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ONE INTERVAL IN THE LATTICE OF PARTIAL HYPERCLONES

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Abstract. In this paper the structure of the interval $[O_A, Hp_A]$ in the lattice of partial hyperclones is determined, where O_A is the clone of all total operations and Hp_A is the clone of all partial hyperoperations on A.

Keywords: clone, hyperoperation, hyperalgebra, hyperclone

MSC 2000: 08A40

1. Preliminaries

Let A be a nonempty set. For a positive integer n, a function from A^n to the family $P(A)$ of all subsets of A is called a *partial n-hyperoperation on A*. Denote by $Hp_A^{(n)}$ the set of all partial n-hyperoperations on A and by Hp_A the set of all partial hyperoperations on A, i.e. $Hp_A = \bigcup\limits_{n\geqslant 0} Hp_A^{(n)}$. A map f from A^n to $P(A)\setminus\{\emptyset\}$ is called a *hyperoperation* [5], and the set of all hyperoperations is denoted by H ($H \subseteq Hp_A$).

Every n-ary operation f from A^n to A can be viewed as a special partial hyperoperation (if we do not make difference between an element $a \in A$ and the corresponding one element subset $\{a\}$ of A). In the same sense, partial operations f from $\mathrm{dom}(f)$ to A, where $\mathrm{dom}(f) \subseteq A^n$, are also special partial hyperoperations (if $(x_1,\ldots,x_n) \notin \mathrm{dom}(f)$, for $(x_1,\ldots,x_n) \in A^n$ and $f \in Hp_A^{(n)}$ we can put $f(x_1,\ldots,x_n) = \emptyset$). Namely, $f \in Hp_A$ with $|f(x)| \leqslant 1$ for each $x \in A^n$ is de facto a partial operation on A (if A is a set, then $|A|$ is the cardinality of A). The set of all operations and the set of all partial operations on A are denoted by O_A and P_A, respectively.

For a positive integer n and for $1 \leqslant i \leqslant n$, e_i^n is a *partial n-hyperprojection* if $e_i^n(x_1, \ldots, x_n) = \{x_i\}$ for all $x_1, \ldots, x_n \in A$. The set of all hyperprojections is denoted by J_A.

For positive integers n and m, $f \in Hp_A^{(n)}$ and $g_1, \ldots, g_n \in Hp_A^{(m)}$, *the composition of f and $g_1, \ldots, g_n$*, denoted by $f(g_1, \ldots, g_n) \in Hp_A^{(m)}$, is defined by $f(g_1, \ldots, g_n)(x_1, \ldots, x_m) = \bigcup\{f(y_1, \ldots, y_n)\colon\ y_i \in g_i(x_1, \ldots, x_m),\ 1 \leqslant i \leqslant n\}$ for each $(x_1, \ldots, x_m) \in A^m$.

The set $C \subseteq Hp_A$ is a *clone of partial hyperoperations* on A or a *partial hyperclone* if C is composition closed and C contains all partial n-hyperprojections for each positive integer n.

For $F \subseteq Hp_A$, $\langle F\rangle$ stands for the clone of partial hyperoperations generated by F. The set F of partial hyperoperations is *complete* if $\langle F\rangle = Hp_A$.

A partial hyperclone C_1 on A is *covered by* a partial hyperclone C_2 if $C_1 \subset C \subset C_2$ holds for no partial hyperclone C. A maximal partial hyperclone on A is a partial hyperclone covered by Hp_A.

We say that an operation $f \in O_A^{(n)}$ *depends on* its i-th variable if there are $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$ such that $h \in O_A^{(1)}$, defined by $h(x) := f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ for every $x \in A$, is non-constant. An n-ary operation $f \in O_A^{(n)}$ on A is *essential* if it depends on at least two variables and $\operatorname{im} f = A$.

