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THE DUAL GROUP OF A DENSE SUBGROUP

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Abstract. Throughout this abstract, G is a topological Abelian group and $\widehat{G}$ is the space of continuous homomorphisms from G into the circle group $\mathbb{T}$ in the compact-open topology. A dense subgroup D of G is said to *determine* G if the (necessarily continuous) surjective isomorphism $\widehat{G} \twoheadrightarrow \widehat{D}$ given by $h \mapsto h\big|D$ is a homeomorphism, and G is *determined* if each dense subgroup of G determines G. The principal result in this area, obtained independently by L. Außenhofer and M. J. Chasco, is the following: Every metrizable group is determined. The authors offer several related results, including these.

1. There are (many) nonmetrizable, noncompact, determined groups.

2. If the dense subgroup D_i determines G_i with G_i compact, then $\bigoplus_i D_i$ determines $\prod_i G_i$. In particular, if each G_i is compact then $\bigoplus_i G_i$ determines $\prod_i G_i$.

3. Let G be a locally bounded group and let G^+ denote G with its Bohr topology. Then G is determined if and only if G^+ is determined.

4. Let $\mathrm{non}(\mathcal{N})$ be the least cardinal κ such that some $X \subseteq \mathbb{T}$ of cardinality κ has positive outer measure. No compact G with $w(G) \geqslant \mathrm{non}(\mathcal{N})$ is determined; thus if $\mathrm{non}(\mathcal{N}) = \aleph_1$ (in particular if CH holds), an infinite compact group G is determined if and only if $w(G) = \omega$.

Question. Is there in ZFC a cardinal κ such that a compact group G is determined if and only if $w(G) < \kappa$? Is $\kappa = \mathrm{non}(\mathcal{N})$? $\kappa = \aleph_1$?

Keywords: Bohr compactification, Bohr topology, character, character group, Außenhofer-Chasco Theorem, compact-open topology, dense subgroup, determined group, duality, metrizable group, reflexive group, reflective group

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This paper is based on the third chapter of the second-listed author's doctoral dissertation [42], written at Wesleyan University, Middletown, Conn., under the direction of the first-listed author.

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0. Terminology, notation and preliminaries

For X a set and κ a cardinal, we write $[X]^\kappa = \{A \subseteq X\colon\ |A| = \kappa\}$.

For every topological space $X = (X, \mathcal{T})$ considered in this paper, whether or not Hausdorff, we write

$$\mathcal{K}(X) := \{K \subseteq X\colon\ K \text{ is compact}\}.$$

All groups considered here, whether or not equipped with a topology, are Abelian groups written additively. The identity of a group G is denoted 0 or 0_G, and the torsion subgroup of G is denoted $\mathrm{tor}(G)$. The reals, rationals, and integers are denoted $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$, respectively, and the "unit circle" group $\mathbb{T}$ is the group $(-\frac{1}{2}, \frac{1}{2}]$ with addition $\bmod 1$. Except when we specify otherwise, these groups carry their usual metrizable topology.

The symbol $\mathbb{P}$ denotes the set of positive prime integers.

The set of homomorphisms $h\colon G \to \mathbb{T}$, a group under pointwise operation, is denoted $\mathrm{Hom}(G, \mathbb{T})$. For a subgroup A of $\mathrm{Hom}(G, \mathbb{T})$ we denote by $(G, \mathcal{T}_A)$ the group G with the topology induced by A. Evidently $(G, \mathcal{T}_A)$ is a Hausdorff topological group if and only if A separates points of G. The topology $\mathcal{T}_A$ is the coarsest topology on G for which the homomorphism $e_A\colon G \to \mathbb{T}^A$ given by $(e_A x)_h = h(x)$ ($x \in G$, $h \in A$) is continuous. When $G = (G, \mathcal{T})$ is a topological group, the set of $\mathcal{T}$-continuous functions in $\mathrm{Hom}(G, \mathbb{T})$ is a subgroup of $\mathrm{Hom}(G, \mathbb{T})$ denoted $\widehat{G}$ or $\widehat{(G, \mathcal{T})}$; in this case the topology $\mathcal{T}_{\widehat{G}}$ is the *Bohr topology* associated with $\mathcal{T}$, and $(G, \mathcal{T}_{\widehat{G}})$ is denoted G^+ or $(G, \mathcal{T})^+$. When $\widehat{(G, \mathcal{T})}$ separates points we say that G is a *maximally almost periodic* group and we write $G = (G, \mathcal{T}) \in \mathbf{MAP}$. Whether or not $(G, \mathcal{T}) \in \mathbf{MAP}$, the closure of $e[G]$ in $\mathbb{T}^{\widehat{G}}$, denoted $b(G)$ or $b(G, \mathcal{T})$, is the *Bohr compactification* of $(G, \mathcal{T})$.

It is useful to note that, with an inconsequential abuse of notation, for every $h \in \widehat{G}$ the projection $\pi_h\colon \mathbb{T}^{\widehat{G}} \to \mathbb{T} = \mathbb{T}_h$ satisfies $\pi_h\big|e[G] = h$. Thus every $h \in \widehat{G^+}$ "lifts continuously" to $b(G)$, and for $h \in \mathrm{Hom}(G, \mathbb{T})$ we have:

$$h \in \widehat{G^+} \text{ if and only if } h \circ e_{\widehat{G}} \in \widehat{(G, \mathcal{T})}.$$

The Bohr compactification $b(G)$ of a topological group G is characterized by the condition that each continuous homomorphism from G into a compact Hausdorff group extends continuously to a homomorphism from $b(G)$. From this and the uniform continuity of continuous homomorphisms it follows that if D is a dense subgroup of G then $b(D) = b(G)$. It is conventional to suppress mention of the function $e_{\widehat{G}}$ and to write simply $\widehat{G} = \widehat{G^+}$. When $G \in \mathbf{MAP}$ we write $G^+ \subseteq b(G) \subseteq \mathbb{T}^{\widehat{G}}$, the inclusions being both algebraic and topological.

A group G with its discrete topology is denoted G_d. For notational convenience, and following van Douwen [18], for every (Abelian) group G we write $G^\# = (G_d)^+ \subseteq b(G_d)$.

For remarks on the history and development of the Bohr topology and the Bohr compactification, both in the context of topological Abelian groups and in broader contexts, the reader might consult Heyer [29, V§4]. The paper [13] concerns topological groups of the form $(G, \mathcal{T}_A)$ for (point-separating) subgroups A of $\mathrm{Hom}(G, \mathbb{T})$. See also [7] and [45], [46].

A subset S of a topological group G is said to be *bounded* in G if for every nonempty open $V \subseteq G$ there is finite $F \subseteq G$ such that $S \subseteq F + V$; G is *locally bounded* (resp., *totally bounded*) if some nonempty open subset of G is bounded (resp., G itself is bounded). It is a theorem of Weil [62] that each locally bounded group G embeds as a dense topological subgroup of a locally compact group $W(G)$, unique in the obvious sense; the group $W(G)$ is compact if and only if G is totally bounded. We denote by **LCA** (resp., **LBA**) the class of locally compact (resp., locally bounded) Hausdorff Abelian groups. The relation $\mathbf{LCA} \subseteq \mathbf{MAP}$ is a well known consequence of the Gel'fand-Raĭkov Theorem (cf. [26, (22.17)]); since each subgroup $S \subseteq G \in \mathbf{MAP}$ clearly satisfies $S \in \mathbf{MAP}$, we have in fact the relations $\mathbf{LCA} \subseteq \mathbf{LBA} \subseteq \mathbf{MAP}$. In Section 5 we will consider certain noncomplete non-locally bounded groups.

From time to time we will invoke the following lemma.

0.1. Lemma. *Let S be a subgroup of $G \in \mathbf{LBA}$. Then*

(a) *S is* dual-embedded *in G in the sense that each $h \in \widehat{S}$ extends to an element of $\widehat{G}$;*

(b) *if $h \in \widehat{S}$ and $x \in G \backslash \overline{S}^G$, the extension $k \in \widehat{G}$ of h may be chosen so that $k(x) \neq 0$.*

Proof. Being uniformly continuous on $S \subseteq G \subseteq W(G)$, h extends continuously over $\overline{S}^{W(G)}$. From $x \in G \backslash \overline{S}^G$ follows $x \in W(G) \backslash \overline{S}^{W(G)}$, so (a) and (b) both follow from [26, (24.12)]. □

From Lemma 0.1 (a) it follows for each subgroup S of a group $G \in \mathbf{LBA}$ that the topology of S^+ coincides with the topology inherited by S from G^+. This observation validates the following notational convention. For $S \subseteq G \in \mathbf{LBA}$, S not necessarily a subgroup of G, we denote by S^+ the set S with the topology inherited from G^+. When G is discrete, so that $G^+ = G^\#$, we write $S^\#$ in place of S^+ when $S \subseteq G$.

We remark in passing that a closed subgroup S of an Abelian **MAP** group G can fail to be dual-embedded [31], [51], [4]. In this case the topology of S^+ and the topology inherited by S from G^+ necessarily differ. This explains why we use the symbol S^+ for (arbitrary) $S \subseteq G$ only when $G \in \mathbf{LBA}$, rather than in the broader setting $S \subseteq G \in \mathbf{MAP}$.

The following two nontrivial results are basic to our investigation.

