), let $u, v, x \in W$, and let $(u, x, v), (v, u, u) \in T$. By (Ax. 1), $(x, u, u) \in T$. As follows from (Ax. 6), $(u, v, x) \in T$.

We will prove that if a signpost system (W, T) satisfies (Ax. 4)–(Ax. 7) and its underlying graph is connected, then (W, T) is a step system. But first, we will need one definition, two lemmas and three corollaries; the lemmas were proved in [7].

Let (W, T) be a signpost system. By a *process* in (W, T) (in short: a process) we mean a sequence

$$(u_0, \dots, u_k), \tag{3}$$

where $k \geqslant 1, u_0, u_1, \dots, u_k \in W$ and

$$(u_i, u_{i+1}, u_k) \in T \text{ for each } i, \quad 0 \leqslant i < k.$$

If (3) is a process, $u_0 = u$ and $v = u_k$, then we say that (3) is an $u - v$ process. The notion of a process was introduced in [7]. Let $k \geqslant 2$. As immediately follows from the definition, (3) is a process in (W, T) if and only if $(u_0, u_1, u_k) \in T$ and $(u_1, \dots, u_k)$ a process in (T, W). By virtue of (Ax. 1) and (1), every process in (W, T) is a walk in the underlying graph of (W, T).

Lemma 1. *Let* (W, T) *be a signpost system satisfying* (Ax. 4) *and* (Ax. 5), *and let* $u_0, \dots, u_k \in W$, *where* $k \geqslant 1$. *If* $(u_0, \dots, u_k)$ *is a process in* (W, T), *then* $(u_k, \dots, u_0)$ *is also a process in* (W, T).

P r o o f. See the proof of Lemma 2 in [7]. □

Corollary 1. *Let* (W,T) *be a signpost system satisfying* (Ax. 4) *and* (Ax. 5), *let* $(u_0,\dots,u_k)$ *be a process in* (W,T), *and let* $0 \leqslant i < j \leqslant k$. *Then* $(u_i,u_{i+1},\dots,u_j)$ *and* $(u_j,u_{j-1},\dots,u_i)$ *are also processes in* (W,T).

Proof. By Lemma 1, $(u_k,\dots,u_0)$, $(u_j,\dots,u_0)$, $(u_0,\dots,u_j)$, $(u_i,\dots,u_j)$ and $(u_j,\dots,u_i)$ are processes in (W,T). □

Corollary 2. *Let* (W,T) *be a signpost system satisfying* (Ax. 4) *and* (Ax. 5), *and let* G *denote the underlying graph of* (W,T). *Then every process in* (W,T) *is a nontrivial path in* G.

Proof. Combining Corollary 1 and (1), we get the result. □

Corollary 3. *Let* (W,T) *be a signpost system satisfying* (Ax. 4) *and* (Ax. 5), *and let* $u_0,\dots,u_k \in W$, *where* $k \geqslant 2$. *If* $(u_0,\dots,u_{k-1})$ *is a process in* (W,T) *and* $(u_k,u_{k-1},u_0) \in T$, *then* $(u_0,\dots,u_{k-1},u_k)$ *is a process in* (W,T).

Proof. By Lemma 1, $(u_{k-1},\dots,u_0)$ is a process in (W,T). Since $(u_k,u_{k-1},u_0) \in T$, we see that $(u_k,u_{k-1},\dots,u_0)$ is a process in (W,T). Lemma 1 implies that $(u_0,\dots,u_{k-1},u_k)$ is a process in (W,T), too. □

Lemma 2. *Let* (W,T) *be a signpost system satisfying* (Ax. 4), (Ax. 5) *and* (Ax. 7), *and let the underlying graph of* (W,T) *be connected. Then for every distinct* $u,v \in W$ *there exists an* $u-v$ *process in* (W,T).

Proof. See the proof of Lemma 5 in [7]. □

The following theorem improves both Theorem 1 in [6] and Theorem 3 in [8]. But the main intention for presenting it here is its new proof, which is shorter and simpler than the proofs of the above-mentioned theorems in [6] and [8].

Let (W,R) be a signpost system, and let G denote its underlying graph. Assume that G is connected. Let d denote the distance function of G. Then for every $j \geqslant 0$, we denote

$$R_j = \{(u,x,v) \in R;\ d(u,v) = j\}.$$

This definition will be applied to two signpost systems considered in the proof of the following theorem.

