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# NUMERICAL SEMIGROUPS WITH A MONOTONIC APÉRY SET

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*Abstract.* We study numerical semigroups $S$ with the property that if $m$ is the multiplicity of $S$ and $w(i)$ is the least element of $S$ congruent with $i$ modulo $m$, then $0 < w(1) < \ldots < w(m-1)$. The set of numerical semigroups with this property and fixed multiplicity is bijective with an affine semigroup and consequently it can be described by a finite set of parameters. Invariants like the gender, type, embedding dimension and Frobenius number are computed for several families of this kind of numerical semigroups.

*Keywords*: 20M14, 13H10

*MSC 2000*: numerical semigroups, Apéry sets, symmetric numerical semigroups, affine semigroups, proportionally modular Diophantine inequality

## 0. Introduction

A *numerical semigroup* is a subset of $\mathbb{N}$ closed under addition, containing the zero element and generating $\mathbb{Z}$ as a group (here $\mathbb{N}$ and $\mathbb{Z}$ denote the set of nonnegative integers and the set of integers, respectively). If $S$ is a numerical semigroup, it can be deduced that the set $\mathrm{H}(S) = \mathbb{N} \setminus S$ is finite (see for instance [3] or [20]; the elements of $\mathrm{H}(S)$ are the gaps of $S$). Thus it makes sense to consider the greatest element in $\mathbb{Z}$ not belonging to $S$, which is usually called the *Frobenius number* and we will denote it by $\mathrm{g}(S)$. The tight relation between numerical semigroups and monomial curves (see for instance [4], [6], [8], [13], [14], [24]) made the terminology used in Algebraic Geometry be translated to numerical semigroups. Along this line, the least positive integer belonging to a numerical semigroup $S$ is its *multiplicity*, denoted by $\mathrm{m}(S)$. Given $n \in S \setminus \{0\}$, the *Apéry set* (called so after [2]) of $S$ with

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respect to $n$ is defined by $\mathrm{Ap}(S,n) = \{s \in S\colon\ s - n \notin S\}$ and it can be proved that if we choose $w(i)$ to be the least element in $S$ congruent with $i$ modulo $n$, then $\mathrm{Ap}(S,n) = \{0, w(1), \ldots, w(n-1)\}$. The set $\mathrm{Ap}(S,n)$ determines completely the semigroup $S$, since $S = \langle \mathrm{Ap}(S,n) \cup \{n\}\rangle$ (here $\langle A\rangle$ denotes the monoid generated by $A$). Moreover, $\mathrm{Ap}(S,n)$ contains in general more information that an arbitrary set of generators of $S$; for instance, $\mathrm{g}(S) = \max(\mathrm{Ap}(S,n)) - n$, and for every $s \in S$ there exist unique $k \in \mathbb{N}$ and $w \in \mathrm{Ap}(S,n)$ such that $s = kn + w$. Thus one could say that the best way to describe a numerical semigroup is by means of the Apéry set of any of its elements, and of course the element in $S$ for which the Apéry set has the least possible number of elements is $\mathrm{m}(S)$.

We say that a numerical semigroup $S$ has a *monotonic Apéry set* if $w(1) < w(2) < \ldots < w(\mathrm{m}(S)-1)$, with $\{0, w(1), \ldots, w(\mathrm{m}(S)-1)\} = \mathrm{Ap}(S, \mathrm{m}(S))$. Our main goal in this paper is to study the set $\mathscr{C}(m)$ of numerical semigroups with a monotonic Apéry set and multiplicity $m$. We show that there is a one-to-one correspondence between $\mathscr{C}(m)$ and a finitely generated subsemigroup of $\mathbb{N}^{m-1}$. This semigroup will be denoted by $\mathscr{A}(m)$ and will be called the affine semigroup associated to $\mathscr{C}(m)$. It turns out that $\mathscr{A}(m) \cup \{(0,\ldots,0)\}$ is a finitely generated commutative monoid and that its minimal system of generators can be explicitly computed as explained in [21]. This will permit us to describe completely $\mathscr{C}(m)$ as shown in Section 1, that is, knowing a system of generators for $\mathscr{A}(m)$ allows us to know the whole set of numerical semigroups with a monotonic Apéry set and multiplicity $m$.

A particular kind of numerical semigroup of special interest for Algebraic Geometry is that of symmetric semigroups, since the semigroup rings associated to them are Gorenstein (see [14]). A numerical semigroup is *symmetric* provided that for every $x \in \mathbb{Z}$, if $x \notin S$, then $\mathrm{g}(S) - x \in S$. If we denote $\mathscr{C}_{\mathrm{sy}}(m) = \{S \in \mathscr{C}(m)\colon\ S \text{ is symmetric}\}$, then we show that $\mathscr{C}_{\mathrm{sy}}(m)$ is isomorphic to a subsemigroup of $\mathscr{A}(m)$ which we denote by $\mathscr{A}_{\mathrm{sy}}(m)$. We prove that if $A$ is a system of generators for $\mathscr{A}(m)$, then $\mathscr{A}_{sy}(m)$ is generated by the elements in $A$ belonging to $\mathscr{A}_{\mathrm{sy}}(m)$, which allows us to use the results and description obtained for $\mathscr{C}(m)$ in order to describe $\mathscr{C}_{\mathrm{sy}}(m)$.

Some families of numerical semigroups with monotonic Apéry sets are given in Section 3. Given two integers $a$ and $b$ (with $b \neq 0$), we denote by $a \pmod b$ the remainder of the division of $a$ by $b$. The set $\{x \in \mathbb{N}\colon\ ax \pmod b \leqslant cx\}$ with $a$, $b$, $c$ positive integers turns out to be a numerical semigroup. In the last section we deal with the numerical semigroups with monotonic Apéry set that are solutions to Diophantine inequalities of the form $ax \pmod b \leqslant cx$ for some positive integers $a$, $b$ and $c$. This study yields other examples and families of numerical semigroups with monotonic Apéry sets for which we explicitly compute their type, Frobenius number and gender (number of gaps).

The following notation will be used throughout the paper. For a rational number $x$, $\lceil x\rceil$ denotes the least integer greater than or equal to $x$, and $\lfloor x\rfloor$ the greatest integer less than or equal to $x$. We denote the fact $m \mid (a-b)$ by $a \equiv b \pmod{m}$. If $a$ and $b$ are positive integers, then there exist unique $q$ and $r$ such that $a = qb + r$ with $0 \leqslant r < b$. Hence $q = \lfloor a/b\rfloor$ and, as pointed out above, for $r$ we use the notation $r = a \pmod{b}$.

## 1. The affine semigroup associated to $\mathscr{C}(m)$

The main result in this section is Theorem 4, and for its proof we need three lemmas, of which the first is a direct consequence of [17, Lemma 3.3].