0.2. Theorem (Glicksberg [24]). *Let $K \subseteq G \in \mathbf{LBA}$. Then $K \in \mathcal{K}(G)$ if and only if $K^+ \in \mathcal{K}(G^+)$. Hence if $K \in \mathcal{K}(G)$, then K and K^+ are homeomorphic.*

0.3. Theorem (Flor [20]. See also Reid [44]). *Let $G \in \mathbf{LBA}$ and let $x_n \to p \in b(G) = b(W(G))$ with each $x_n \in G^+ \subseteq (W(G))^+ \subseteq b(G)$. Then*

(a) *$p \in (W(G))^+$, and*

(b) *not only $x_n \to p$ in $(W(G))^+ \subseteq b(G)$ but also $x_n \to p$ in $W(G)$.*

0.4. Remarks. (a) From Theorems 0.2 and 0.3 it follows for $G \in \mathbf{LBA}$ that $b(G)$ is metrizable if and only if G is totally bounded and metrizable. For if $b(G) = b(W(G))$ is metrizable then $(W(G))^+$, being sequentially closed in $b(G)$, coincides with $b(G)$, so $(W(G))^+$ is compact and metrizable, hence $W(G)$ is compact and metrizable, i.e., G is totally bounded and metrizable. Conversely if G is totally bounded and metrizable then $W(G)$ is compact metric, so its continuous image $(W(G))^+$ is compact metric [19, (3.1.22)] and hence $(W(G))^+ = b(G)$ is metrizable.

(b) It is worth noting that an **LCA** group G may contain homeomorphic subspaces S_0 and S_1 such that S_0^+ and S_1^+ are not homeomorphic. For example, take $S_0 = (-1, +1) \subseteq S_1 = \mathbb{R} = G$. The identity function from G onto G^+ is a homeomorphism on each $K \in \mathcal{K}(G)$ and hence on each set with compact closure, so S_0^+ is metrizable, but according to Remark (a) above the space $G^+ = \mathbb{R}^+$ is not metrizable. In this connection see also [58, §3].

(c) Strictly speaking, the papers cited above in connection with Theorems 0.2 and 0.3 deal with groups $G \in \mathbf{LCA}$. Our modest generalization to the case $G \in \mathbf{LBA}$, which will be useful to us below, is justified by 0.2 and 0.3 as originally given and by these facts about $G \in \mathbf{LBA}$: (i) G is a (dense) topological subgroup of $W(G) \in \mathbf{LCA}$; (ii) G^+ is a (dense) topological subgroup of $(W(G))^+$; and (iii) $b(G) = b(W(G))$.

(d) In any space, a convergent sequence together with its limit point constitute a compact set. Thus Theorem 0.3 (b) is a consequence of 0.3 (a) and 0.2.

(e) Theorem 0.2 for G discrete was given by Leptin [39]. See [17, (3.4.3)] for a succinct proof of Theorem 0.2 for $G \in \mathbf{LCA}$, and see [58], [59] for an alternative approach and for generalizations in several directions.

In what follows, groups of the form $\widehat{G}$ will be given the *compact-open* topology. This is defined as usual: the family $\{U(K, \varepsilon)\colon\ K \in \mathcal{K}(G),\ \varepsilon > 0\}$ is a base at $0 \in \widehat{G}$, where for $A \subseteq G$ one writes

$$U(A, \varepsilon) = \{h \in \widehat{G}\colon\ x \in A \Rightarrow |h(x)| < \varepsilon\}.$$

We have noted already that for $G \in \mathbf{MAP}$ the groups $\widehat{G}$ and $\widehat{G^+}$ are identical; that is, $\widehat{G} = \widehat{G^+}$ as groups. Our principal interest in Theorem 0.2 is that for $G \in \mathbf{LBA}$ it gives a topological consequence, as follows.

0.5. Corollary. *Let $G \in \mathbf{LBA}$. Then $\widehat{G} = \widehat{G^+}$ as topological groups. That is, the compact-open topology on $\widehat{G}$ determined by $\mathcal{K}(G)$ coincides with the compact-open topology on $\widehat{G}$ determined by $\mathcal{K}(G^+)$.*

1. The groups $\widehat{G}$ for G metrizable

We have noted already that if D is a dense subgroup of an Abelian topological group $G = (G, \mathcal{T})$ then every $h \in \widehat{D}$, being not only continuous on D but indeed uniformly continuous, extends (uniquely) to an element of $\widehat{G}$; and of course, each $h \in \widehat{G}$ satisfies $h\big|D \in \widehat{D}$. Accordingly, abusing notation slightly, we have $\widehat{D} = \widehat{G}$ as groups. (The "abuse" derives from the fact that when the inclusion $D \subseteq G$ is proper, each $h \in \widehat{D}$ has domain$(h) = D$, while each $h \in \widehat{G}$ has domain$(h) = G \neq D$.) Since groups of the form $\widehat{G}$ carry the compact-open topology, it is natural to inquire whether the identity $\widehat{G} = \widehat{D}$ is topological as well as algebraic. Informally: do $\mathcal{K}(G)$ and $\mathcal{K}(D)$ induce the same topology on the set $\widehat{G} = \widehat{D}$? The question provokes this definition.

1.1. Definition. Let G be an Abelian topological group.

(a) Let D be a dense subgroup of G. Then D *determines* G (alternatively: G is *determined* by D) if $\widehat{G} = \widehat{D}$ as topological groups.

(b) G is *determined* if every dense subgroup of G determines G.

1.2. Remarks. (a) It is a theorem of Kaplan [34, (2.9)] (cf. Banasczyzk [5, (1.3)] and Raczkowski [42, §3.1] for alternative treatments, and Außenhofer [2] and [3, (3.4)] for a generalization) that for each G the family $\{U\big(K, \frac{1}{4}\big) \colon K \in \mathcal{K}(G)\big\}$ is basic at $0 \in \widehat{G}$. (For notational simplicity, henceforth we write $U(K) := U(K, \frac{1}{4}) \subseteq \widehat{G}$ for $K \in \mathcal{K}(G)$.) Thus the condition that a group G is determined by its dense subgroup D reduces to (i.e., is equivalent to) the condition that $\mathcal{K}(D)$ is cofinal in $\mathcal{K}(G)$ in the sense that for each $K \in \mathcal{K}(G)$ there is $E \in \mathcal{K}(D)$ such that $U(E) \subseteq U(K)$.

(b) Let D and S be dense subgroups of a topological group G such that $D \subseteq S \subseteq G$. Then since $\mathcal{K}(D) \subseteq \mathcal{K}(S) \subseteq \mathcal{K}(G)$ it is clear that D determines G if and only if D determines S and S determines G. In particular, a dense subgroup of a determined group is determined.

(c) Our focus here is principally on **LBA** groups. If D is a nondense subgroup of $G \in \mathbf{LBA}$, say with $x \in G\backslash\overline{D}^G$, then from Lemma 0.1 (b) the map $\widehat{G} \to \widehat{D}$ given by $h \mapsto h|D$ is not a bijection, indeed there is $h \in \widehat{G}$ such that $h \equiv 0$ on D and $|h(x)| > \frac{1}{4}$; clearly in this case no $K \in \mathcal{K}(D)$ can satisfy $U(K) \subseteq U(\{x\})$. This explains why we have chosen to define the relation "D determines G" only when D is a dense subgroup of G.

(d) Let D be a dense subgroup of $G \in$ **MAP**. Since $\mathcal{K}(D) \subseteq \mathcal{K}(G)$, the isomorphism from $\widehat{G}$ onto $\widehat{D}$ given by $h \mapsto h|D$ is continuous and the condition that D determines G is equivalent to the condition that this restriction map is an open function.

(e) The principal theorem in this corner of mathematics is the following result, obtained independently by Außenhofer [3, (4.3)] and Chasco [9]. This is the point of departure of the present inquiry.

1.3. Theorem. *Every metrizable, Abelian group is determined.*

1.4. Discussion. Is every topological group determined? Is every **MAP** group determined? Are there nonmetrizable, determined groups? Is every closed (or, open) subgroup of a determined group itself determined? Is the class of determined groups closed under passage to continuous homomorphisms? Continuous isomorphisms? The formation of products? These are some of the questions we address in this paper.

2. Determined groups: G vs. G^+

2.1. Lemma. *Let D be a subgroup of $G \in$ **LBA**. Then D is dense in G if and only if D^+ is dense in G^+.*

P r o o f. ($\Rightarrow$) The isomorphism $G \twoheadrightarrow G^+$ given by $x \mapsto x$ is continuous.

($\Leftarrow$) If D is not dense in G then by Lemma 0.1 (b) there is $h \in \widehat{G}$ such that $h|D \equiv 0$ and $h \neq 0 \in \widehat{G}$. Then $h \in \widehat{G^+}$, $h|D^+ \equiv 0$, and $h \neq 0 \in \widehat{G^+}$. □

2.2. Corollary. *Let D be a subgroup of $G \in$ **LBA**. Then D determines G if and only if D^+ determines G^+.*

P r o o f. In view of 1.2 (c) and Lemma 2.1, we assume that D and D^+ are dense in G and G^+ respectively. The required statement is now obvious from Theorem 0.2: Given $K \subseteq G$ and $E \subseteq D$ we have $K \in \mathcal{K}(G)$ if and only if $K^+ \in \mathcal{K}(G^+)$, and also $E \in \mathcal{K}(D)$ if and only if $E^+ \in \mathcal{K}(D^+)$; and clearly $U(E) \subseteq U(K) \subseteq \widehat{G}$ if and only if $U(E^+) \subseteq U(K^+) \subseteq \widehat{G^+}$. □

2.3. Theorem. *Let $G \in$ **LBA**. Then G is determined if and only if G^+ is determined.*

Proof. This is immediate from Lemma 2.1 and Corollary 2.2. $\square$

2.4. Theorem. *Let G be an* **LBA** *group such that G^+ determines $b(G)$. Then*
(a) *G is totally bounded (and hence $G = G^+$); and*
(b) *if also $G \in$* **LCA** *then G is compact (and hence $G = G^+ = b(G)$).*

Proof. (a) is equivalent to the condition that the **LCA** group $W(G)$ is compact. If this fails then $(W(G))^\frown$ is not discrete (cf. [26, (23.17)], so $((W(G))^+)^\frown$ is not discrete by Corollary 0.5. Thus $(W(G))^+$ does not determine $b(W(G))$, so its subgroup G^+ does not determine $b(W(G)) = b(G)$.