Theorem 1. *Let* (W,T) *be a signpost system. Then* (W,T) *is a step system if and only if its underlying graph is connected and* (W,T) *satisfies* (Ax. 4)–(Ax. 7).

Proof. Let $\mathbf{P}$ denote the set of all processes in (W,T), and G denote the underlying graph of (W,T).

Assume that (W, T) is a step system. By Proposition 2, (W, T) is the step system of G. Thus the underlying graph of (W, T) is connected. Moreover, (W, T) satisfies (Ax. 4)–(Ax. 7). (As was said above, verifications of (Ax. 4)–(Ax. 6) and of a stronger version of (Ax. 7) can be found in [6]).

Conversely, let G be connected, and let (W, T) satisfy axioms (Ax. 4)–(Ax. 7). We denote by d and S the distance function of G and the set of all steps in G respectively. We will prove that (W, T) is the step system of G. Since $V(G) = W$, it is sufficient to prove that $T = S$. Suppose, to the contrary, that $T \neq S$. Then there exists $n \geqslant 0$ such that $T_n \neq S_n$ and

$$T_f = S_f \quad \text{for all } f,\ 0 \leqslant f < n. \tag{4}$$

Obviously, $T_0 = \emptyset = S_0$. Thus $n \geqslant 1$. We distinguish two cases.

Case 1. Let $S_n - T_n \neq \emptyset$. As follows from the definition of the underlying graph, $n \geqslant 2$. Then there exist $u, v, x \in W$ such that $d(u, v) = n$ and $(u, x, v) \in S - T$. Clearly, there exist $v_0, \ldots, v_{n-1},\ v_n \in W$ such that $v_0 = v$, $v_{n-1} = x$, $v_n = u$ and $(v_n, v_{n-1}, \ldots, v_0)$ is a geodesic in G. Lemma 2 implies that there exist $u_0, \ldots, u_m \in W$ such that $m \geqslant 1$, $u_0 = u$, $u_m = v$ and $(u_0, \ldots, u_m)$ is an $u - v$ process in (W, T). By Corollary 2, $(u_0, \ldots, u_m)$ is a path in G. This means that $m \geqslant n$ and

$$d(u_j, v_j) \leqslant n \quad \text{for each } j,\ 0 \leqslant j \leqslant n. \tag{5}$$

Put $u_{-1} = v_{n-1}$. Since $u_{-1} = x$, we have $(u_0, u_{-1}, v_0) \notin T$. Define

$$\alpha_i = (u_i, \ldots, u_m = v_0, \ldots, v_i) \quad \text{for each } i,\ 0 \leqslant i \leqslant n.$$

We see that $d(u_0, v_0) = n$ and $\alpha_0 \in \mathbf{P}$. Since $(u_0, \ldots, u_m) \in \mathbf{P}$ and $m \geqslant n$, we have, by Corollary 1, $(u_n, \ldots, u_0) \in \mathbf{P}$. Hence $(u_n, u_{n-1}, v_n) \in T$. Recall that $(u_0, u_{-1}, v_0) \notin T$. There exists h, $0 \leqslant h \leqslant n-1$, such that

$$d(u_h, v_h) = n, \alpha_h \in \mathbf{P} \quad \text{and} \quad (u_h, u_{h-1}, v_h) \notin T, \tag{6}$$

and

$$d(u_{h+1}, v_{h+1}) \neq n \quad \text{or} \quad \alpha_{h+1} \notin \mathbf{P} \quad \text{or} \quad (u_{h+1}, u_h, v_{h+1}) \in T. \tag{7}$$