2. Results

Let

$$\begin{aligned} H_A &= \bigcup_{n\in\mathbb{N}} \{f \in Hp_A^{(n)}\colon\ |f(\mathbf{x})| \geqslant 1 \text{ for every } \mathbf{x} \in A^n\},\\ M &= \Big\langle \bigcup_{n\in\mathbb{N}} \{f \in Hp_A^{(n)}\colon\ |f(\mathbf{x})| < 1 \text{ for every } \mathbf{x} \in A^n\} \Big\rangle,\\ &= \bigcup_{n\in\mathbb{N}} \{f\colon\ A^n \to \{\emptyset\}\} \cup J_A,\\ O_A &= \bigcup_{n\in\mathbb{N}} \{f \in Hp_A^{(n)}\colon\ |f(\mathbf{x})| = 1 \text{ for every } \mathbf{x} \in A^n\} \end{aligned}$$

and

$$P_A = \bigcup_{n\in\mathbb{N}} \{f \in Hp_A^{(n)}\colon\ |f(\mathbf{x})| \leqslant 1 \text{ for every } \mathbf{x} \in A^n\}.$$

It is clear that these sets are clones of partial hyperoperations.

The next lemma follows from [3].

Lemma 2.1. *The clone O_A is covered by $O_A \cup M$ and the clone $O_A \cup M$ is covered by P_A.*

Lemma 2.2. *If $f \in H_A \setminus O_A$, then $\langle O_A \cup f\rangle = H_A$, i.e. the clone O_A is covered by H_A.*

Proof. If for an arbitrary $n \geqslant 1$, $f \in (H_A)^{(n)} \setminus O_A$, then there is at least one n-tuple $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ such that $f(\mathbf{a}) = \{c_0, \ldots, c_{p-1}\}$ and $p \geqslant 2$.

The statement $\langle O_A \cup \{f\}\rangle \subseteq H_A$ is obvious because H_A is a partial hyperclone and $f \in H_A$ and $O_A \subseteq H_A$. Now, we shall prove that the statement $\langle O_A \cup \{f\}\rangle \supseteq H_A$ is also correct. Let h be an arbitrary m-ary hyperoperation from H_A. Let us define maps $f_1, \ldots, f_n \in O_A^{(m)}$ and $g \in O_A^{(l+m)}$ in the following way. If $h(y_1, \ldots, y_m) = \{d_0, d_1, \ldots, d_{q-1}\}$ for some $q \geqslant 1$, then $(f_1(y_1, \ldots, y_m), \ldots, f_n(y_1, \ldots, y_m)) = (\{a_1\}, \ldots, \{a_n\})$ and

$$\begin{aligned} g(y_1, \ldots, y_m, c_0, \ldots, c_0, c_0) &= \{d_0\} \\ g(y_1, \ldots, y_m, c_0, \ldots, c_0, c_1) &= \{d_1\} \\ &\vdots \\ g(y_1, \ldots, y_m, c_{p-1}, \ldots, c_{p-1}, c_{p-1}) &= \{d_{q-1}\} \end{aligned}$$

where $l \in \mathbb{N}$ is the number such that $p^{l-1} < \max\limits_{(y_1,\ldots,y_m)\in A^m} |h(y_1, \ldots, y_m)| \leqslant p^l$.

Precisely, $g(y_1, \ldots, y_m, c_{i_1}, \ldots, c_{i_{l-1}}, c_{i_l}) = \{d_i\}$ where

$$i = \begin{cases} i_1p^{l-1} + i_2p^{l-2} + \ldots + i_lp^0 & \text{if } i_1p^{l-1} + i_2p^{l-2} + \ldots + i_lp^0 \leqslant q-1, \\ q-1 & \text{else.} \end{cases}$$