A totally bounded **LCA** group is compact. The bijection $G \twoheadrightarrow G^+$ given by $x \mapsto x$ is continuous from G onto the dense subgroup G^+ of $b(G)$, so $G = G^+ = b(G)$ if G is compact. $\square$

2.5. Corollary. *Let $G \in$* **LBA**. *Then $b(G)$ is determined if and only if $W(G)$ is compact and determined; in this case $W(G) = b(G)$.*

Proof. $(W(G))^+$ determines $b(W(G)) = b(G)$, so $W(G) = (W(G))^+ = b(W(G)) = b(G)$ by Theorem 2.4 (b) applied to the **LCA** group $W(G)$. $\square$

2.6. Corollary. *Let $G \in$* **LCA**. *If G is noncompact then G^+ does not determine $b(G)$ (and hence $b(G)$ is not determined).*

2.7. Remark. We put Corollary 2.2 into broader perspective. Let us say, following [58], [59], that a topological group G *respects compactness* if each $K \subseteq G$ satisfies $K \in \mathcal{K}(G)$ if and only if $K^+ \in \mathcal{K}(G^+)$; we say further that a subgroup H of a topological group G is *dual-closed* (in G) if for each $x \in G \backslash H$ there is $h \in \widehat{G}$ such that $h|H \equiv 0$ and $h(x) \neq 0$. An examination of the proof of Corollary 2.2 shows that if G is an Abelian topological group such that (i) G respects compactness and (ii) each closed subgroup of G is dual-closed, then a dense subgroup D of G determines G if and only if D^+ determines G^+. (Note in this connection that from (ii) it follows that G and G^+ share the same dense subgroups.) Now **LBA** groups satisfy (i) and (ii). Indeed, the product of (arbitrarily many) **LBA** groups, and each closed subgroup of such a product, satisfies (i) and (ii). (See [47, (2.1)] for (i), and Theorem 2 of [35] for (ii) in the **LCA** case which can be lifted without much trouble to the present case. Notice, however, as proved by Higasikawa [30], correcting a statement of Noble [41], that the product of two groups with (ii) may fail to satisfy (ii).) We now have the following result.

2.8. Theorem. *Let G be a closed subgroup of a product of* **LBA** *groups. Then a dense subgroup D of G determines G if and only D^+ determines G^+. Thus G is determined if and only if G^+ is determined.*

3. Determined groups: some examples

3.1. Theorem. *There are totally bounded, nonmetrizable, determined groups.*

Proof. Let G be an arbitrary determined **LBA** group such that G is not totally bounded. (Appealing to Theorem 1.3, one might choose $G \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.) That G^+ is as required follows from three facts: (a) G^+ is determined (Theorem 2.3); (b) a group with a dense metrizable subgroup is itself metrizable [8, Prop. IX, §2.1.1]; (c) $b(G)$ is not metrizable (Remark 0.4 (a)). □

3.2. Theorem. *A nondetermined group may have a dense, determined subgroup.*

Proof. Let G be a noncompact, determined **LCA** group. Then $b(G)$ is not determined (by Corollary 2.6), but its dense subgroup G^+ is determined by Theorem 2.3. □

3.3. Theorem. *The image of a nondetermined group under a continuous homomorphism may be determined.*

Proof. We show in Corollary 4.16 below that compact groups of weight $\geqslant \mathfrak{c}$ are nondetermined. Each such group maps by a continuous homomorphism onto either the group $\mathbb{T}$ or a group of the form $(\mathbb{Z}(p))^\omega$ ($p \in \mathbb{P}$) (cf. 4.15 below), and such groups are determined by Theorem 1.3. □

3.4. Discussion. Obviously an **LBA** group with no proper dense subgroup is vacuously determined. We mention three classes of such groups.

(i) Discrete groups.

(ii) Groups of the form $G^\# = (G_d)^+$. (It is well known [14, (2.1)] that every subgroup of such a group is closed.) For additional examples related to groups of the form $G^\#$, the reader may consult [16].

(iii) **LCA** groups of the type given by Rajagopalan and Subrahmanian [43]. Specifically, let $\kappa \geqslant \omega$, fix $p \in \mathbb{P}$, and topologize the group $G := (\mathbb{Z}(p^\infty))^\kappa$ so that its subgroup $H := (\mathbb{Z}(p))^\kappa$ in its usual compact topology is open-and-closed in G. (In detail, retaining always our additive notation: A set $U \subseteq G$ is open if and only if $(U - x) \cap H$ is open in H for each $x \in U$.) If D is a dense subgroup of G and $x \in G$ then since G is divisible there is $y \in G$ such that $py = x$, and with $z \in (y + H) \cap D$, say $z = y + h \in D$ with $h \in H$, we have $x = py = py + ph = pz \in D$.

3.5. Theorem.

(a) *A determined group may contain a nondetermined open-and-closed subgroup.*

(b) *There are non-totally bounded, nonmetrizable, determined* **LBA** *groups.*

P r o o f. Let $\kappa \geqslant \mathfrak{c}$ and $p \in \mathbb{P}$ and topologize $G := (\mathbb{Z}(p^\infty))^\kappa$ as in 3.4 (iii). We note below in Theorem 4.12 that $H = (\mathbb{Z}(p))^\kappa$ is not determined, so G is as required in both (a) and (b). $\square$

Although compact groups of the form K^κ with $\kappa \geqslant \mathfrak{c}$ are not determined, we see next (in Corollary 3.12 below) that such groups do contain nontrivial determining subgroups.

3.6. Notation. Let $\{G_i\colon\ i \in I\}$ be a set of groups, let $S_i \subseteq G_i$, and let $p \in G := \prod_{i\in I} G_i$. Then

(i) $s(p) = \{i \in I\colon\ p_i \neq 0_i\}$;

(ii) $\bigoplus_{i\in I} G_i = \{x \in G\colon\ |s(x)| < \omega\}$; and

(iii) $\bigoplus_{i\in I} S_i = (\prod_{i\in I} S_i) \cap (\bigoplus_{i\in I} G_i)$.

In this context we often identify S_i with the subset $S_i \times \{0_{I\setminus\{i\}}\}$ of G. In particular we write $G_i \subseteq G$ and we identify $\widehat{G_i}$ with $\{h\big|G_i\colon\ h \in \widehat{G}\}$.

We will use the following property to find some determining subgroups of certain (nondetermined) products.

3.7. Definition. A topological group G has the *cofinally zero* property if for all $K \in \mathcal{K}(G)$ there is $F \in \mathcal{K}(G)$ such that every $h \in U(F)$ satisfies $h\big|K \equiv 0$.

3.8. Remark. We record two classes of groups with the cofinally zero property.

(i) G is a determining subgroup of a compact Abelian group. (There is $F \in \mathcal{K}(G)$ such that $U(F) = \{0\}$, so each $h \in U(F)$ satisfies $h\big|K \equiv 0$ for all $K \in \mathcal{K}(G)$.)

(ii) G is a torsion group of bounded order. (Given $K \in \mathcal{K}(G)$, let $n > 4$ satisfy $nx = 0$ for all $x \in G$ and use Remark 1.2 (a) to choose $F \in \mathcal{K}(G)$ such that $U(F) \subseteq U(K, 1/n)$.)

3.9. Lemma. *Let $\{G_i\colon\ i \in I\}$ be a set of* **LBA** *groups with the cofinally zero property and let $G = \prod_{i\in I} G_i$. If D_i is a dense, determining subgroup of G_i, then $D := \bigoplus_{i\in I} D_i$ determines G.*

P r o o f. Given $K \in \mathcal{K}(G)$ we must find $E \in \mathcal{K}(D)$ such that $U(E) \subseteq U(K)$. We assume without loss of generality that $0 \in K$, we set $K_i = \pi_i[K]$, and we choose $F_i \in \mathcal{K}(G_i)$ such that each $h \in U(F_i)$ satisfies $h\big|K_i \equiv 0$. Since D_i determines G_i there is $E_i \in \mathcal{K}(D_i)$ such that $U(E_i) \subseteq U(F_i) \subseteq \widehat{G_i}$ and hence $U(E_i) \subseteq U(F_i) \subseteq \widehat{G}$.

Let $E := \{x \in \prod_{i\in I} E_i\colon\ |s(x)| \leqslant 1\}$. Since E is closed in $\prod_{i\in I} E_i$ and $E \subseteq \bigoplus_{i\in I} E_i \subseteq D$, we have $E \in \mathcal{K}(D)$. From $E_i \subseteq E$ follows $U(E) \subseteq U(E_i) \subseteq U(F_i)$, so if $h \in U(E) \subseteq \widehat{G}$ then $h \equiv 0$ on each K_i, hence on $\bigoplus_{i\in I} K_i$, hence on $\prod_{i\in I} K_i$, hence on $K \subseteq \prod_{i\in I} K_i$. $\square$

We note two consequences of Lemma 3.9.