Since $\alpha_h \in \mathbf{P}$,

$$(u_h, u_{h+1}, v_h) \in T. \tag{8}$$

Since $d(u_h, v_h) = n$, $(u_h, \ldots, u_0 = v_n, \ldots, v_{h+1}, v_h)$ is a geodesic in G. Thus

$$d(u_h, v_{h+1}) = n - 1 \tag{9}$$

and $(u_h, u_{h-1}, v_{h+1}) \in S$. As follows from (4) and (9), $(u_h, u_{h-1}, v_{h+1}) \in T$. If $(v_h, v_{h+1}, u_h) \in T$, then, by (Ax. 4), $(u_h, u_{h-1}, v_h) \in T$, which contradicts (6). Hence

$$(v_h, v_{h+1}, u_h) \notin T. \tag{10}$$

Let $(u_h, u_{h+1}, v_{h+1}) \in T$. By (4) and (9), $(u_h, u_{h+1}, v_{h+1}) \in S$. Thus $d(u_{h+1}, v_{h+1}) = n - 2$. By (6), $d(u_h, v_h) = n$. This implies that $d(u_{h+1}, v_h) = n - 1$ and $(v_h, v_{h+1}, u_{h+1}) \in S$. By (4), $(v_h, v_{h+1}, u_{h+1}) \in T$. Combining (8) with (Ax. 4), we get $(v_h, v_{h+1}, u_h) \in T$, which contradicts (10). Thus $(u_h, u_{h+1}, v_{h+1}) \notin T$. Obviously, $(v_h, v_{h+1}, v_{h+1}) \in T$. Combining (8) and (10) with (Ax. 7), we get

$$(v_{h+1}, v_h, u_{h+1}) \in T. \tag{11}$$

Let $d(u_{h+1}, v_{h+1}) < n$. By (4) and (11), $(v_{h+1}, v_h, u_{h+1}) \in S$. Therefore, $d(u_{h+1}, v_h) < n - 1$. This means that $d(u_h, v_h) < n$, which contradicts (6). Thus, by virtue (5), we get $d(u_{h+1}, v_{h+1}) = n$.

Assume that $(u_{h+1}, u_h, v_{h+1}) \in T$. Combining (8), (11) and (Ax. 6), we get $(v_h, v_{h+1}, u_h) \in T$, which contradicts (10). Thus $(u_{h+1}, u_h, v_{h+1}) \notin T$.

By (6), $\alpha_h \in \mathbf{P}$. Hence $(u_{h+1}, \ldots, u_m = v_0, \ldots, v_h) \in \mathbf{P}$. Combining (11) with Corollary 3, we get $\alpha_{h+1} = (u_{h+1}, \ldots, u_m = v_0, \ldots, v_h, v_{h+1}) \in \mathbf{P}$. Thus (7) does not hold, which is a contradiction.

Case 2. Let $S_n \subseteq T_n$. Then $T_n - S_n \neq \emptyset$. There exist $u, v, x \in W$ such that $d(u, v) = n$ and $(u, x, v) \in T - S$. If $x = v$, then $n = 1$ and $(u, x, v) \in S$; a contradiction. Thus $x \neq v$. Since G is connected, Lemma 2 implies that there exist $u_1, \ldots, u_m \in W$ such that $m \geqslant 2$, $u_1 = x$, $u_m = v$ and $(u_1, \ldots, u_m) \in \mathbf{P}$. Put $u_0 = u$. Since $(u_0, u_1, u_m) \in T$, we get $(u_0, u_1, \ldots, u_m) \in \mathbf{P}$. Clearly, there exist $v_0, \ldots, v_n$ such that $v_0 = v$, $v_n = u$ and $(v_n, \ldots, v_0)$ is a geodesic in G. Thus

$$d(v_0, v_j) = j \quad \text{for each } j,\ 0 \leqslant j \leqslant n. \tag{12}$$

It is obvious that (5) holds. Clearly, $m > n$. If $m = n$, then $(u_0, u_1, \ldots, u_m)$ is a geodesic in G and thus $(u, x, v) \in S$; a contradiction. Hence, $m > n$.

Define α_i, $0 \leqslant i \leqslant n$, in the same way as in Case 1. Assume that $\alpha_n \in \mathbf{P}$. Then $(u_n, u_{n+1}, v_n) \in T$. Recall that $u_0 = v_n$. By Corollary 1, $(u_{n+1}, u_n, \ldots, u_0) \in \mathbf{P}$. We get $(u_{n+1}, u_n, v_n) \in T$, which contradicts (Ax. 2). Thus $\alpha_n \notin \mathbf{P}$. Simultaneously we see that $d(u_0, v_0) = n$ and $\alpha_0 \in \mathbf{P}$. There exists h, $0 \leqslant h \leqslant n - 1$, such that

$$d(u_h, v_h) = n \quad \text{and} \quad \alpha_h \in \mathbf{P}, \tag{13}$$

and

$$d(u_{h+1}, v_{h+1}) \neq n \quad \text{or} \quad \alpha_{h+1} \notin \mathbf{P}. \tag{14}$$

Similarly as in Case 1, we can show that (8) and (9) hold.