**Lemma 1.** *Let $m$ be an integer greater than one and let*

$$X = \{0 = w(0), w(1), \ldots, w(m-1)\}$$

*be a subset of $\mathbb{N}$ with $m$ elements such that $w(i) \equiv i \pmod{m}$) and $m < w(i)$ for all $i \in \{1, \ldots, m-1\}$. Let $S$ be the submonoid of $\mathbb{N}$ generated by $X \cup \{m\}$. Then $S$ is a numerical semigroup with multiplicity $m$. Furthermore, $\mathrm{Ap}(S, m) = X$ if and only if for all $i, j \in \{1, \ldots, m-1\}$ there exist $k \in \{0, \ldots, m-1\}$ and $t \in \mathbb{N}$ such that $w(i) + w(j) = w(k) + tm$.*

With this we obtain the other two lemmas (compare with Lemmas 8 and 9 in [21]).

**Lemma 2.** *Let $m$ be an integer greater than one and let $(k_1, \ldots, k_{m-1}) \in \mathbb{N}^{m-1}$ with $1 \leqslant k_1 \leqslant \ldots \leqslant k_{m-1}$ and $k_i + k_j \geqslant k_{i+j}$ for all $i, j \in \{1, \ldots, m-1\}$ such that $i + j \leqslant m-1$. Then there exists a numerical semigroup $S$ with multiplicity $m$ and $\mathrm{Ap}(S, m) = \{0, k_1m + 1, k_2m + 2, \ldots, k_{m-1}m + m - 1\}$.*

Proof. We make use of Lemma 1 with

$$X = \{0 = w(0), k_1m + 1 = w(1), \ldots, k_{m-1}m + m - 1 = w(m-1)\}.$$

Then the monoid $S = \langle X \cup \{m\}\rangle$ is a numerical semigroup of multiplicity $m$. Now we have to check that for $i, j \in \{1, \ldots, m-1\}$ there exist $k \in \{0, \ldots, m-1\}$ and $t \in \mathbb{N}$ such that $w(i) + w(j) = w(k) + tm$. For given $i, j \in \{1, \ldots, m\}$ we distinguish three cases.

(1) If $i + j \leqslant m-1$, then $w(i) + w(j) = tm + w(i+j)$ with $t = k_i + k_j - k_{i+j} \in \mathbb{N}$ (here the condition arises $k_i + k_j \geqslant k_{i+j}$).
(2) If $i + j = m$, then $w(i) + w(j) = tm + w(0)$ with $t = k_i + k_j + 1 \in \mathbb{N}$.

(3) If $i+j > m$, then $i \geqslant i+j-m \geqslant 1$, whence $k_i m+i \geqslant k_{i+j-m}m+i+j-m$ and this leads to $k_i m+i+k_j m+j \geqslant k_{i+j-m}m+i+j-m$. Since $k_i m+i+k_j m+j \equiv k_{i+j-m}m+i+j-m \pmod m$, we deduce that there exist $t \in \mathbb{N}$ such that $w(i)+w(j) = tm+w(i+j-m)$. □

Observe that the condition $1 \leqslant k_1 \leqslant \ldots \leqslant k_{m-1}$ implies that $k_1 m+1 < k_2 m+2 < \ldots < k_{m-1}m+m-1$ and consequently the semigroup $S$ given in Lemma 2 has a monotonic Apéry set, that is, belongs to $\mathscr{C}(m)$. Actually, the next lemma proves that all semigroups with monotonic Apéry sets are of this form.

**Lemma 3.** *Let $S$ be a numerical semigroup with $\mathrm{m}(S) = m$ and*

$$\mathrm{Ap}(S, m) = \{0 = w(0) < w(1) < \ldots < w(m-1)\},$$

*where $w(i) \equiv i \pmod m$ for all $i \in \{0, \ldots, m-1\}$. For $i \in \{1, \ldots, m-1\}$, set $k_i \in \mathbb{N}$ to be the element such that $w(i) = k_i m+i$ (observe that $w(i) > i$, since $i < m$ and $m = \mathrm{m}(S)$). Then $1 \leqslant k_1 \leqslant \ldots \leqslant k_{m-1}$ and $k_i+k_j \geqslant k_{i+j}$ for all $i, j \in \{1, \ldots, m-1\}$ such that $i+j \leqslant m-1$.*

Proof. Since $S$ is a numerical semigroup of multiplicity $m$ and $w(1) \in S \setminus \{0\}$, we have that $w(1) > m$ and thus $k_1 \geqslant 1$. As $w(1) < \ldots < w(m-1)$, we obtain $1 \leqslant k_1 \leqslant \ldots \leqslant k_{m-1}$. Now for $i, j \in \{1, \ldots, m-1\}$ such that $i+j \leqslant m-1$, Lemma 1 states that $w(i)+w(j) = tm+w(l)$ for some $t \in \mathbb{N}$ and $l \in \{0, \ldots, m-1\}$. Observe that $w(i+j) \equiv i+j \equiv w(i)+w(j) \equiv w(l) \pmod m$ and this forces $l$ to be $i+j$, whence $k_i+k_j \geqslant k_{i+j}$. □

With these lemmas we have proved the following result.

**Theorem 4.** *Let*

$$\mathscr{C}(m) = \left\{ \begin{matrix} S \text{ numerical} \\ \text{semigroup} \end{matrix} \;\middle|\; \begin{matrix} \mathrm{m}(S) = m, \\ S \text{ has a monotonic Apéry set} \end{matrix} \right\}$$

*and*

$$\mathscr{A}(m) = \left\{ (k_1, \ldots, k_{m-1}) \in \mathbb{N}^{m-1} \;\middle|\; \begin{matrix} 1 \leqslant k_1 \leqslant \ldots \leqslant k_{m-1}, \\ k_i+k_j \geqslant k_{i+j} \text{ for } 2 \leqslant i+j \leqslant m-1 \end{matrix} \right\}.$$

*Then the map $\varphi\colon \mathscr{A}(m) \to \mathscr{C}(m)$ defined by*

$$\varphi(k_1, \ldots, k_{m-1}) = \langle m, k_1 m+1, \ldots, k_{m-1}m+m-1 \rangle$$

*is bijective. Moreover,*

$$\mathrm{Ap}(\varphi(k_1, \ldots, k_{m-1}), m) = \{0, k_1 m+1, \ldots, k_{m-1}m+m-1\}.$$

Let $S$ be a numerical semigroup with a monotonic Apéry set and multiplicity $m$. If $(k_1,\ldots,k_{m-1})=\varphi^{-1}(S)$, then we can determine some properties of $S$ from the integers $k_1,\ldots,k_{m-1}$. Recall that $\mathrm{H}(S)=\mathbb{N}\setminus S$ and that $\mathrm{g}(S)$ is the greatest integer not in $S$.

**Proposition 5.** *Let $S$ be a numerical semigroup with a monotonic Apéry set and multiplicity $m$. Assume that $\varphi^{-1}(S)=(k_1,\ldots,k_{m-1})$. Then*
(1) $\mathrm{g}(S)=k_{m-1}m-1$,
(2) $\#\mathrm{H}(S)=k_1+\ldots+k_{m-1}$.