3.10. Corollary. *Let $\{G_i\colon i \in I\}$ be a set of determined* **LBA** *groups with the cofinally zero property and let $G = \prod_{i\in I} G_i$. If D_i is a dense subgroup of G_i, then $\bigoplus_{i\in I} D_i$ determines G.*

Proof. D_i determines G_i, so Lemma 3.9 applies. □

3.11. Corollary. *Let $\{G_i\colon i \in I\}$ be a set of* **LBA** *groups with the cofinally zero property and let $G = \prod_{i\in I} G_i$. Then $\bigoplus_{i\in I} G_i$ determines G.*

Proof. G_i determines G_i, so Lemma 3.9 applies. □

3.12. Corollary. *Let $\{G_i\colon i \in I\}$ be a set of compact Hausdorff groups and let $G = \prod_{i\in I} G_i$. Then $\bigoplus_{i\in I} G_i$ determines G.*

Proof. Surely, as noted in Remark 3.8 (a), the groups G_i have the cofinally zero property, so Corollary 3.11 applies. □

3.13. Lemma. *Let $G, H \in$* **LBA** *and let $\varphi\colon G \twoheadrightarrow H$ be a continuous, surjective homomorphism such that $\varphi^{-1}(K) \in \mathcal{K}(G)$ whenever $K \in \mathcal{K}(H)$. If D is a dense, determining subgroup of G then $\varphi[D]$ determines H.*

Proof. Given $K \in \mathcal{K}(H)$ we must find $E \in \mathcal{K}(\varphi[D])$ such that $U(E) \subseteq U(K) \subseteq \widehat{H}$. Since $\varphi^{-1}(K) \in \mathcal{K}(G)$ and D determines G there is $F \in \mathcal{K}(D)$ such that $U(F) \subseteq U(\varphi^{-1}(K)) \subseteq \widehat{G}$, and it is easy to see that $E := \varphi[F]$ is as required. □

3.14. Corollary. *Let $G, H \in$* **LBA** *and let $\varphi\colon G \twoheadrightarrow H$ be a continuous, open, surjective homomorphism such that $\varphi^{-1}(K) \in \mathcal{K}(G)$ whenever $K \in \mathcal{K}(H)$. If G is determined then H is determined.*

Proof. Since φ is open each dense $D \subseteq H$ has $\varphi^{-1}(D)$ dense in G, so $D = \varphi[\varphi^{-1}(D)]$ determines H by Lemma 3.13. □

3.15. Corollary. *The image under a continuous homomorphism of a compact determined group is determined.*

Proof. It is well known (see for example [26, (5.29)]) that such a homomorphism is an open map; hence Corollary 3.14 applies. □

3.16. Remark. It is easily checked that if a locally compact space X is σ-compact then it is *hemicompact*, i.e., some countable subfamily $\{K_n\colon n < \omega\}$ of $\mathcal{K}(X)$ is cofinal in $\mathcal{K}(X)$ in the sense that for each $K \in \mathcal{K}(X)$ there is $n < \omega$ such that $K \subseteq K_n$. It follows that if an **LCA** group G is σ-compact (equivalently: Lindelöf) then $w(\widehat{G}) \leqslant \omega$, so $\widehat{G}$ in this case is determined by Theorem 1.3.

The principal result of this section is that compact Abelian groups of weight $\geqslant \mathfrak{c}$ are nondetermined. We begin with four preliminary results.

4.1. Lemma. *Let D be a proper dense subgroup of a group $G \in$ **LBA** such that either*
(i) *each $K \in \mathcal{K}(D)$ is finite or*
(ii) $\cup\{\overline{\langle K\rangle}^G : K \in \mathcal{K}(D)\} \neq G$.
Then D does not determine G.

Proof. In (i) we fix $x \in G\backslash D$, in (ii) we fix $x \in G\backslash \cup \{\overline{\langle K\rangle}^G : K \in \mathcal{K}(D)\}$. We show that no $K \in \mathcal{K}(D)$ satisfies $U(K) \subseteq U(\{x\})$. In (i) we assume without loss of generality, replacing G by $W(G)$ if necessary, that $G \in$ **LCA**. By Lemma 0.1 (b) applied to the discrete group G_d, there is a homomorphism $h\colon G \to \mathbb{T}$ such that $h|K \equiv 0$ and $h(x) \neq 0$; clearly there is $n \in \mathbb{Z}$ such that $|nh(x)| > \frac{1}{4}$—say $|nh(x)| = \frac{1}{4} + \varepsilon$ with $0 < \varepsilon \leqslant \frac{1}{4}$. By [26, (26.16)] there is $k \in \widehat{G}$ such that $|k(y) - nh(y)| < \varepsilon$ for all $y \in K \cup \{x\}$, and then $k \in U(K)\backslash U(\{x\})$. In (ii) we have $x \notin \overline{\langle K\rangle}^G$, so by Lemma 0.1 (b) applied with $S = \langle K\rangle$ and $h = 0 \in \widehat{S}$ there is $k \in \widehat{G}$ such that $k|\langle K\rangle \equiv 0$ and $k(x) \neq 0$. Then with $n \in \mathbb{Z}$ chosen so that $|nk(x)| > \frac{1}{4}$ we have $nk \in U(K)\backslash U(\{x\})$. $\square$

We showed in Theorem 2.4 (b) that if G^+ determines $b(G)$ with $G \in$ **LCA**, then G is compact (in fact $G = G^+ = b(G)$). Lemma 4.1 allows a more direct proof in the case that G is discrete.

4.2. Corollary. *Let G be an infinite Abelian group. Then $G^{\#}$ does not determine $b(G_d)$.*

Proof. Each $K \in \mathcal{K}(G^{\#})$ is finite by Theorem 0.2, so Lemma 4.1(i) applies. $\square$

4.3. Therem ([13]). *Let G be an Abelian group.*
(a) *If A is a point-separating subgroup of* $\mathrm{Hom}(G, \mathbb{T})$, *then $(G, \mathcal{T}_A)$ is a totally bounded, Hausdorff topological group with $\widehat{(G, \mathcal{T}_A)} = A$;*
(b) *for every totally bounded Hausdorff topological group topology $\mathcal{T}$ on G the subgroup $A := \widehat{(G, \mathcal{T})}$ of* $\mathrm{Hom}(G, \mathcal{T})$ *is point-separating and satisfies $\mathcal{T} = \mathcal{T}_A$.*

4.4. Discussion. It is easily checked that for each Abelian group G the set $\mathrm{Hom}(G, \mathbb{T})$ is closed in the compact space $\mathbb{T}^G$. Thus $\mathrm{Hom}(G, \mathbb{T})$, like every Hausdorff (locally) compact group, carries a Haar measure. Our convention here is that Haar measure is complete, so in particular every subset of a measurable set of measure 0 is itself measurable (and of measure 0); see in this connection [26, (11.21) (ii)& (11.24)].

Concerning Haar measure λ on a **LCA** group G we will appeal frequently in what follows to this statement, which we call the *Steinhaus-Weil Theorem*: *If $S \subseteq G$ is λ-measurable and $\lambda(S) > 0$, then the difference set $S - S := \{x - y\colon x, y \in S\}$ contains a nonempty open subset of G; thus S, if a subgroup of G, is open in G.* See Steinhaus [55] or Hewitt and Stromberg [28, (10.43)] for the classical case; see Weil [63, p. 50] or Stromberg [56] for proofs in full generality.

4.5. Lemma ([15, (3.10)]). *Let G be an Abelian group, let $\{x_n\colon n < \omega\}$ be a faithfully index sequence in G, and let*

$$S := \{h \in \mathrm{Hom}(G, \mathbb{T})\colon\ h(x_n) \to 0 \in \mathbb{T}\}.$$

Let λ be the Haar measure of $\mathrm{Hom}(G, \mathbb{T})$. *Then S is a λ-measurable subgroup of* $\mathrm{Hom}(G, \mathbb{T})$, *with $\lambda(S) = 0$.*

P r o o f. For the measurability of S it is enough to set

$$A_{n,m} = \left\{h \in \mathrm{Hom}(G, \mathbb{T})\colon\ |h(x_n)| \leqslant \frac{1}{m}\right\} \quad \text{for } 0 < m, n < \omega$$

and to note that $S = \bigcap\limits_{m<\omega} \bigcup\limits_{N\geqslant m} \bigcap\limits_{n\geqslant N} A_{n,m}$, a ($\lambda$-measurable) $F_{\sigma\delta}$-set in $\mathrm{Hom}(G, \mathbb{T})$. If $\lambda(S) > 0$ then since S is a subgroup of $\mathrm{Hom}(G, \mathbb{T})$ it follows from the Steinhaus-Weil Theorem that S is open in $\mathrm{Hom}(G, \mathbb{T})$. We here complete the proof (that $\lambda(S) = 0$) only in the special case that G is torsion-free, a condition equivalent to the condition that the group $\mathrm{Hom}(G, \mathbb{T}) = \widehat{G_d}$ is connected (cf. [26, (24.35)]); a more delicate analysis covering the general case is given in [15, (3.10)]. When $\mathrm{Hom}(G, \mathbb{T})$ is connected then from the condition that S is open-and-closed in $\mathrm{Hom}(G, \mathbb{T})$ it follows that $S = \mathrm{Hom}(G, \mathbb{T})$, so $x_n \to 0$ in $G^{\#}$; then since $K := \{x_n\colon n < \omega\} \cup \{0\} \in \mathcal{K}(G^{\#})$ we have also $K \in \mathcal{K}(G_d)$ by Theorem 0.2, a contradiction. □

4.6. Theorem. *Let X be a compact Hausdorff space such that $|X| < 2^{\aleph_1}$. Then*
(a) ([25]) *X contains a closed, countably infinite subspace; and*
(b) *X contains a nontrivial convergent sequence.*

P r o o f. To derive (b) from (a), let C be a closed subset of X such that $|C| = \omega$. Being (locally) compact and with each point a G_δ-point, C is first countable at each of its points, hence second countable, hence metrizable. □

4.7. Theorem. *Let G be an Abelian group such that $|G| < 2^{\aleph_1}$ and let A be a dense subgroup of* $\mathrm{Hom}(G, \mathbb{T})$ *such that either*

(i) *A is non-Haar measurable, or*

(ii) *A is Haar measurable, with $\lambda(A) > 0$.*

Then $(G, \mathcal{T}_A)$ does not determine $W(G, \mathcal{T}_A)$.