Obviously, $h \leqslant m-2$. Consider an arbitrary f, $h \leqslant f \leqslant m-2$. Assume that $d(u_{f+1}, v_h) < n-(f-h)$. As follows from (13), $(u_{f+1}, u_{f+2}, v_h) \in T$. By (4), $(u_{f+1}, u_{f+2}, v_h) \in S$ and thus $d(u_{f+2}, v_h) < n-(f+1-h)$. This means that if $d(u_{h+1}, v_h) < n$, then $d(u_m, v_h) < n-((m-1)-h) = (n-(m-1))+h \leqslant h$. Since $u_m = v_0$, we have $d(v_0, v_h) < h$, which contradicts (12). Thus

$$d(u_{h+1}, v_h) \geqslant n. \tag{15}$$

Let $d(u_{h+1}, v_{h+1}) < n$. If $n = 1$, then $h = 0$ and therefore, $x = u$; a contradiction. Thus $n \geqslant 2$. It follows from (15) that $d(u_{h+1}, v_h) = n$ and

$$d(u_{h+1}, v_{h+1}) = n-1.$$

Thus $(v_h, v_{h+1}, u_{h+1}) \in S$. Since $S_n \subseteq T_n$, we get $(v_h, v_{h+1}, u_{h+1}) \in T$. Combining (8) and (Ax. 5), we get $(u_h, u_{h+1}, v_{h+1}) \in T$. Combining (4) and (9), we get $(u_h, u_{h+1}, v_{h+1}) \in S$; therefore $d(u_{h+1}, v_{h+1}) = n-2$; a contradiction. As follows from (5), $d(u_{h+1}, v_{h+1}) = n$.

By virtue of (13), $d(u_h, v_h) = n$ and therefore, $(v_h, v_{h+1}, u_h) \in S$. Since $d(u_{h+1}, v_{h+1}) = n$, we have $(u_{h+1}, u_h, v_{h+1}) \in S$. Since $S_n \subseteq T_n$, we have $(v_h, v_{h+1}, u_h), (u_{h+1}, u_h, v_{h+1}) \in T$. Combining (8) with (Ax. 6), we see that (11) holds (cf. Remark 1 if $n = 1$).

Using (11), similarly as in Case 1 we obtain that $\alpha_{h+1} \in \mathbf{P}$, which contradicts (14).

Thus $T = S$, which completes the proof. □

2. Geodetically smooth graphs

We will say that a graph G is *geodetically smooth*—or, in short, *smooth*—if G is connected and

> if (u, x, v) and (v, y, z) are steps in G and (u, x, y) is not a step in G,
> then (u, x, z) is not a step in G

for all $u, v, x, y, z \in V(G)$.

It is easy to see that all cycles and all complete graphs are smooth.

Proposition 3. *Let G be a connected graph. Then G is smooth if and only if each block of G is smooth.*

Proof. Obviously, if G is smooth, then each block of G is also smooth.

Conversely, let each block of G be smooth. Let i denote the number of blocks of G. We will prove that G is smooth. We proceed by induction on i. The case when $i = 1$ is obvious. Let $i \geqslant 2$. Then there exist induced subgraphs G_1 and G_2 of G and a cut-vertex t of G such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = \{t\}$. By the induction hypothesis, G_1 and G_2 are smooth.

Consider arbitrary $u, v, x, y, z \in V(G)$ such that (u, x, v) and (v, y, z) are steps in G and (u, x, y) is not a step in G. It is sufficient to prove that (u, x, z) is not a step in G.

Clearly, $ux, vy \in E(G)$. Assume that there exists distinct $f, g \in \{1, 2\}$ such that $ux \in E(G_f)$ and $vy \in E(G_g)$. Since (u, x, v) is a step in G, we see that (u, x, t) is also a step in G. This implies that (u, x, y) is a step in G; a contradiction. Thus there exists $h \in \{1, 2\}$ such that $ux, vy \in E(G_h)$. Without loss of generality, let $h = 1$. Then (u, x, v) is a step in G_1 and (u, x, y) is not a step in G_1. Recall that G_1 is smooth.