Proof. (1) Clearly $\max(\mathrm{Ap}(S,m))=k_{m-1}m+m-1$, whence $\mathrm{g}(S)=k_{m-1}m+m-1-m=k_{m-1}m-1$.

(2) From [23] we know that

$$\begin{aligned}\#\mathrm{H}(S)&=\frac{1}{m}(k_1m+1+\ldots+k_{m-1}m+m-1)-\frac{m-1}{2}\\&=\frac{1}{m}(k_1m+\ldots+k_{m-1}m)+\frac{1}{m}(1+\ldots+m-1)-\frac{m-1}{2}\\&=k_1+\ldots+k_{m-1}+\frac{1}{m}\frac{m(m-1)}{2}-\frac{m-1}{2}\\&=k_1+\ldots+k_{m-1}.\end{aligned}$$

□

An *affine semigroup* is a finitely generated subsemigroup of $\mathbb{N}^k$ for some nonnegative integer $k$. Clearly $\mathscr{A}(m)$ is a subsemigroup of $\mathbb{N}^{m-1}$. The next result shows that it is in fact finitely generated. Assume that $A=\{a_1,\ldots,a_r\}$ is a system of generators of $\mathscr{A}(m)$ with $a_i=(a_{i_1},\ldots,a_{i_{m-1}})$ for $i\in\{1,\ldots,r\}$. Then, if $a\in\mathscr{A}(m)$, there exist $\lambda_1,\ldots,\lambda_r\in\mathbb{N}$ such that $a=\sum\limits_{i=1}^{r}\lambda_ia_i$. Thus, in view of Theorem 4,

$$\mathscr{C}(m)=\Big\{\Big\langle m,\Big(\sum_{i=1}^{r}\lambda_ia_{i_1}\Big)m+1,\ldots,\Big(\sum_{i=1}^{r}\lambda_ia_{i_{m-1}}\Big)m+m-1\Big\rangle:\\ (\lambda_1,\ldots,\lambda_r)\in\mathbb{N}^r\setminus\{0\}\Big\}.$$

**Proposition 6.** *Let $m$ be an integer greater than one. Then $\mathscr{A}(m)$ is finitely generated.*

Proof. Let $M(m)$ be the submonoid of $\mathbb{Z}^{m-1}$ defined by $(x_1,\ldots,x_{m-1})\in M(m)$ if $x_1\geqslant 0$, $x_{i+1}-x_i\geqslant 0$ for all $i\in\{1,\ldots,m-2\}$ and $x_i+x_j-x_{i+j}\geqslant 0$ for all $i,j\in\{1,\ldots,m-1\}$ with $i+j\leqslant m-1$. Let $(x_1,\ldots,x_{m-1})$ be an element of $M(m)$.

Since $x_1 \geqslant 0$ and $x_{i+1} \geqslant x_i$ for all $i \in \{1, \ldots, m-2\}$, we have that $(x_1, \ldots, x_{m-1}) \in \mathbb{N}^{m-1}$. Thus $M(m) \subseteq \mathbb{N}^{m-1}$. Furthermore, if $(x_1, \ldots, x_{m-1}) \in M(m)$ and $x_1 = 0$, then $(x_1, \ldots, x_{m-1}) = (0, \ldots, 0)$ (note that $x_1 + x_1 - x_2 \geqslant 0$ forces $x_2$ to be zero; then we use repeatedly $x_1 + x_i - x_{i+1} \geqslant 0$ to obtain that $(x_1, \ldots, x_{m-1}) = (0, \ldots, 0)$). Hence $M(m) = \mathscr{A}(m) \cup \{0\}$ and in particular this implies that if $M(m)$ is finitely generated as a monoid, then so is $\mathscr{A}(m)$ as a semigroup. By [21, Lemma 5], we know that $M(m)$ is finitely generated, and this concludes the proof. $\square$

In [21] a procedure to find a minimal system of generators of $M(m)$ using slack variables is explained (this can be used together with the method given in [7]). In [1] an alternative method for finding the minimal system of generators of $M(m)$ is given without using slack variables.

An alternative proof of Proposition 6 can be given by using [12, Theorem 15.11].

**Example 7.**

$$M(5) = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{c} x_1 \geqslant 0,\ x_2 \geqslant x_1,\ x_3 \geqslant x_2,\ x_4 \geqslant x_3 \\ 2x_1 \geqslant x_2,\ x_1 + x_2 \geqslant x_3,\ x_1 + x_3 \geqslant x_4 \\ 2x_2 \geqslant x_4 \end{array} \right\}.$$

Computing a minimal system of generators as explained in [21] or [1], we obtain that

$$M(5) = \big\langle (1,2,2,2), (1,1,1,1), (1,1,1,2), (1,2,2,3), (1,1,2,2), (1,2,3,3), (1,2,3,4) \big\rangle.$$

Theorem 4 states that $\mathscr{C}(5)$ consists of all numerical semigroups of the form

$$\langle 5, k_1 5 + 1, k_2 5 + 2, k_3 5 + 3, k_4 5 + 4 \rangle$$

such that $(k_1, k_2, k_3, k_4) \in \mathscr{A}(5)$, or in other words,

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} + \lambda_4 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix} + \lambda_5 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} + \lambda_6 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix} + \lambda_7 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix},$$

for some $(\lambda_1, \ldots, \lambda_7) \in \mathbb{N}^7 \setminus \{0\}$. $\square$

We conclude this section by giving the minimal systems of generators for $\mathscr{A}(m)$ with $m \in \{2, \ldots, 8\}$, that is, we describe the sets of numerical semigroups with monotonic Apéry sets and multiplicity up to 8.

- $\mathscr{A}(2)$ is generated by $\{1\}$,
- $\mathscr{A}(3)$ is generated by $\{(1,1),(1,2)\}$,
- $\mathscr{A}(4)$ is generated by $\{(1,1,1),(1,1,2),(1,2,2),(1,2,3)\}$,
- a system of generators for $\mathscr{A}(5)$ is given in Example 7,
- $\mathscr{A}(6)$ is generated by

$$\begin{aligned}\{&(1,2,2,2,2),(1,1,1,1,1),(1,2,2,2,3),(1,1,1,1,2),(1,1,1,2,2),\\ &(1,2,2,3,3),(1,2,2,3,4),(1,2,3,3,3),(1,1,2,2,2),(1,1,2,2,3),\\ &(1,2,3,3,4),(1,2,3,4,4),(1,2,3,4,5)\},\end{aligned}$$