Proof. (The density hypothesis guarantees that $\mathcal{T}_A$ is a Hausdorff topology.) If the assertion fails then by Lemma 4.1, with $(G, \mathcal{T}_A)$ and $W(G, \mathcal{T}_A)$ in place of D and G respectively, some $K \in \mathcal{K}(G, \mathcal{T}_A)$ is infinite. Then by Lemma 4.6 (b) some nontrivial sequence converges in $K \subseteq (G, \mathcal{T}_A)$, so some nontrivial sequence x_n converges to 0 in $(G, \mathcal{T}_A)$, so with S defined as in Lemma 4.5 we have $A \subseteq S$ and hence A is measurable with $\lambda(A) = 0$, a contradiction. $\square$

4.8. Remarks. (a) While our proof of the general result Theorem 4.7 depends in part on [15, (3.10)], our proof of the important special case Theorem 4.9 is complete and self-contained since $\mathbb{Z}$ is torsion-free.

(b) The rest of this Section is devoted to showing that under CH a compact group is determined if and only if it is metrizable (Corollary 4.17); hence under CH a product $\prod_{i \in I} G_i$ of nontrivial compact groups is determined if and only if each G_i is determined and $|I| \leqslant \omega$ (Corollary 4.18). To achieve this, we show by direct arguments that the groups $\mathbb{T}^{\mathfrak{c}}$ and $F^{\mathfrak{c}}$ (F finite, $|F| > 1$) are nondetermined. Strictly speaking these arguments are redundant since a more delicate analysis in Section 6 yields a more subtle result.

4.9. Theorem. *The group $\mathbb{T}^{\mathfrak{c}}$ is not determined.*

Proof. We claim that there is a nonmeasurable subgroup A of $\mathbb{T}$ with A algebraically of the form $A = \bigoplus_{\mathfrak{c}} \mathbb{Z}$. (Our construction of such a subgroup parallels the usual construction of a Bernstein subset of $\mathbb{R}$.) Let $\{F_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of all uncountable, closed subsets of $\mathbb{T}$. Choose nontorsion $p_0, q_0 \in F_0$ with $q_0 \notin \langle p_0 \rangle$. Recursively, if $\xi < \mathfrak{c}$ and p_η, q_η have been chosen for all $\eta < \xi$, choose nontorsion $p_\xi, q_\xi \in F_\xi$ with

$$\langle p_\xi \rangle \cap \langle \{p_\eta \colon \eta < \xi\} \cup \{q_\eta \colon \eta < \xi\} \rangle = \{0\},$$
$$\langle q_\xi \rangle \cap \langle \{p_\eta \colon \eta \leqslant \xi\} \cup \{q_\eta \colon \eta < \xi\} \rangle = \{0\}.$$

The availability of such p_ξ, q_ξ derives from the fact that $|F_\xi| = \mathfrak{c}$ while $|\mathrm{tor}(\mathbb{T})| = \omega$. Clearly the group $A := \langle \{p_\xi \colon \xi < \mathfrak{c}\} \rangle$ is isomorphic with $\bigoplus_{\mathfrak{c}} \mathbb{Z}$. The group A is not λ-measurable in $\mathrm{Hom}(\mathbb{Z}, \mathbb{T})$ because (a) $\lambda(A) > 0$ is impossible, since the inequality implies that the subgroup A is open-and-closed in $\mathbb{T}$ by the Steinhaus-Weil Theorem and then $A = \mathbb{T}$, (b) $\lambda(A) = 0$ is impossible, since then $\mathbb{T} \backslash A$ is measurable and from $\lambda(\mathbb{T}\backslash A) > 0$ it follows that there is an uncountable compact subset $F = F_\xi$ of $\mathbb{T}\backslash A$ with $\lambda(F) > 0$, so that $p_\xi \in F \subseteq (\mathbb{T}\backslash A)$ and $p_\xi \in A$. Since the elements of $\{p_\xi\colon \xi < \mathfrak{c}\}$ are algebraically independent, the embedding $e_A\colon \mathbb{Z} \to \mathbb{T}^A = \mathbb{T}^{\mathfrak{c}}$ given by $e_A(n)(h) = h(n)$ $(n \in \mathbb{Z},\, h \in A)$ takes $\mathbb{Z}$ onto a dense subgroup of the compact group $\mathbb{T}^A = \mathbb{T}^{\mathfrak{c}}$. (This is a version of the classical Kronecker approximation theorem. See [26, (26.15)] for details, taking $G = \mathbb{Z}_d$, $\Gamma = A$, and $f \in \widehat{A_d} = \mathbb{T}^{\mathfrak{c}}$ there.) Suppressing mention of the function e_A, we have the dense inclusion $(\mathbb{Z}, \mathcal{T}_A) \subseteq W(\mathbb{Z}, \mathcal{T}_A) = \mathbb{T}^A = \mathbb{T}^{\mathfrak{c}}$, and Theorem 4.7 (i) applies. □

We turn next to a consideration of groups of the form $F^{\mathfrak{c}}$ with F finite. Our first statement is perhaps known to aficionadoes, though we have not located a statement and proof in the literature.

4.10. Theorem. *Let $\kappa \geqslant \omega$ and let F be a finite Abelian group. Then $b((\bigoplus_\kappa F)_d) = F^{2^\kappa}$.*

Proof. It is noted in [26, (26.12)] that the Bohr compactification of a discrete Abelian group G_d is realized by the relation $b(G_d) = ((\widehat{G_d})_d)\widehat{\ }$; the isomorphism $i = i_G\colon G \twoheadrightarrow G^\# \subseteq b(G_d)$ is here given by $i_G(x) = \widehat{x} \in ((\widehat{G_d})_d)\widehat{\ }$ with $\widehat{x}\colon \widehat{G_d} \to \mathbb{T}$ defined by $\widehat{x}(h) = h(x)$.

Now fix an isomorphism $\psi\colon \bigoplus_{2^\kappa} F \twoheadrightarrow F^\kappa$. (The existence of such ψ is well known. See [22, p. 44] or [26, (A.25)] or [27, (4.5)] for fundamental special cases, or see [11, (3.1)] for the full argument.) Equating as usual $((\bigoplus_\kappa F)_d)\widehat{\ }$ with F^κ and $((\bigoplus_{2^\kappa} F)_d)\widehat{\ }$ with F^{2^κ} (cf. [26, (23.22)], and with $\varphi := \widehat{\psi}\colon ((F^\kappa)_d)\widehat{\ } \twoheadrightarrow ((\bigoplus_{2^\kappa} F)_d)\widehat{\ }$ denoting the topological isomorphism adjoint to ψ, we have, taking $G := (\bigoplus_\kappa F)_d$, the dense inclusion $(G_d)^\# \subseteq b(G) = ((\widehat{G_d})_d)\widehat{\ } = ((F^\kappa)_d)\widehat{\ }$ and then the dense inclusion $\varphi[(G_d)^\#] \subseteq \varphi[b(G)] = ((\bigoplus_{2^\kappa} F)_d)\widehat{\ } = F^{2^\kappa}$. The following diagram captures the argument:

$$
\begin{tikzcd}
(\bigoplus_\kappa F)_d \arrow[r, "b"] \arrow[d, "\widehat{\ }"'] & (\bigoplus_\kappa F)^\# \arrow[d, "\widehat{\ }"] & & \\
F^\kappa \arrow[r, equal] \arrow[d, "\widehat{\ }"'] & F^\kappa \arrow[d, "(*)"'] & (F^\kappa)_d \arrow[l, "\mathrm{id}"] \arrow[d, "\widehat{\ }"] & (\bigoplus_{2^\kappa} F)_d \arrow[l, "\psi"] \arrow[d, "\widehat{\ }"] \\
(\bigoplus_\kappa F)_d \arrow[r, "b"'] & (\bigoplus_\kappa F)^\# \arrow[r, "i"'] & b(\bigoplus_\kappa F)_d \arrow[r, "\varphi"'] & F^{2^\kappa}
\end{tikzcd}
$$

□

4.11. Remarks. (a) Given $\kappa \geqslant \omega$ and a finite Abelian group F, there is to the best of the authors' knowledge no "canonical" or "natural" dense copy of the group

$G = \bigoplus_\kappa F$ inside F^{2^κ} which realizes the identity $F^{2^\kappa} = b(G) \supseteq G^\#$. Alternatively stated: we know of no natural, functorial dense embedding $G_d \twoheadrightarrow G^\# \subseteq F^{2^\kappa}$. The construction of Theorem 4.10 departs from the isomorphism $\psi\colon \bigoplus_{2^\kappa} F \twoheadrightarrow F^\kappa$, which even in the basic case $F = \mathbb{Z}(p)$ or $F = \mathbb{Z}(p^r)$ depends on the axiom of choice and is nonconstructive.

(b) The isomorphism $\bigoplus_{2^\kappa} F = F^\kappa$ can fail when F is infinite, as Specker [53] has shown in the case $F = \mathbb{Z}$, $\kappa = \omega$.

4.12. Theorem. *Let $\kappa \geqslant \omega$ and let F be a finite Abelian group. Then the group F^{2^κ} is not determined.*

Proof. With $G := \bigoplus_\kappa F$ we have by Theorem 4.10 the dense inclusion $G^\# \subseteq b(G_d) = F^{2^\kappa}$, so Corollary 4.2 applies. □

4.13. Remark. For clarity, we emphasize a feature of the preceding discussion. Let $\kappa \geqslant \omega$, let F be a finite Abelian group, and set $G_0 := \bigoplus_\kappa F$ and $G_1 := \bigoplus_{2^\kappa} F$. Then the compact group F^{2^κ} contains a copy of $(G_0)^\#$ as a dense topological subgroup, and a dense copy of G_1, such that

(i) $(G_0)^\#$ does not determine F^{2^κ} (by Corollary 4.2 and Theorem 4.10) and

(ii) G_1 does determine F^{2^κ} (by Corollary 3.12).