First, let $z \in V(G_1)$. Then (u, x, z) is a step in G_1. Since G_1 is smooth, we see that (u, x, z) is not a step in G_1. This implies that (u, x, z) is not a step in G. Now, let $z \notin V(G_1)$. Then (v, y, t) is a step in G and, therefore, in G_1. This implies that (u, x, t) is not a step in G_1. Therefore, (u, x, z) is not a step in G.

Thus G is smooth. □

We will show that the class of smooth graphs involves an interesting subclass: the class of median graphs. We say that a graph G is a *median* graph if it is connected and for every ordered triple (u, v, w) of vertices of G there exists exactly one vertex z of G such that

$$
\begin{aligned}
d(u, v) &= d(u, z) + d(z, v), \\
d(v, w) &= d(v, z) + d(z, w)
\end{aligned}
$$

and

$$d(u, w) = d(u, z) + d(z, w),$$

where d denotes the distance function of G. All trees and all n-cubes ($n \geqslant 0$) are median graphs. Median graphs and also the connections between them and other mathematial structures have been intensively studied; see the survey [3].

Proposition 4. *Every median graph is smooth.*

P r o o f. We use a result proved in the book [4]. In Section 3.2 of that book, 16 facts on median graphs are proved. Fact (8) stated on p. 80 of [4] can be reformulated as follows: Let G be a median graph, let $r, r', s, s' \in V(G)$, let (r, r', s') and (r', r, s) be steps in G, and let $ss' \in E(G)$. Then

(16) (s, s', t) is a step in G if and only if (r, r', t) is a step in G for every $t \in V(G)$.

Let G be a median graph. It is easy to see that G is bipartite. Consider arbitrary $u, v, x, y, z \in G$ such that (u, x, v) and (v, y, z) are steps in G and (u, x, y) is not a step in G. Since (v, y, z) is a step in G, we get $\{v, y\} \in E(G)$. Since (u, x, y) is not a step in G, Proposition 1 implies that (x, u, y) is a step in G. Since (v, y, z) is a step in G, (16) implies that (x, u, z) is also a step in G. By Proposition 1, (u, x, z) is not a step in G.

Thus G is smooth. $\square$

Let (W, T) be the step system of a smooth graph. It is obvious that (W, T) satisfies the following axiom:

(Ax. 8) if $(u, x, v), (v, y, z) \in T$ and $(u, x, y) \notin T$, then $(u, x, z) \notin T$ for all $u, v, x, y, z \in W$.

The next theorem is the main result of this section.

Theorem 2. *Let* (W, T) *be a signpost system, and let* G *denote its underlying graph. Then* G *is a smooth graph and* (W, T) *is the step system of* G *if and only if* (W, T) *satisfies* (Ax. 4)–(Ax. 8).

P r o o f. If G is smooth and (W, T) is the step system of G, then (W, T) satisfies (Ax. 8) and, by Theorem 1, also (Ax. 4)–(Ax. 7).

Conversely, let (W, T) satisfy (Ax. 4)–(Ax. 8). Let F be an arbitrary component of G. Define

$$R = \{(u, x, v) \in T;\ \ u, v, x \in V(H)\}.$$

It is obvious that $(V(F), R)$ satisfies (Ax. 1) and (Ax. 2). The definition of the underlying graph of a signpost system implies that $(V(F), R)$ satisfies also (Ax. 3). Thus $(V(F), R)$ is a signpost system. Clearly, F is its underlying graph. Moreover, it is obvious that $(V(F), R)$ satisfies (Ax. 4)–(Ax. 8). Since F is connected, Theorem 1 implies that $(V(F), R)$ is the step system of F. Since $(V(F), R)$ satisfies (Ax. 8), we see that F is smooth.