- $\mathscr{A}(7)$ is generated by

$$\begin{aligned}\{&(1,2,2,2,2,2),(1,1,1,1,1,1),(1,2,2,2,2,3),(1,1,1,1,1,2),(1,2,2,3,3,3),\\ &(1,1,1,1,2,2),(1,2,2,2,3,4),(1,2,2,3,3,3),(1,1,1,2,2,2),(1,2,2,3,3,4),\\ &(1,2,2,3,4,4),(1,2,3,3,3,3),(1,1,2,2,2,2),(1,2,3,3,3,4),(1,1,2,2,2,3),\\ &(1,2,3,3,4,4),(1,1,2,2,3,3),(1,2,3,3,4,4),(1,2,3,4,4,4),(2,2,3,4,4,5),\\ &(1,2,3,4,4,5),(2,3,4,6,6,8),(2,2,3,4,5,6),(1,2,3,4,5,5),(1,2,3,4,5,6)\},\end{aligned}$$

- $\mathscr{A}(8)$ is generated by

$$\begin{aligned}\{&(1,2,3,3,3,3,3),(1,2,2,2,2,2,2),(1,1,2,2,2,2,2),(1,1,1,1,1,1,1),\\ &(1,2,2,2,2,2,3),(1,1,1,1,1,1,2),(1,2,3,3,3,3,4),(1,1,2,2,2,2,3),\\ &(1,2,2,2,2,3,3),(1,1,1,1,1,2,2),(1,2,2,2,2,3,4),(1,2,3,3,3,4,4),\\ &(1,1,2,2,2,3,3),(1,2,3,3,3,4,5),(1,2,2,2,3,3,3),(1,1,1,1,2,2,2),\\ &(1,2,2,2,3,3,4),(1,2,2,2,3,4,4),(1,2,3,3,4,4,4),(1,1,2,2,3,3,3),\\ &(1,2,3,3,4,4,5),(1,1,2,2,3,3,4),(1,2,3,3,4,5,5),(1,2,3,3,4,5,6),\\ &(1,2,3,4,4,4,4),(1,2,2,3,3,3,3),(1,1,1,2,2,2,2),(1,2,2,3,3,3,4),\\ &(1,1,1,2,2,2,3),(1,2,3,4,4,4,5),(2,2,3,4,4,6,6),(1,2,2,3,3,4,4),\\ &(1,2,2,3,3,4,5),(1,2,3,4,4,5,5),(1,2,3,4,4,5,6),(2,3,4,6,6,8,8),\\ &(2,3,4,6,6,8,9),(2,2,3,4,5,6,6),(2,2,3,4,5,6,7),(1,2,2,3,4,4,4),\\ &(1,2,2,3,4,4,5),(1,2,3,4,5,5,5),(1,2,3,4,5,5,6),(1,2,3,4,5,6,6),\\ &(1,2,3,4,5,6,7)\}.\end{aligned}$$

## 2. The symmetric elements of $\mathscr{C}(m)$

If $S$ is an element of $\mathscr{C}(m)$ and $\mathrm{Ap}(S,m)=\{0<k_1m+1<\ldots<k_{m-1}m+m-1\}$, then from [18, Lemma 1.1] we deduce that $S$ is symmetric if and only if $k_i+k_{m-1-i}=k_{m-1}$ for all $i\in\{1,\ldots,m-2\}$. As a consequence of this fact and Theorem 4 we obtain the following result.

**Proposition 8.** *Let $m$ be an integer greater than one, let $\mathscr{C}_{\mathrm{sy}}(m)$ be the subset of symmetric semigroups of $\mathscr{C}(m)$ and $\mathscr{A}_{\mathrm{sy}}$ the set of elements in $\mathscr{A}(m)$ such that $k_i+k_{m-1-i}=k_{m-1}$ for all $i\in\{1,\ldots,m-2\}$. Then the map $\varphi\colon\ \mathscr{A}_{\mathrm{sy}}(m)\to\mathscr{C}_{\mathrm{sy}}(m)$ defined by*

$$\varphi(k_1,\ldots,k_{m-1})=\langle m,k_1m+1,\ldots,k_{m-1}m+m-1\rangle$$

*is bijective. Moreover,*

$$\mathrm{Ap}(\varphi(k_1,\ldots,k_{m-1}),m)=\{0,k_1m+1,\ldots,k_{m-1}m+m-1\}.$$

Clearly, $\mathscr{A}_{\mathrm{sy}}(m)$ is a subsemigroup of $\mathscr{A}(m)$. Next we show that if $A$ generates $\mathscr{A}(m)$, then $\mathscr{A}_{\mathrm{sy}}(m)\cap A$ generates $\mathscr{A}_{\mathrm{sy}}(m)$ (this in particular means by Proposition 6 that $\mathscr{A}_{\mathrm{sy}}(m)$ is finitely generated). This fact can be deduced from the following result.

**Proposition 9.** *Let $m$ be an integer greater than one and let $x=(x_1,\ldots,x_{m-1})$, $y=(y_1,\ldots,y_{m-1})$ be in $\mathscr{A}(m)$. Then $x+y\in\mathscr{A}_{\mathrm{sy}}(m)$ implies $x,y\in\mathscr{A}_{\mathrm{sy}}(m)$.*

Proof. If $x\notin\mathscr{A}_{\mathrm{sy}}(m)$, then $x_i+x_{m-1-i}>x_{m-1}$ (recall that $x,y\in\mathscr{A}(m)$) for some $i\in\{1,\ldots,m-2\}$, whence $x_i+y_i+x_{m-1-i}+y_{m-1-i}>x_{m-1}+y_{m-1}$ and thus $x+y\notin\mathscr{A}_{\mathrm{sy}}(m)$. □

**Example.** A system of generators for $\mathscr{A}(5)$ is

$$A=\{(1,2,2,2),(1,1,1,1),(1,1,1,2),(1,2,2,3),(1,1,2,2),(1,2,3,3),(1,2,3,4)\}.$$

Hence $A\cap\mathscr{A}_{\mathrm{sy}}(5)=\{(1,1,1,2),(1,2,3,4)\}$. By Proposition 8 this means that the symmetric numerical semigroups with monotonic Apéry sets and multiplicity 5 are of the form

$$\langle 5,k_15+1,k_25+2,k_35+3,k_45+4\rangle$$

with

$$\begin{pmatrix}k_1\\k_2\\k_3\\k_4\end{pmatrix}=\lambda_1\begin{pmatrix}1\\1\\1\\2\end{pmatrix}+\lambda_2\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

for some $(\lambda_1,\lambda_2)\in\mathbb{N}^2\setminus\{0\}$.

As we did before, we now give the minimal generating sets for $\mathscr{A}_{\mathrm{sy}}(m)$ with $m \in \{2, \ldots, 8\}$.