The Weil completions $W((G_0)^\#)$ and $W(G_1)$ are both equal to F^{2^κ}, and the character groups $\widehat{(G_0)^\#}$ and $\widehat{G_1}$ are both isomorphic to the group $\bigoplus_{2^\kappa} F$, but the topological group $\widehat{G_1}$ is discrete while $\widehat{(G_0)^\#}$ is not discrete.

4.14. Discussion. It is known ([26, (24.15)]) that a compact Abelian group G satisfies $w(G) = |\widehat{G}|$. If in addition $w(G) = |\widehat{G}| > \omega$ then the torsion-free rank $\kappa_0 = r_0(\widehat{G})$ and the p-ranks $\kappa_p = r_p(\widehat{G})$ $(p \in \mathbb{P})$ satisfy the relation

$$(*) \qquad w(G) = |\widehat{G}| = r(\widehat{G}) = \kappa_0 + \Sigma_{p\in\mathbb{P}}\,\kappa_p$$

(cf. [22, §16]), so that $\widehat{G}$ contains algebraically the group

$$\Big(\bigoplus_{\kappa_0} \mathbb{Z}\Big) \oplus \Big(\bigoplus_{p\in\mathbb{P}}\Big(\bigoplus_{\kappa_p} \mathbb{Z}(p)\Big)\Big).$$

Using Pontrjagin duality and familiar techniques from [26, §24] it follows (always assuming $w(G) = \kappa > \omega$) that there is a continuous epimorphism $\varphi\colon G \twoheadrightarrow \mathbb{T}^{\kappa_0} \times \prod_{p\in\mathbb{P}} (\mathbb{Z}(p))^{\kappa_p}$. Hence if α is a cardinal such that $w(G) = \kappa \geqslant \alpha \geqslant \mathrm{cf}(\alpha) > \omega$ then from $(*)$ either $\kappa_0 \geqslant \alpha$ or $\kappa_p \geqslant \alpha$ for some $p \in \mathbb{P}$ and (projecting $\mathbb{T}^{\kappa_0}$ onto $\mathbb{T}^\alpha$ or $(\mathbb{Z}(p))^{\kappa_p}$ onto $(\mathbb{Z}(p))^\alpha$) we have the following familiar result (see for example [12, (5.4)].

4.15. Theorem. *Let G be a compact Abelian group and let α be a cardinal such that $w(G) \geqslant \alpha \geqslant \operatorname{cf}(\alpha) > \omega$. Then there is a continuous epimorphism $\varphi\colon G \twoheadrightarrow K^\alpha$ with either $K = \mathbb{T}$ or $K = \mathbb{Z}(p)$ for some $p \in \mathbb{P}$.*

4.16. Corollary. *Let G be a compact Abelian group. If $w(G) \geqslant \mathfrak{c}$ then G is not determined.*

Proof. From $\mathfrak{c} = \mathfrak{c}^\omega < \mathfrak{c}^{\operatorname{cf}(\mathfrak{c})}$ it follows that $\operatorname{cf}(\mathfrak{c}) > \omega$. The groups $\mathbb{T}^\mathfrak{c}$ and $(\mathbb{Z}(p))^\mathfrak{c}$ are nondetermined by Theorems 4.9 and 4.12 respectively, so Corollary 3.15 and Theorem 4.15 apply. □

4.17. Corollary [CH]. *Let G be a compact Abelian group. Then G is determined if and only if G is metrizable.*

Proof. Use Theorem 1.3 and Corollary 4.16. □

4.18. Corollary [CH]. *Let $\{G_i\colon i \in I\}$ be a set of compact Abelian groups with each $|G_i| > 1$, and let $G = \prod_{i\in I} G_i$. Then G is determined if and only if $|I| \leqslant \omega$ and each G_i is determined.*

We close this section with an example indicating that the intersection of dense, determining subgroups may be dense and nondetermining.

4.19. Theorem. *There are dense, determining subgroups D_i $(i = 0, 1)$ of $\mathbb{T}^\mathfrak{c}$ such that $D_0 \cap D_1$ is dense in $\mathbb{T}^\mathfrak{c}$ and does not determine $\mathbb{T}^\mathfrak{c}$.*

Proof. Let Z be a dense, cyclic, nondetermining subgroup of $\mathbb{T}^\mathfrak{c}$ (as furnished by the proof of Theorem 4.9), let A_i $(i = 0, 1)$ be dense, torsion subgroups of $\mathbb{T}$ such that $A_0 \cap A_1 = \{0_\mathbb{T}\}$, and set $D_i := Z + \bigoplus_\mathfrak{c} A_i \subseteq \mathbb{T}^\mathfrak{c}$ $(i = 0, 1)$. Then A_i determines $\mathbb{T}$ by Theorem 1.3 so $\bigoplus_\mathfrak{c} A_i$ determines $\mathbb{T}^\mathfrak{c}$ by Lemma 3.9, so D_i determines $\mathbb{T}^\mathfrak{c}$ $(i = 0, 1)$; but the dense subgroup $Z = D_0 \cap D_1$ of $\mathbb{T}^\mathfrak{c}$ does not determine $\mathbb{T}^\mathfrak{c}$. □

5. Concerning topological linear spaces

5.1. Remark. Let κ be a cardinal number and denote by l^1_κ the space of real κ-sequences $x = \{x_\xi\colon \xi < \kappa\}$ such that $\|x\|_1 := \sum\limits_{\xi<\kappa} |x_\xi| < \infty$. The additive topological group l^1_κ respects compactness (cf. [48]).

We claim that $\widehat{(l^1_\kappa)^+}$ is not discrete, so the Weil completion $W((l^1_\kappa)^+)$ is another example of a compact nondetermined group. As usual, since l^1_κ respects compactness, the character groups of l^1_κ and $(l^1_\kappa)^+$ are topologically isomorphic. By [61, Theorem 2], the dual (as a LCS) of l^1_κ, equipped with the compact-open topology, is topologically isomorphic to the character group of l^1_κ, which means that the latter is actually a LCS, and therefore cannot be discrete.

Remark 5.1 generalizes Example 105 of [42].

Definition. A topological group G is *(group) reflective* if the evaluation mapping $\Omega_G\colon G \to \widehat{\widehat{G}}$ defined by $\Omega_G(x)(h) := h(x)$ for $x \in G$, $h \in \widehat{G}$ is a topological isomorphism of G onto $\widehat{\widehat{G}}$.

The Raĭkov completion $R(G)$ of an Abelian topological group G is the completion of G when equipped with its left uniformity; it is known (cf. for example [49, (10.15)]) that $R(G)$ is a complete Abelian topological group. When G is locally bounded, $R(G) = W(G)$. To know more about the subject and pertinent references the reader is invited to consult the paper of Galindo and Hernández [23], who additionally have constructed a **MAP** group G such that $R(G)$ is not **MAP**. We show now that such a group cannot be reflective.

5.2. Theorem. *Let G be a noncomplete, reflective group and let $R(G)$ be its completion. Then $R(G) \in$ **MAP** and G does not determine $R(G)$.*

Proof. Since G is reflective, it has a base at the identity consisting of quasiconvex sets which are in turn closed in G^+ (again by [23, Note]), hence Corollary 1 of [23] yields the first assertion. Assume that the restriction map $\varphi\colon \widehat{R(G)} \to \widehat{G}$ is a topological isomorphism and let f be the inverse of φ. Then its adjoint map [26, (24.37)] $\widehat{f}\colon \widehat{\widehat{R(G)}} \to \widehat{\widehat{G}}$ is a topological isomorphism as well (the proof in 26](24.38) also works in this case). Notice that, since $R(G) \in$ **MAP**, the function $F := \widehat{f} \circ \Omega_{R(G)}\colon R(G) \hookrightarrow \widehat{\widehat{G}}$ is a (possibly discontinuous) injective homomorphism. Notice as well however, that $(\widehat{f} \circ \Omega_{R(G)})\big|G\colon G \hookrightarrow \widehat{\widehat{G}}$ equals the surjective evaluation mapping $\Omega_G\colon G \twoheadrightarrow \widehat{\widehat{G}}$, which is a topological isomorphism. Hence the injectivity of F is impossible. $\square$

Example 5.4 *infra* illustrates Theorem 5.2.

5.3. Definition. A reflexive locally convex vector space (LCS) in which every closed bounded subset is compact is called a *Montel space*.

Reflexivity and boundedness ([50, §I.5, §IV.5]) are meant here in the sense of topological vector spaces. By a *Montel* group we mean the underlying (additive) topological group of a Montel space. Since by definition these are reflexive LCS, Montel groups are reflective as proven in [52].

5.4. Example. Kōmura [37] and Amemiya and Kōmura [1] construct by induction three different noncomplete Montel spaces, the completion of each being a "big product" of copies of $\mathbb{R}$, and one of them being exactly $\mathbb{R}^{\mathfrak{c}}$. These groups indicate that Theorem 5.2 is not vacuous. One of the spaces constructed in [1] is separable. Thus in particular, again by Theorem 5.2, we see that $\mathbb{R}^{\mathfrak{c}}$ has a countable dense subgroup which does not determine $\mathbb{R}^{\mathfrak{c}}$.

The remarks above show yet again that the property of being determined is not $\mathfrak{c}$-productive.

6. Cardinals κ such that $\omega < \kappa \leqslant \mathfrak{c}$

It is well known (cf. for example [38, (2.18)] or [10, (8.2.4)]) that under Martin's Axiom [MA] every cardinal κ with $\omega \leqslant \kappa < \mathfrak{c}$ satisfies $2^{\kappa} = \mathfrak{c}$. In particular under MA $+\neg$CH it follows from Theorem 4.6 (b) that every compact Hausdorff space X such that $|X| < 2^{\aleph_1} = \mathfrak{c}$ contains a nontrivial convergent sequence. Malykhin and Šapirovskiĭ [40] have achieved a nontrivial extension of this result: Under MA, every compact Hausdorff space X with $|X| \leqslant \mathfrak{c}$ contains a nontrivial convergent sequence. This furnishes the following result, whose statement and proof closely parallel those of Theorem 4.7 above.