Now, we can prove that $h = g\bigl(e_1^m, \ldots, e_m^m, f(f_1, \ldots, f_n), \ldots, f(f_1, \ldots, f_n)\bigr)$, which implies $h \in \langle O_A \cup f\rangle$. For $h(y_1, \ldots, y_m) = \{d_0, d_1, \ldots, d_{q-1}\}$, we have $g\bigl(e_1^m, \ldots, e_m^m, f(f_1, \ldots, f_n), \ldots, f(f_1, \ldots, f_n)\bigr)(y_1, \ldots, y_m) = g(\{y_1\}, \ldots, \{y_m\}, \{c_0, \ldots, c_{p-1}\}, \ldots, \{c_0, \ldots, c_{p-1}\}) = g(y_1, \ldots, y_m, c_0, \ldots, c_0) \cup g(y_1, \ldots, y_m, c_0, \ldots, c_0, c_1) \cup \ldots \cup g(y_1, \ldots, y_m, c_{p-1}, \ldots, c_{p-1}, c_{p-1}) = \{d_0, d_1, \ldots, d_{q-1}\} = h(y_1, \ldots, y_m)$. □

Lemma 2.3. *If $f \in M$, then $\langle H_A \cup \{f\}\rangle = H_A \cup M$, i.e. the clone H_A is covered by $H_A \cup M$.*

Proof. It is obvious that $\langle H_A \cup \{f\}\rangle \subseteq H_A \cup M$.

We only have to prove that for an arbitrary $h \in M$, $h \in \langle H_A \cup \{f\}\rangle$ holds. It is easy to see that $h = f(g_1, \ldots, g_n)$ because $f(g_1, \ldots, g_n)(y_1, \ldots, y_m) = \emptyset$ for each $g_1, \ldots, g_n \in H_A^{(m)}$. □

Corollary 2.1. *If $f \in H_A \setminus O_A$, then $\langle O_A \cup M \cup \{f\}\rangle = H_A \cup M$, i.e. the clone $O_A \cup M$ is covered by $H_A \cup M$.*

Lemma 2.4. *If $f \in Hp_A \setminus (P_A \cup H_A)$, then $\langle O_A \cup \{f\}\rangle = Hp_A$.*

Proof. It is obvious that $\langle O_A \cup \{f\}\rangle \subseteq Hp_A$. It remains to prove that $Hp_A \subseteq \langle O_A \cup \{f\}\rangle$.

Since $f \in Hp_A \setminus (P_A \cup H_A)$, there is $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ such that $f(\mathbf{a}) = \emptyset$ and $\mathbf{b} = (b_1, \ldots, b_n) \in A^n$ such that $|f(\mathbf{b})| \geqslant 2$. We can suppose that $f(\mathbf{b}) = f(b_1, \ldots, b_n) = \{c_0, \ldots, c_{p-1}\}$ and $p \geqslant 2$.

Let h be an arbitrary map from $Hp_A^{(m)}$.

Let us define $f_1, \ldots, f_n \in O_A^{(m)}$ and $e \in O_A^{(m+l)}$ as follows. If $h(\mathbf{y}) = \emptyset$, where $\mathbf{y} = (y_1, \ldots, y_m) \in A^n$, then $(f_1(\mathbf{y}), \ldots, f_n(\mathbf{y})) = (\{a_1\}, \ldots, \{a_n\})$. If $h(\mathbf{y}) = \{d_0, d_1, \ldots, d_{q-1}\}$ for some $q \geqslant 1$, then $(f_1(\mathbf{y}), \ldots, f_n(\mathbf{y})) = (\{b_1\}, \ldots, \{b_n\})$ and $e(y_1, \ldots, y_m, c_{i_1}, \ldots\ c_{i_{l-1}}, c_{i_l}) = \{d_i\}$, where $l \in \mathbb{N}$ satisfies $p^{l-1} < \max\limits_{(y_1,\ldots,y_m)\in A^m} |h(\mathbf{y}| \leqslant p^l$ and

$$i = \begin{cases} i_1p^{l-1} + i_2p^{l-2} + \ldots + i_lp^0 & \text{if } i_1p^{l-1} + i_2p^{l-2} + \ldots + i_lp^0 \leqslant q-1, \\ q-1 & \text{else.} \end{cases}$$

Now, we can prove that $h = e(e_1^m, \ldots, e_m^m, f(f_1, \ldots, f_n) \ldots, f(f_1, \ldots, f_n))$, i.e. h belongs to $\langle O_A \cup \{f\}\rangle$.