(1) $\mathscr{A}_{\mathrm{sy}}(2)$ is generated by $\{1\}$,
(2) $\mathscr{A}_{\mathrm{sy}}(3)$ is generated by $\{(1,2)\}$,
(3) $\mathscr{A}_{\mathrm{sy}}(4)$ is generated by $\{(1,1,2),(1,2,3)\}$,
(4) $\mathscr{A}_{\mathrm{sy}}(5)$ is generated by $\{(1,1,1,2),(1,2,3,4)\}$,
(5) $\mathscr{A}_{\mathrm{sy}}(6)$ is generated by $\{(1,1,1,1,2),(1,2,2,3,4),(1,1,2,2,3),(1,2,3,4,5)\}$,
(6) $\mathscr{A}_{\mathrm{sy}}(7)$ is generated by $\{(1,1,1,1,1,2),(1,2,2,2,3,4),(1,2,3,4,5,6)\}$,
(7) $\mathscr{A}_{\mathrm{sy}}(8)$ is generated by

$$\big\{(1,1,1,1,1,1,2),(1,2,2,2,2,3,4),(1,1,2,2,3,3,4),(1,2,3,3,4,5,6),\\ (1,1,1,2,2,2,3),(1,2,2,3,3,4,5),(1,2,3,4,5,6,7)\big\}.$$

## 3. Some families of numerical semigroups with monotonic Apéry sets

In this section we study some families of numerical semigroups with monotonic Apéry sets, computing explicitly some relevant invariants for them. The (Cohen-Macaulay) type of a numerical semigroup is one of these invariants. We recall its definition.

Let $S$ be a numerical semigroup with multiplicity $m$. The *type* of $S$ is the cardinality of the set

$$\mathrm{T}(S) = \{x \in \mathbb{Z} \setminus S\colon\ x + s \in S \quad \text{for all}\ \ s \in S \setminus \{0\}\}$$

and we will denote it by $\mathrm{t}(S)$. Observe that $\mathrm{g}(S)$ is always in $\mathrm{T}(S)$. It can be shown that $S$ is symmetric if and only if $\mathrm{t}(S) = 1$ (see for instance [3]).

On $S$ we can define the following order relation: for $a, b \in S$, $a \leqslant_S b$ if $b - a \in S$. It is straightforward to prove (see for instance [19]) that

$$\mathrm{T}(S) = \{x - m\colon\ x \in \mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m))\}$$

and therefore

$$\mathrm{t}(S) = \#\,\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)).$$

The *embedding dimension* of a numerical semigroup $S$, denoted by $\mathrm{e}(S)$, is the cardinality of its minimal system of generators.

A numerical semigroup $S$ is *arithmetic* if there exist positive integers $a$ and $b$ such that $S = \langle a, a+1, \ldots, a+b\rangle$. The next result proves that every arithmetic semigroup has a monotonic Apéry set. Recall that $\varphi$ was the bijection defined in Theorem 4. In the sequence $(n_1, \dots, n_{k-1}, n, \overset{(a}{\dots}, n, n_{k+a}, \dots, n_p)$, the notation $n, \overset{(a}{\dots}, n$ is used to express that $n$ appears $a$times.

**Proposition 11.** *Let $S$ be a numerical semigroup generated by $\{a, a+1, \dots, a+b\}$ for some positive integers $a$ and $b$ such that $b < a$. Then $S$ has a monotonic Apéry set,*

$$\varphi^{-1}(S) = (1, \overset{(b}{\dots}, 1, 2, \overset{(b}{\dots}, 2, \dots, \lfloor (a-1)/b \rfloor, \overset{(b}{\dots}, \lfloor (a-1)/b \rfloor, \\ \lfloor (a-1)/b \rfloor + 1, \overset{(a-1) \pmod b}{\dots\dots\dots}, \lfloor (a-1)/b \rfloor + 1)$$

*and*

(1) $\mathrm{e}(S) = b + 1$,
(2) $\#\mathrm{H}(S) = \frac{1}{2}(\lfloor (a-1)/b \rfloor + 1)(a - 1 + (a-1) \pmod b)$,
(3) $\mathrm{g}(S) = \lceil (a-1)/b \rceil a - 1$,
(4) $\mathrm{t}(S) = \begin{cases} b & \text{if } b \mid a-1, \\ (a-1) \pmod b & \text{otherwise.} \end{cases}$

P r o o f. Since $a + i \notin \langle \{a, a+1, \dots, a+b\} \setminus \{a+i\} \rangle$ for all $i \in \{0, \dots, b\}$, the set $\{a, a+1, \dots, a+b\}$ is the minimal system of generators of $S$. Thus $\mathrm{e}(S) = b + 1$. By [11, Corollary 4], we know that

$$\mathrm{Ap}(S, a) = \{0, a+1, \dots, a+b, 2a+b+1, \dots, 2a+2b, \\ \dots, ta + (t-1)b + 1, \dots, ta + tb, (t+1)a + tb + 1, \dots, (t+1)a + tb + r\}$$

where $t = \lfloor (a-1)/b \rfloor$ and $r = (a-1) \pmod b$. Hence $S$ has a monotonic Apéry set. Moreover, it follows easily that

$$\varphi^{-1}(S) = (1, \overset{(b}{\dots}, 1, 2, \overset{(b}{\dots} 2, \dots, \lfloor (a-1)/b \rfloor, \overset{(b}{\dots}, \lfloor (a-1)/b \rfloor, \\ \lfloor (a-1)/b \rfloor + 1, \overset{((a-1) \pmod b)}{\dots\dots\dots}, \lfloor (a-1)/b \rfloor + 1).$$

By using this fact and Proposition 5, after some algebraic manipulation, we get (2). Note that (1) is trivial, and that (3) follows again by Proposition 5 (this is in fact a well known result; see for instace [5] and the references given in [15], [16]).

Finally, (4) can be found in [9, Corollary 5], and can be obtained by taking into account that $\mathrm{t}(S) = \#\,\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S, a))$. □

If $S$ is a numerical semigroup with multiplicity $m$, then the $(m-1)$-tuple $\varphi^{-1}(S)$ completely determines $S$. Some $(m-1)$-tuples trivially fulfill the conditions required for $S$ to have a monotonic Apéry set. Next we pick two families of these "simple" $(m-1)$-tuples and compute several invariants for the resulting numerical semigroupss.

**Proposition 12.** *Let $m$ and $a$ be positive integers such that $a \leqslant (m-2)/2$ and let*

$$S = \langle m, m+1, \dots, m+a, 2m+2a+1, \dots, 2m+(m-1)\rangle.$$

*Then $S$ is a numerical semigroup with a monotonic Apéry set,*

$$\varphi^{-1}(S) = (1, \overset{(a}{\dots}, 1, 2, \overset{(m-a-1}{\dots\dots}, 2)$$

*and*
(1) $\mathrm{e}(S) = m - a$,
(2) $\#\mathrm{H}(S) = 2m - a - 2$,
(3) $\mathrm{g}(S) = 2m - 1$,
(4) $\mathrm{t}(S) = m - a - 1$.