6.1. Theorem [MA]. *Let G be a group with $|G| \leqslant 2^{\omega}$, and let A be a dense nonmeasurable subgroup of $\widehat{G_d}$. Then every compact subset of $(G, \mathcal{T}_A)$ is finite, so its completion $W(G, \mathcal{T}_A)$ is not determined.*

We denote by λ the usual Haar measure on $\mathbb{T}$, and by λ^* the associated outer measure. The existence of a nonmeasurable subset X of $\mathbb{T}$ (with $|X| = \mathfrak{c}$) is well known, so the case $\kappa = \mathfrak{c}$ of the following theorem recaptures parts of the argument of Theorem 4.9.

6.2. Theorem. *Let $\omega < \kappa \leqslant \mathfrak{c}$. If there is $X \in [\mathbb{T}]^{\kappa}$ such that $\lambda^*(X) > 0$, then there is a nonmeasurable, free Abelian subgroup A of $\mathbb{T}$ algebraically of the form $A = \bigoplus_{\kappa} \mathbb{Z}$.*

P r o o f. Let D be a minimal divisible extension in $\mathbb{T}$ of $\langle X \cup (\mathrm{tor}(\mathbb{T}))\rangle$ and write $D = \mathrm{tor}(\mathbb{T}) \times X_0$ (cf. [26, (A.8)]) for a suitable (necessarily torsion-free) subgroup X_0 of $\mathbb{T}$; evidently $|X_0| = |D| = |X| = \kappa$ and X_0, a homomorphic image of D, is itself divisible. Since $\lambda^*(D) \geqslant \lambda^*(X) > 0$ and $|D/X_0| = \omega$, the countable subadditivity of λ^* ([26, (11.21 (iv))]) gives $\lambda^*(X_0) > 0$. Let B_0 be a maximal independent subset of X_0. The existence of B_0 is guaranteed by the Kuratowski-Zorn Lemma [26, (A.11)]. For $n < \omega$ set $B_n := \{x/n!\colon x \in B_0\}$ and notice that the maximality of B_0

implies that each $B_n \in [\mathbb{T}]^\kappa$ and is independent. Then algebraically $\langle B_n \rangle = \bigoplus_\kappa \mathbb{Z}$ for $n < \omega$, and $\langle B_n \rangle \subseteq \langle B_{n+1} \rangle$. Since $X_0 = \bigcup\limits_{n<\omega} \langle B_n \rangle$ and $\lambda^*(X_0) > 0$, the countable subadditivity of λ^* again implies that there is $n_0 < \omega$ such that $\lambda^*(\langle B_{n_0} \rangle) > 0$. If the group $\langle B_{n_0} \rangle$ were measurable then by the Steinhaus-Weil theorem it would be open, so $\kappa = \mathfrak{c}$ and $\langle B_{n_0} \rangle = \mathbb{T}$ and we have the absurdity $\emptyset = \mathrm{tor}(\langle B_{n_0} \rangle) = \mathrm{tor}(\mathbb{T}) \neq \emptyset$. Thus $A := \langle B_{n_0} \rangle = \bigoplus_\kappa \mathbb{Z}$ is as required. $\square$

Responding to a question on a closely related matter, Stevo Todorčević [57] proposed and proved the above result for $\kappa = \aleph_1$. In this case his proof (not given here) additionally yields that $X \setminus \mathrm{tor}(\mathbb{T})$ can be broken into ω-many pairwise disjoint independent sets, each of cardinality $\aleph_1$.

For torsion groups of prime order, we obtain the following.

6.3. Theorem. *Let F be a finite group of prime order p, and let κ_1, κ_2 be infinite cardinals such that $\kappa_1 \leqslant 2^{\kappa_2}$. Denote by λ the Haar measure of F^{κ_2}. If there is $X \in [F^{\kappa_2}]^{\kappa_1}$ such that $\lambda^*(X) > 0$, then there is a nonmeasurable subgroup A of F^{κ_2} algebraically of the form $A = \bigoplus_{\kappa_1} F$.*

P r o o f. As indicated in the proof of Theorem 4.10, we have algebraically the isomorphism $F^{\kappa_2} = \bigoplus_{2^{\kappa_2}} F$. Clearly this group has (many) subgroups A algebraically of the form $\bigoplus_{2^{\kappa_2}} F$ for which $|F^{\kappa_2}/A| = |(\bigoplus_{2^{\kappa_2}} F)/A| = \omega$, and (because λ is countably additive and $\lambda(F^{\kappa_2}) = 1$) such a group A cannot be λ-measurable. We assume therefore that $\kappa_1 < 2^{\kappa_2}$. By the Kuratowski-Zorn lemma, there is a maximal independent subset of X, say X_0 [26, (A11)]. We set $A := \langle X_0 \rangle$. We show that $X \subseteq A$ (and hence $X_0 \in [F^{\kappa_2}]^{\kappa_1}$). Indeed if $b \in X \setminus \{0\}$ then there are finitely many elements in X_0, say $b_1, \ldots, b_n$, and elements $s, s_1, \ldots, s_n \in \mathbb{Z}(p) := \{0, 1, \ldots, p-1\}$, such that $sb + s_1 b_1 + \ldots + s_n b_n = 0$ but $sb \neq 0$ and at least one $s_i b_i \neq 0$. Then since the equation $ys = 1$ has a solution $\overline{y}$ in the field $\mathbb{Z}(p)$, it follows that $b = \overline{y}(s_1 b_1 + \ldots + s_n b_n) \in A$, as required. Hence from $\lambda^*(X) > 0$ follows $\lambda^*(A) > 0$. Note that A is algebraically of the form $A = \bigoplus_{\kappa_1} F$. If A is λ-measurable then (by the Steinhaus-Weil theorem) A is open in F^{κ_2} and we have the contradiction $2^{\kappa_2} = |A| = \kappa_1 < 2^{\kappa_2}$. $\square$

6.4. Discussion. As usual for an ideal $\mathcal{I}$ of subsets of a set S we write

$$\mathrm{add}(\mathcal{I}) = \min\Big\{|\mathcal{J}|\colon \mathcal{J} \subseteq \mathcal{I}, \bigcup \mathcal{J} \notin \mathcal{I}\Big\},$$

and

$$\mathrm{non}(\mathcal{I}) = \min\{|Y|\colon Y \subseteq S, Y \notin \mathcal{I}\}.$$

Let F be a finite group ($|F| > 1$), let $\lambda_\mathbb{T}$ and λ_{F^ω} denote completed Haar measure on $\mathbb{T}$ and F^ω respectively, and let $\mathcal{N}(\mathbb{T})$ and $\mathcal{N}(F^\omega)$ denote the σ-algebra of $\lambda_\mathbb{T}$-

and λ_{F^ω}-measurable sets of measure zero. As with any two compact metric spaces of equal cardinality equipped with atomless ("continuous") probability measures, the spaces $\mathbb{T}$ and F^ω are Borel-isomorphic in the sense that there is a bijection $\varphi\colon \mathbb{T} \twoheadrightarrow F^\omega$ such that the associated bijection $\overline{\varphi}\colon \mathcal{P}(\mathbb{T}) \twoheadrightarrow \mathcal{P}(F^\omega)$ carries the Borel algebra $\mathcal{B}(\mathbb{T})$ onto the Borel algebra $\mathcal{B}(F^\omega)$ in such a way that $\lambda_{F^\omega}(\overline{\varphi}(B)) = \lambda_{\mathbb{T}}(B)$ for each $B \in \mathcal{B}(\mathbb{T})$. (See [36, (17.41)] or [54, (3.4.23)] for a proof of this "Borel isomorphism Theorem for measures".) It is helpful to remark that the inclusions $\mathcal{N}(\mathbb{T}) \subseteq \mathcal{B}(\mathbb{T})$ and $\mathcal{N}(F^\omega) \subseteq \mathcal{B}(F^\omega)$ fail; indeed we have $|\mathcal{B}(\mathbb{T})| = \mathfrak{c}$, but since there are $N \in \mathcal{N}(\mathbb{T}) \cap [\mathbb{T}]^{\mathfrak{c}}$ and $\mathcal{P}(N) \subseteq \mathcal{N}(\mathbb{T})$ we have $|\mathcal{N}(\mathbb{T})| = 2^{\mathfrak{c}}$ (and similarly $|\mathcal{B}(F^\omega)| = \mathfrak{c} < 2^{\mathfrak{c}} = |\mathcal{N}(F^\omega)|$).

In the following lemma we retain the notation of the previous paragraph.