For $h(y_1, \ldots, y_m) = \emptyset$ the statement is obvious and for $h(y_1, \ldots, y_m) = \{d_0, d_1, \ldots, d_{q-1}\}$, for some $q \geqslant 1$, we have $e\big(e_1^m, \ldots, e_m^m, f(f_1, \ldots, f_n), \ldots, f(f_1, \ldots, f_n)\big)(y_1, \ldots, y_m) = e\big(\{y_1\}, \ldots, \{y_m\}, f(\mathbf{a}), \ldots, f(\mathbf{a})\big) = e(\{y_1\}, \ldots, \{y_m\}, \{c_0, \ldots, c_{p-1}\}, \ldots, \{c_0, \ldots, c_{p-1}\}) = e(y_1, \ldots, y_m, c_0, \ldots, c_0) \cup e(y_1, \ldots, y_m, c_0, \ldots, c_0, c_1) \cup \ldots \cup e(y_1, \ldots, y_m, c_{p-1}, \ldots, c_{p-1}, c_{p-1}) = \{d_0, d_1, \ldots, d_{q-1}\} = h(y_1, \ldots, y_m)$. □

Lemma 2.5. *If $f \in Hp_A \setminus (H_A \cup M)$, then $\langle H_A \cup M \cup \{f\}\rangle = Hp_A$, i.e. the clone $H_A \cup M$ is covered by Hp_A.*

Proof. If $f \in P_A$, the statement is a consequence of Lemma 2.1, or else (if $f \notin P_A$) of Lemma 2.4. □

Lemma 2.6. *If $f \in Hp_A \setminus P_A$, then $\langle P_A \cup \{f\}\rangle = Hp_A$, i.e. P_A is covered by Hp_A.*

Proof. If $f \in H_A$, then the statement follows from Lemma 2.2. Otherwise, if $f \notin H_A$, then it follows from Lemma 2.4. □

Theorem 2.1. *The structure of the interval*

$$[O_A, Hp_A] = \{O_A, H_A, O_A \cup M, H_A \cup M, P_A, Hp_A\}$$

is described in the following figure.

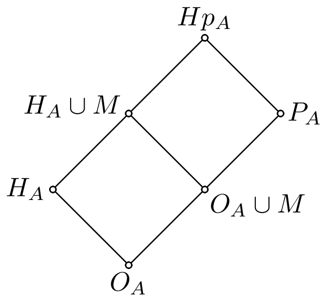

Figure. The interval $[O_A, Hp_A]$.

Theorem 2.2 (Slupecki-type criterion)**.** *Let A be finite. If $F \subseteq Hp_A$ satisfies the following three conditions:* (i) *F contains an essential operation,* (ii) *F generates all unary operations and* (iii) *F contains the partial hyperoperation $f \in Hp_A \backslash (P_A \cup H_A)$, then F is complete.*

P r o o f. By Slupecki criterion, (i) and (ii) imply $O_A \subseteq \langle F \rangle$. From (iii) and Lemma 2.4 we obtain $Hp_A = \langle O_A \cup \{f\} \rangle \subseteq \langle F \rangle \subseteq Hp_A$, i.e. $\langle F \rangle = Hp_A$. □

Corollary 2.2. *Let A be finite and $f \in Hp_A$. Then $\{f\}$ is complete iff $f \in Hp_A \setminus (P_A \cup H_A)$ and $\langle \{f\} \rangle$ contains all unary operations and at least one essential operation.*

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