P r o o f. Clearly

$$\mathrm{Ap}(S, m) = \{0, m+1, \dots, m+a, 2m+a+1, \dots, 2m+2a, 2m+2a+1, \dots, 2m+(m-1)\}.$$

Hence $S$ is a numerical semigroup with a monotonic Apéry set. It is straightforward to see that $\{m, m+1, \dots, m+a, 2m+2a+1, \dots, 2m+(m-1)\}$ is a minimal set of generators for $S$ and consequently $\mathrm{e}(S) = a+1+m-1-(2a+1)+1 = m-a$. From the shape of $\mathrm{Ap}(S,a)$ we have that $\varphi^{-1}(S) = (1, \overset{(a}{\dots}, 1, 2, \overset{(m-a-1}{\dots\dots}, 2)$. By Proposition 5 we get that $\#\mathrm{H}(S) = a + 2(m-a-1) = 2m - a - 2$ and $\mathrm{g}(S) = 2m-1$. Finally, since

$$\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)) = \{2m+a+1, \dots, 2m+(m-1)\},$$

we conclude that $\mathrm{t}(S) = m - 1 - (a+1) + 1 = m - a - 1$. □

**Proposition 13.** *Let $m$ and $a$ be positive integers and let $b$ be a nonnegative integer such that $4a + b \leqslant m - 2$. Set*

$$S = \langle m, m+1, \dots, m+a, 2m+2a+1, \dots, 2m+3a+b, 3m+4a+b+1, \dots, 3m+(m-1)\rangle.$$

*Then $S$ is a numerical semigroup with a monotonic Apéry set,*

$$\varphi^{-1}(S) = (1, \overset{(a}{\dots}, 1, 2, \overset{(2a+b}{\dots\dots}, 2, 3, \overset{(m-1-3a-b}{\dots\dots\dots\dots}, 3)$$

*and*
(1) $\mathrm{e}(S) = m - 2a$,
(2) $\#\mathrm{H}(S) = 3m - 4a - b - 3$,
(3) $\mathrm{g}(S) = 3m - 1$,
(4) $\mathrm{t}(S) = m - 2a - 1$.

P r o o f. The set $\{m, m+1, \ldots, m+a, 2m+2a+1, \ldots, 2m+3a+b, 3m+4a+b+1, \ldots, 3m+(m-1)\}$ is the minimal system of generators of $S$. This implies that $\mathrm{e}(S) = a+1+3a+b-2a+m-1-4a-b = m-2a$. Clearly

$$\mathrm{Ap}(S, m) = \{0, m+1, \ldots, m+a, 2m+a+1, \ldots, 2m+3a+b,\\ 3m+3a+b+1, \ldots, 3m+(m-1)\}.$$

Hence $\varphi^{-1}(S) = (1, \overset{(a}{\ldots}, 1, 2, \overset{(2a+b}{\ldots}, 2, 3, \overset{(m-1-3a-b}{\ldots}, 3)$. From Proposition 5 we deduce that $\#\mathrm{H}(S) = a+2(2a+b)+3(m-1-3a-b) = 3m-4a-b-3$ and $\mathrm{g}(S) = 3m-1$. Since

$$\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S, m)) = \{2m+a+1, \ldots, 2m+2a+b, 3m+3a+b+1, \ldots,\\ 3m+(m-1)\},$$

we conclude that $\mathrm{t}(S) = a+b+m-1-3a-b = m-2a-1$. □

## 4. Proportionally modular and numerical semigroups with monotonic Apéry sets

Recall that for $a$, $b$ and $c$ positive integers, the set $\{x \in \mathbb{N}\colon\ ax \pmod b \leqslant cx\}$ is a numerical semigroup. We say that a numerical semigroup $S$ is *proportionally modular* if there exist positive integers $a$, $b$ and $c$ such that

$$S = \{x \in \mathbb{N}\colon\ ax \pmod b \leqslant cx\}.$$

In this setting, we say that $S$ is given by the Diophantine inequality $ax \pmod b \leqslant cx$. Note that if $S \neq \mathbb{N}$, then we can choose $a > c$. By [22] we know that if $T$ is the submonoid of $\mathbb{R}_0^+$ generated by the closed interval $[b/a, b/(a-c)]$, then $S = T \cap \mathbb{N}$.

The usual way to represent a numerical semigroup is by one of its systems of generators or simply by its minimal system of generators. Proportionally modular numerical semigroups have the advantage that they can be represented by a single inequality, which depends only on three parameters (the integers $a$, $b$ and $c$). The membership problem becomes trivial for these semigroups. However, the problem of finding a formula in terms of these three parameters for the largest integer not belonging to one of these semigroups (its Frobenius number) remains unsolved. In this section we introduce a family of proportionally modular numerical semigroups with monotonic Apéry sets for which a formula for their Frobenius number can be obtained.

**Proposition 14.** *Let $a$, $b$ and $m$ be positive integers. Then*

$$S = \{x \in \mathbb{N} \colon\ (bm + a)x \pmod{m(bm + a)} \leqslant ax\}$$

*is a numerical semigroup with multiplicity $m$ and a monotonic Apéry set.*

Proof. We know that $S = T \cap \mathbb{N}$ with $T$ the submonoid of $\mathbb{R}_0^+$ generated by $[m, m+a/b]$. Hence $S = \bigcup_{k\in\mathbb{N}} ([km, km+ka/b]\cap\mathbb{N})$. It follows that $m$ is the multiplicity of $S$. In order to show that $S$ has a monotonic Apéry set, we prove that if $km+i \in S$ for some $k \in \mathbb{N} \setminus \{0\}$ and $i \in \{2, \ldots, m-1\}$, then $\{km+1, \ldots, km+i\} \subset S$. If $km+i \in S$ for some $k \in \mathbb{N} \setminus \{0\}$ and $i \in \{2, \ldots, m-1\}$, then there exists $l \in \mathbb{N}$ such that $km+i \in [lm, lm+la/b]$. As $i \in \{2, \ldots, m-1\}$, we have that $l \leqslant k$. Hence $\{km+1, \ldots, km+i\} \subseteq [lm, lm+la/b]$. □

**Remark 15.** There are numerical semigroups with monotonic Apéry sets that are not proportionally modular. For instance, let $S = \langle 4, 9, 10, 11\rangle$. Then $\mathrm{Ap}(S, 4) = \{0, 2\times 4+1, 2\times 4+2, 2\times 4+3\}$. By [22] we know that $S$ is not proportionally modular.

**Proposition 16.** *Let $a$, $b$ and $m$ be positive integers and let*

$$S = \{x \in \mathbb{N} \colon\ (bm + a)x \pmod{m(bm + a)} \leqslant ax\}.$$

*Then*

$$\mathrm{Ap}(S, m) = \{0, k_1m + 1, \ldots, k_{m-1}m + m - 1\}$$

*where $k_i = \lceil ib/a\rceil$ for all $i \in \{1, \ldots, m-1\}$.*

Proof. Let $S'$ be the numerical semigroup generated by

$$\{m, k_1m + 1, \ldots, k_{m-1}m + (m-1)\}.$$

As $k_i > 0$ for all $i$, we have that $\mathrm{m}(S') = m$. Since $k_i + k_j \geqslant k_{i+j}$ for all $i, j$ such that $2 \leqslant i + j \leqslant m-1$, in view of Lemma 1 we have that $\mathrm{Ap}(S', m) = \{0, k_1m + 1, \ldots, k_{m-1}m + (m-1)\}$. Now we prove that $S' = S$. The element $x$ belongs to $S'$ if and only if $\lfloor x/m\rfloor \geqslant k_{x \pmod m}$, and this is equivalent to $\lfloor x/m\rfloor \geqslant \lceil (x \pmod m)b/a\rceil$. This last condition can be formulated as $\lfloor x/m\rfloor \geqslant (x \pmod m)b/a$, or as $(x - (x \pmod m))/m \geqslant (x \pmod m)b/a$. Hence $x \in S'$ if and only if $(bm + a)(x \pmod m) \leqslant ax$, or in other words, $(bm + a)x \pmod{m)(bm + a)} \leqslant ax$. □

**Corollary 17.** *Let $a$, $b$ and $m$ be positive integers and let*

$$S = \{x \in \mathbb{N} \colon \; (bm+a)x \pmod{m(bm+a)} \leqslant ax\}.$$

*Then* $\mathrm{g}(S) = \lceil b(m-1)/a \rceil m - 1$.