6.5. Lemma. *The cardinals* $\mathrm{non}(\mathcal{N}(\mathbb{T}))$ *and* $\mathrm{non}(\mathcal{N}(F^\omega))$ *are equal.*

Proof. We have $\mathrm{non}(\mathcal{N}(\mathbb{T})) = \min\{|X|\colon X \subseteq \mathbb{T},\ \lambda^*_{\mathbb{T}}(X) > 0\}$, where $\lambda^*_{\mathbb{T}}\colon \mathcal{P}(\mathbb{T}) \to [0,1]$ is the outer measure associated with $\lambda_{\mathbb{T}}$. Let X be as indicated, say with $\lambda^*_{\mathbb{T}}(X) = \varepsilon > 0$. Since $\lambda^*_{\mathbb{T}}$ is outer-regular we have

$$\varepsilon = \lambda^*_{\mathbb{T}}(X) = \inf\{\lambda_{\mathbb{T}}(U)\colon U \text{ is open in } \mathbb{T},\ X \subseteq U\}$$

and hence $\varepsilon = \inf\{\lambda_{\mathbb{T}}(B)\colon X \subseteq B \in \mathcal{B}(\mathbb{T})\}$. Since φ is measure-preserving and $\overline{\varphi}$ carries $\mathcal{B}(\mathbb{T})$ onto $\mathcal{B}(F^\omega)$ we then have that the outer measure $\lambda^*_{F^\omega}\colon \mathcal{P}(F^\omega) \to [0,1]$ associated with λ_{F^ω} satisfies

$$\lambda^*_{F^\omega}(\varphi[X]) = \inf\{\lambda_{F^\omega}(C)\colon \varphi[X] \subseteq C \in \mathcal{B}(F^\omega)\} = \varepsilon > 0$$

and hence $\mathrm{non}(\mathcal{N}(F^\omega)) \leqslant |\varphi[X]| = |X| = \mathrm{non}(\mathcal{N}(\mathbb{T}))$; the reverse inequality follows similarly. □

It is easy to see that $\mathrm{add}(\mathcal{N}(\mathbb{T}))$ is regular and that $\mathrm{cf}(\mathrm{non}(\mathcal{N}(\mathbb{T}))) > \omega$; indeed one has $\mathrm{cf}(\mathrm{non}(\mathcal{N}(\mathbb{T}))) \geqslant \mathrm{add}(\mathcal{N}(\mathbb{T})) > \omega$ ([6, (2.1.5(2)]). For more information on these cardinals and their relation to other familiar "small cardinals" the reader may consult [21], or [6] and [60] and the diagram given there.

For notational simplicity in what follows we write $\mathrm{non}(\mathcal{N}) := \mathrm{non}(\mathcal{N}(\mathbb{T})) = \mathrm{non}(\mathcal{N}(F^\omega))$, a definition justified by Lemma 6.5.

6.6. Theorem. *Let G be a compact Abelian group such that $w(G) \geqslant \mathrm{non}(\mathcal{N})$. Then G is nondetermined.*

Proof. Since $\mathrm{cf}(\mathrm{non}(\mathcal{N})) > \omega$, there is by Theorem 4.15 a continuous epimorphism from G onto a group K of the form $\mathbb{T}^{\mathrm{non}(\mathcal{N})}$ or $(\mathbb{Z}(p))^{\mathrm{non}(\mathcal{N})}$ for some $p \in \mathbb{P}$.

By Corollary 3.15 it then suffices to prove that such groups K are nondetermined. We handle the two cases separately.

Case 1. $K = \mathbb{T}^{\mathrm{non}(\mathcal{N})}$. By Theorem 6.2 with $\kappa = \mathrm{non}(\mathcal{N})$ there is a nonmeasurable subgroup A of $\mathbb{T}$ such that A is algebraically of the form $A = \bigoplus_{\mathrm{non}(\mathcal{N})} \mathbb{Z}$. Arguing much as in the proof of Theorem 4.9 and again using Kronecker's theorem, we see that the evaluation map $e_A\colon\ \mathbb{Z} \to \mathbb{T}^A$ takes $\mathbb{Z}$ onto a dense subgroup (denoted simply $(\mathbb{Z}, \mathcal{T}_A)$) of $\mathbb{T}^A$. Then $K = W(\mathbb{Z}, \mathcal{T}_A)$ and $(\mathbb{Z}, \mathcal{T}_A)$ does not determine K by Theorem 4.7.

Case 2. $K = (\mathbb{Z}(p))^{\mathrm{non}(\mathcal{N})}$. It is immediate from Theorem 6.3, taking $F = \mathbb{Z}(p)$, $\kappa_1 = \mathrm{non}(\mathcal{N})$ and $\kappa_2 = \omega$, that there is a nonmeasurable subgroup A of $(\mathbb{Z}(p))^\omega$ such that A is algebraically of the form $A = \bigoplus_{\mathrm{non}(\mathcal{N})} \mathbb{Z}(p)$. Then

$$A = \bigoplus_{\mathrm{non}(\mathcal{N})} \mathbb{Z}(p) \subseteq (\mathbb{Z}(p))^\omega = \mathrm{Hom}\Big(\bigoplus_{\omega} \mathbb{Z}(p), \mathbb{T}\Big) = \mathrm{Hom}\Big(\bigoplus_{\omega} \mathbb{Z}(p), \mathbb{Z}(p)\Big)$$

and the evaluation map $e_A\colon\ \bigoplus_\omega \mathbb{Z}(p) \to (\mathbb{Z}(p))^A$ defined as in §0 takes $\bigoplus_\omega \mathbb{Z}(p)$ onto a subgroup (again denoted simply $(\bigoplus_\omega \mathbb{Z}(p), \mathcal{T}_A)$) of $(\mathbb{Z}(p))^A = K$. Taking $G := \bigoplus_\omega \mathbb{Z}(p)$ and $\Gamma := A$ in [26, (26.15)], and using the fact that $\widehat{A_d} = (\mathbb{Z}(p))^{\mathrm{non}(\mathcal{N})}$, we see that $(\bigoplus_\omega \mathbb{Z}(p), \mathcal{T}_A)$ is dense in K, hence $K = W(\bigoplus_\omega \mathbb{Z}(p), \mathcal{T}_A)$. That $(\bigoplus_\omega \mathbb{Z}(p), \mathcal{T}_A)$ does not determine K follows as before from Theorem 4.7. $\square$

7. Questions

7.1. Question. Is there a compact group G with a countable dense subgroup D such that $w(G) > \omega$ and D determines G?

7.2. Question. If $\{G_i\colon\ i \in I\}$ is a set of topological Abelian groups and D_i is a dense determining subgroup of G_i, must $\bigoplus_{i\in I} D_i$ determine $\prod_{i\in I} G_i$? In particular, does $\bigoplus_{i\in I} G_i$ determine $\prod_{i\in I} G_i$? In particular, does $\bigoplus_{\mathfrak{c}} \mathbb{R}$ determine $\mathbb{R}^{\mathfrak{c}}$?

7.3. Discussion. Consider the following cardinals:

(a) $\mathfrak{m}_{\mathbb{T}}$:= the least cardinal κ such that $\mathbb{T}^\kappa$ is nondetermined;

(b) $\mathfrak{m}_{f\exists}$ [resp., $\mathfrak{m}_{f\forall}$] := the least cardinal κ such that some [resp., each] finite group F has F^κ nondetermined;

(c) $\mathfrak{m}_{c\exists}$ [resp., $\mathfrak{m}_{c\forall}$] := the least cardinal κ such that some [resp., each] compact abelian group of weight κ is nondetermined;

(d) $\mathfrak{m}_{p\exists}$ [resp., $\mathfrak{m}_{p\forall}$] := the least cardinal κ such that some [resp., each] product of κ-many compact determined groups is nondetermined.

It follows from Theorems 1.3 and 6.6 that each $\mathfrak{m}_x$, with the possible exception of $\mathfrak{m}_{p\exists}$, satisfies $\aleph_1 \leqslant \mathfrak{m}_x \leqslant \mathrm{non}(\mathcal{N})$. Further if $\mathrm{non}(\mathcal{N}) = \aleph_1$, then all seven cardinals $\mathfrak{m}_x$ are equal to $\aleph_1$. The condition $\mathrm{non}(\mathcal{N}) = \aleph_1$ is clearly consistent with CH, and it has been shown to be consistent as well with $\neg$CH (see for example [6], [21] and [33, Example 1, p. 568]), so in particular there are models of ZFC + $\neg$CH in which every compact (Abelian) group G satisfies: G is determined if and only if G is metrizable. (Without appealing to the cardinal $\mathrm{non}(\mathcal{N})$, Michael Hrušák [32] in informal conversation suggested the existence of models of ZFC + $\neg$CH in which $\{0,1\}^{\aleph_1}$ is nondetermined.)

The following (related) questions are ripe for investigation.

7.4. Question. Are the various cardinal numbers $\mathfrak{m}_x$ equal in ZFC? Are they equal to one of the familiar "small cardinals" conventionally noted in the Cichoń diagram (cf. [6], [60])? Is each $\mathfrak{m}_x = \mathrm{non}(\mathcal{N})$? Is each $\mathfrak{m}_x = \aleph_1$? Is each $\mathrm{cf}(\mathfrak{m}_x) > \omega$?

We emphasize that we know of no models of ZFC in which $\mathbb{T}^{\aleph_1}$, or some group of the form $F^{\aleph_1}$ (F finite, $|F| > 1$), is determined. Thus restating a part of Question 7.4, we are forced to consider the possibility that the following questions have an affirmative answer.

7.5. Question. Are the following (equivalent) statements theorems of ZFC?

(a) The group $\mathbb{T}^{\aleph_1}$ and groups of the form $F^{\aleph_1}$ (F finite, $|F| > 1$) are nondetermined.

(b) A compact abelian group G is determined if and only if G is metrizable.

The following question, taken from our Abstract, is suggested by those above.

7.6. Question. Is there in ZFC a cardinal κ such that a compact group G is determined if and only if $w(G) < \kappa$?

We have noted already that if $\mathrm{non}(\mathcal{N}) = \aleph_1$, then $\mathfrak{m}_{p\exists} = \aleph_1$. In the absence of such a hypothesis, we can make only the obvious statements about $\mathfrak{m}_{p\exists}$: that $2 < \mathfrak{m}_{p\exists} < \omega$ is impossible, and that $\mathfrak{m}_{p\exists}$, if infinite, is regular. In particular we do not know the answer to these questions.

7.7. Question. Is it consistent with ZFC that $\mathfrak{m}_{p\exists} = 2$? Is it consistent with ZFC that $\mathfrak{m}_{p\exists} = \omega$?

Question 7.7 has analogues in the context of groups which are not assumed to be compact, as follows.

7.8. Question. In ZFC alone or in augmented axiom systems: Is the product of finitely many determined groups necessarily determined? If G is determined, is $G \times G$ necessarily determined?

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