We wish to prove that F is identical with G. Suppose, to the contrary, that there exists $z \in V(G)$ such that $z \notin V(F)$. Consider a vertex $u \in V(F)$. Combining (Ax. 3), (1) and the definition of the underlying graph, we see that there exists an infinite sequence

$$u_0, u_1, u_2, \dots \text{ of vertices in } G$$

such that $u_0 = u$ and

$$(u_i, u_{i+1}, z) \in T \text{ for each } i \geqslant 0.$$

Clearly,

$$u_0u_1, u_1u_2, u_2u_3, \dots \text{ are edges of } F.$$

Since $z \notin V(F)$, (Ax. 2) implies that

$$u_0 \neq u_2 \neq u_4 \neq \dots$$

and

$$u_1 \neq u_3 \neq u_5 \neq \dots.$$

Since $V(F)$ is finite, there exist f and h, $0 \leqslant f \leqslant h - 3$, such that $u_h = u_f$ and the vertices $u_f, u_{f+1}, \dots, u_{g-1}$ are pairwise distinct. We see that the edges

$$u_fu_{f+1}, u_{f+1}u_{f+2}, \dots, u_{h-1}u_h$$

form a cycle of length $h - f$ in F.

Recall that $u_h = u_f$. Clearly, $(u_f, u_{f+1}, u_{f+1}) \in T$ and $(u_f, u_{f+1}, u_h) \notin T$. This means that there exists g, $f + 1 \leqslant g < h$, such that $(u_f, u_{f+1}, u_g) \in T$ and $(u_f, u_{f+1}, u_{g+1}) \notin T$. Simultaneously, $(u_f, u_{f+1}, z), (u_g, u_{g+1}, z) \in T$. This implies that (W, T) does not satisfy (Ax. 8), which is a contradiction.

Thus F is identical with G. We have $R = T$. This means that G is smooth and (W, T) is the step system of G. □

Corollary 4. *A signpost system (W, T) is the step system of a smooth graph if and only if it safisfies* (Ax. 4)–(Ax. 8).

Corollary 5. *A graph G is smooth if and only if G is the underlying graph of a signpost system that satisfies* (Ax. 4)–(Ax. 8).

Remark 2. If we replace "exacly one vertex z" by "at least one vertex z" in our definition of median graphs, we obtain a definition of modular graphs (but similarly as median graphs, modular graphs can be defined with the help of the interval function of a graph; see [1]). Results structurally similar to Theorem 2 were proved for median graphs and modular graphs in [5]. Three axioms were found there, say (Ax. a), (Ax. b) and (Ax. c), and the following results are proved: Let (W, T) be a signpost system, and let G be its underlying graph. Then (1) G is a modular graph and (W, T) is the step system of G if and only if (W, T) satisfies (Ax. 5), (Ax. a), and (Ax. b); and (2) G is a median graph and (W, T) is the step system of G if and only if (W, T) satisfies (Ax. 5), (Ax. a), (Ax. b) and (Ax. c). Note that (Ax. a), (Ax. b) and (Ax. c) can be—similarly as (Ax. 1)–(Ax. 8)—formulated in a language of the first-order logic.

References

[1] *H.-J. Bandelt and H. M. Mulder*: Pseudo-modular graphs. Discrete Math. *62* (1986), 245–260.

[2] *G. Chartrand and L. Lesniak*: Graphs & Digraphs. Third edition. Chapman & Hall, London, 1996.

[3] *S. Klavžar and H. M. Mulder*: Median graphs: characterizations, location theory and related structures. J. Combin. Math. Combin. Comput. *30* (1999), 103–127.

[4] *H. M. Mulder*: The interval function of a graph. Math. Centre Tracts 132. Math. Centre, Amsterdam, 1980.

[5] *H. M. Mulder and L. Nebeský*: Modular and median signpost systems and their underlying graphs. Discuss. Math. Graph Theory *23* (2003), 309–32444.

[6] *L. Nebeský*: Geodesics and steps in a connected graph. Czechoslovak Math. J. *47 (122)* (1997), 149–161.

[7] *L. Nebeský*: An axiomatic approach to metric properties of connected graphs. Czechoslovak Math. J. *50 (125)* (2000), 3–14.

[8] *L. Nebeský*: A theorem for an axiomatic approach to metric properties of graphs. Czechoslovak Math. J. *50 (125)* (2000), 121–133.

[9] *L. Nebeský*: A tree as a finite nonempty set with a binary operation. Math. Bohem. *125* (2000), 455–458.

Author's address: Univerzita Karlova v Praze, Filozofická fakulta, nám J. Palacha 2, 116 38 Praha 1, Czech Republic, e-mail: `Ladislav.Nebesky@ff.cuni.cz`.