P r o o f. By Proposition 16 we know that $\max(\mathrm{Ap}(S,m)) = \lceil b(m-1)/a \rceil m + m - 1$. Hence $\mathrm{g}(S) = \lceil b(m-1)/a \rceil m + m - 1 - m$. □

Fixing $a$ and $b$ in Corollary 17, we obtain a family of numerical semigroups with monotonic Apéry sets. Next we show how to compute invariants for these families in a couple of examples.

**Corollary 18.** *Let $m$ be a positive integer greater than 3 and let*

$$S = \langle m, m+1, 2m+3 \rangle.$$

*Then*
(1) $S = \{x \in \mathbb{N} \colon \; (2m+3)x \pmod{m(2m+3)} \leqslant 3x\}$,
(2) $\mathrm{g}(S) = \lceil \frac{2}{3}(m-1) \rceil m - 1$,
(3) $\#\mathrm{H}(S) = \lfloor \frac{2}{3} m^2 \rfloor$,
(4) $\mathrm{t}(S) = 2$.

P r o o f. Let $S' = \{x \in \mathbb{N} \colon \; (2m+3)x \pmod{m(2m+3)} \leqslant 3x\}$. We prove that $S = S'$. By Proposition 16, with $b = 2$ and $a = 3$, we deduce that $\mathrm{Ap}(S', m) = \{0, k_1 m + 1, \ldots, k_{m-1} m + m - 1\}$ where $k_{3i} = 2i$, $k_{3i+1} = 2i + 1$ and $k_{3i+2} = 2i + 2$.

Clearly $S' = \langle m, k_1 m + 1, \ldots, k_{m-1} + m - 1 \rangle$. Moreover, $k_{3i+1} = k_{3i} + k_1$, $k_{3i+2} = k_{3i} + k_2$ and $k_{3i} = k_{3(i-1)} + k_3$. Hence $S' = \langle m, m+1, 2m+2, 2m+3 \rangle = \langle m, m+1, 2m+3 \rangle$. This proves (1).

Statement (2) follows from Corollary 17 with $b = 2$ and $a = 3$.

For (3), we distinguish three cases.

(i) Assume that $m - 1 = 3c$ for some $c \in \mathbb{N} \setminus \{0\}$. In view of Proposition 5, $\#\mathrm{H}(S) = k_1 + \ldots + k_{m-1}$, whence

$$\begin{aligned} \#\mathrm{H}(S) &= k_3 + k_{2\times 3} + \ldots + k_{c\times 3} + k_1 + k_{3\times 1+1} + \ldots + k_{3\times(c-1)+1} \\ &\quad + k_2 + k_{3\times 1+2} + \ldots + k_{3\times(c-1)+2} \\ &= (2 + 2\times 2 + \ldots + 2c) + (1 + 2\times 1 + 1 + \ldots + 2(c-1) + 1) \\ &\quad + (2 + 2\times 1 + 2 + \ldots + 2(c-1) + 2) \\ &= 2(1 + \ldots + c) + c + 2(1 + \ldots + c - 1) + 2c + 2(1 + \ldots + c - 1) \\ &= 5c + 6\frac{c(c-1)}{2} = 3c^2 + 2c = 3\Big(\frac{m-1}{3}\Big)^2 + 2\frac{m-1}{3} = \frac{m^2-1}{3} = \Big\lfloor \frac{m^2}{3} \Big\rfloor. \end{aligned}$$

(ii) Assume that $m-1=3c+1$ for some $c\in\mathbb{N}\setminus 0$. Arguing as in the preceding case, we get that $\#\mathrm{H}(S)=3c^2+4c+1=\frac{1}{3}(m^2-1)=\lfloor\frac{1}{3}m^2\rfloor$.

(iii) Finally, suppose that $m-1=3c+2$ for some $c\in\mathbb{N}\setminus 0$. Then as above we obtain $\#\mathrm{H}(S)=3c^2+6c+3=\frac{1}{3}m^2=\lfloor\frac{1}{3}m^2\rfloor$.

It is well known (see [10]) that if $S$ is a numerical semigroup with $\mathrm{e}(S)=3$ then $\mathrm{t}(S)\in\{1,2\}$, and that $\mathrm{t}(S)=1$ if and only if $S$ is symmetric. Moreover, $S$ is symmetric if and only if $2\#\mathrm{H}(S)=\mathrm{g}(S)+1$ (see for instance [10]). The reader can check that $\lceil\frac{2}{3}(m-1)\rceil m\neq 2\lfloor\frac{1}{3}m^2\rfloor$. This demonstrates (4) and concludes the proof. □

**Corollary 19.** *Let $m$ be a positive integer greater than or equal to six and let*

$$S=\langle m,m+1,m+2,2m+5\rangle.$$

*Then*

(1) $S=\{x\in\mathbb{N}\colon\ (2m+5)x \pmod{m(2m+5)}\leqslant 5x\}$,

(2) $\mathrm{g}(S)=\lceil\frac{2}{5}(m-1)\rceil m-1$,

(3) $\#\mathrm{H}(S)=\lfloor\frac{1}{5}m(m+1)\rfloor$,

(4) $\mathrm{t}(S)\in\{1,2,3\}$ *and*

  (4.1) $\mathrm{t}(S)=1$ *if and only if* $m\equiv 4 \pmod 5$,

  (4.2) $\mathrm{t}(S)=2$ *if and only if* $m\equiv 0 \pmod 5$ *or* $m\equiv 2 \pmod 5$,

  (4.3) $\mathrm{t}(S)=3$ *if and only if* $m\equiv 1 \pmod 5$ *or* $m\equiv 3 \pmod 5$.

P r o o f. Let $S'=\{x\in\mathbb{N}\colon\ (2m+5)x \pmod{m(2m+5)}\leqslant 5x\}$. We prove that $S'=S$. By Proposition 16, taking $b=2$ and $a=5$, we deduce that $\mathrm{Ap}(S',m)=\{0,k_1m+1,\dots,k_{m-1}m+m-1\}$, where

$$k_{5i}=2i,\ \ k_{5i+1}=2i+1,\ \ k_{5i+2}=2i+1,\ \ k_{5i+3}=2i+2,\ \ k_{5i+4}=2i+2.$$

Clearly $S'=\langle m,k_1m+1,\dots,k_{m-1}m+m-1\rangle$ and

$$k_{5i+1}=k_{5i}+k_1,\ \ k_{5i+2}=k_{5i}+k_2,\ \ k_{5i+3}=k_{5i}+k_3,$$
$$k_{5i+4}=k_{5i}+k_4,\ \ k_{5i}=k_{5(i-1)}+k_5.$$

Hence $S'=\langle m,m+1,m+2,2m+3,2m+4,2m+5\rangle=\langle m,m+1,m+2,2m+5\rangle$. This proves (1).

By using Corollary 17 with $b=2$ and $a=5$, we have that $\mathrm{g}(S)=\lceil\frac{2}{5}(m-1)\rceil m-1$, obtaining in this way (2).

In order to prove (3), we distinguish five cases.

(i) Assume that $m-1=5c$. In view of Proposition 5,

$$\begin{aligned}\#\mathrm{H}(S) &= k_1+\ldots+k_{m-1} = k_5+k_{2\times 5}+\ldots+k_{c\times 5}+k_1+k_{5\times 1+1}+\ldots+k_{5(c-1)+1}\\ &\quad + k_2+k_{5\times 1+2}+\ldots+k_{5(c-1)+2}+k_3+k_{5\times 1+3}+\ldots+k_{5(c-1)+3}\\ &\quad + k_4+k_{5\times 1+4}+\ldots+k_{5(c-1)+4}\\ &= (2+2\times 2+\ldots+2c)+(1+2\times 1+1+\ldots+2(c-1)+1)\\ &\quad +(1+2\times 1+1+\ldots+2(c-1)+1)\\ &\quad +(2+2\times 1+2+\ldots+2(c-1)+2)\\ &\quad +(2+2\times 1+2+\ldots+2(c-1)+2)\\ &= 2(1+\ldots+c)+2(c+2(1+\ldots+c-1))+2(2c+2(1+\ldots+c-1))\\ &= 5c^2+3c = 5\left(\frac{m-1}{5}\right)^2+3\frac{m-1}{5} = \frac{m^2+m-2}{5} = \left\lfloor \frac{m(m+1)}{5}\right\rfloor.\end{aligned}$$

(ii) Assume that $m-1=5c+1$ for a positive integer $c$. Arguing as in (i), we obtain that

$$\#\mathrm{H}(S) = 5c^2+5c+1 = \frac{m^2+m-1}{5} = \left\lfloor \frac{m(m+1)}{5}\right\rfloor.$$

(iii) If $m-1=5c+2$ for a positive integer $c$, then

$$\#\mathrm{H}(S) = 5c^2+7c+2 = \frac{m^2+m-2}{5} = \left\lfloor \frac{m(m+1)}{5}\right\rfloor.$$

(iv) For $m-1=5c+3$ with $c$ a positive integer we obtain

$$\#\mathrm{H}(S) = 5c^2+9c+4 = \frac{m^2+m}{5} = \left\lfloor \frac{m(m+1)}{5}\right\rfloor.$$

(v) Finally, in the case $m-1=5c+4$ we get

$$\#\mathrm{H}(S) = 5c^2+11c+6 = \frac{m^2+m}{5} = \left\lfloor \frac{m(m+1)}{5}\right\rfloor.$$

As with (3), the proof of (4) is splitted in five cases, depending on $(m-1) \pmod 5$.

(i) Assume that $m-1=5c+3$ for some $c\in\mathbb{N}\setminus\{0\}$. Note that

$$(*)\qquad \begin{aligned} k_{5c+3} &= k_{5i}+k_{5(c-i)+3} = k_{5i+1}+k_{5(c-i)+2} = k_{5i+2}+k_{5(c-i)+1}\\ &= k_{5i+3}+k_{5(c-i)} = k_{5i+4}+k_{5(c-i-1)+4}.\end{aligned}$$

Hence $\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)) = \{k_{5+3m+m-1}\} = \{k_{m-1}m+m-1\}$ and $\mathrm{t}(S)=1$.

(ii) Assume that $m-1 = 5c+4$ for a positive integer $c$. According to the equality $(*)$ and as $k_{5c+3} = k_{5c+4}$, we deduce that

$$\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)) = \{k_{m-2}m + m - 2, k_{m-1}m + m - 1\}$$

and $\mathrm{t}(S) = 2$.

(iii) If $m-1 = 5c$ for a positive integer $c$, then by using the equality $(*)$ we deduce that

$$\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)) \subseteq \{k_{5(c-1)+3}m + m - 3, k_{5(c-1)+4}m + m - 2, k_{5c}m + m - 1\},$$

and since $k_{5(c-1)+3} = k_{5(c-1)+4} = k_{5c}$ we conclude that

$$\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)) = \{k_{5(c-1)+3}m + m - 3, k_{5(c-1)+4}m + m - 2, k_{5c}m + m - 1\}$$

and $\mathrm{t}(S) = 3$.

(iv) Now assume that $m-1 = 5c+1$ for a positive integer $c$. Proceeding as in the previous cases we obtain

$$\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)) \subseteq \{k_{5(c-1)+3}m + m - 4, k_{5(c-1)+4}m + m - 3, k_{5c}m + m - 2, k_{5c+1}m + m - 1\}.$$

Note that $k_{5(c-1)+4} + k_2 = k_{5c+1}$ and that $k_{5c} + k_1 = k_{5c+1}$, whence

$$\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)) \subseteq \{k_{5(c-1)+3}m + m - 4, k_{5c+1}m + m - 1\}.$$

Taking into account that $k_{5(c-1)+3} = 2(c-1)+2 = 2c$, $k_{5c+1} = 2c+1$ and $(2c+1)m + m - 1 - (2cm + m - 4) = m + 3 \notin S$, we conclude that

$$\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)) = \{k_{5(c-1)+3}m + m - 4, k_{5c+1}m + m - 1\}$$

and consequently $\mathrm{t}(S) = 2$.

(v) Finally, for $m-1 = 5c+2$ with $c$ a positive integer, we deduce that

$$\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)) \subseteq \{k_{5(c-1)+3}m + m - 5, k_{5(c-1)+4}m + m - 4, k_{5c}m + m - 3, k_{5c+1}m + m - 2, k_{5c+2}m + m - 1\}.$$

By using again the equalities $k_{5c} + k_1 = k_{5c+1} = k_{5(c-1)+4}k + k_2$ we obtain that

$$\mathrm{Max}_{\leqslant_S}(\mathrm{Ap}(S,m)) \subseteq \{k_{5(c-1)+3}m + m - 5, k_{5c+1}m + m - 2, k_{5c+2}m + m - 1\}.$$

Arguing as in (iv), we get the equality and therefore $\mathrm{t}(S) = 3$. □

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