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SOME OSCILLATION THEOREMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper we establish some oscillation or nonoscillation criteria for the second order half-linear differential equation

$$(r(t)\Phi(u'(t)))' + c(t)\Phi(u(t)) = 0,$$

where
(i) $r, c \in C([t_0, \infty), \mathbb{R} := (-\infty, \infty))$ and $r(t) > 0$ on $[t_0, \infty)$ for some $t_0 \geqslant 0$;
(ii) $\Phi(u) = |u|^{p-2}u$ for some fixed number $p > 1$.
We also generalize some results of Hille-Wintner, Leighton and Willet.

Keywords: oscillatory, nonoscillatory, Riccati differential equation, Sturm Comparison Theorem

MSC 2000: 34C10, 34C15

0. Introduction

In this paper we discuss the nonoscillatory property of the solutions of the second order linear differential equation

$$(r(t)u'(t))' + c(t)u(t) = 0 \tag{1}$$

and the second order half-linear differential equation

$$(r(t)\Phi(u'(t)))' + c(t)\Phi(u(t)) = 0, \tag{2}$$

where

(i) $r, c \in C([t_0,\infty), \mathbb{R} := (-\infty,\infty))$ and $r(t) > 0$ on $[t_0,\infty)$ for some $t_0 \geqslant 0$;
(ii) $\Phi(u) = |u|^{p-2}u$ for some fixed number $p > 1$.

Clearly, if $p = 2$, then (2) reduces to (1). By a solution of (2) we will mean a real-valued function $u(t)$ which is not identically zero on $[t_0,\infty)$ and satisfies (2).

Equation (1) or (2) is said to be nonoscillatory on $[t_0,\infty)$ if no solution of equation (1) or (2) vanishes more than once in this interval. The equation (1) or (2) will be said to be oscillatory if one (and therefore all) of its solutions have an infinite number of zeros on $[t_0,\infty)$.

Our main concern will be to obtain nonoscillatory (or oscillatory) criteria for equation (1) or (2), that is, conditions on the functions $r(t)$, $c(t)$ and Φ from which conclusions may be drawn as to the nonoscillatory (or oscillatory) character of equation (1) or (2). There exists an extensive literature on this subject, see, for example, [1]–[19]. In [11], Li and Yeh obtained some nonoscillatory criteria for the second order differential equation (1) by using the substitution $w(t) = u(t)/\sqrt{a(t)}$. In this note, we will first use another method which transforms the second order linear differential equation (1) into a Riccati differential equation and then establish a nonoscillatory characterization for equation (1). Using this result, we improve some results from [5], [6], [11], [14], [16], [18], [19] and we also give an alternative proof of the Hille-Wintner Comparison Theorem for equation (1). In the second section, we extend the Leighton oscillation criterion, the Sturm Comparison Theorem and the Hille-Wintner Comparison Theorem from equation (1) to the second order half-linear differential equation (2). For other related results, we refer to [2], [10] and [12].

1. Oscillation criteria for equation (1)

Let $u(t)$ be a solution of (1). Taking into account the Kummer transformation (see [7] or [19]), we define

$$w(t) = \frac{u(t)}{\sqrt{a(t)}} \quad \text{on } [t_0,\infty),$$

where $a(t) \in C^2([t_0,\infty),(0,\infty))$ is a given function. Then (1) is transformed into

$$(a(t)r(t)w'(t))' + \varphi(t)w(t) = 0, \tag{3}$$

where $\varphi(t) := a(t)[c(t) + r(t)f^2(t) - (r(t)f(t))']$ and $f(t) := -a'(t)/2a(t)$. Hence, equations (1), (3) and the following differential equation are equivalent:

$$(a_1(t)a(t)r(t)v'(t))' + a_1(t)[\varphi(t) + a(t)r(t)g^2(t) - (a(t)r(t)g(t))']v(t) = 0, \tag{4}$$

where $a_1(t) \in C^2([t_0,\infty),(0,\infty))$ and $g(t) = -a_1'(t)/2a_1(t)$ on $[t_0,\infty)$.

Using these equivalent relations, Li and Yeh [11] established the following nonoscillatory characterization for equation (1) as follows:

Theorem A. *Equation* (1) *is nonoscillatory if and only if one of the following conditions holds:*

(a) *There exists a function* $f \in C([T,\infty),\mathbb{R})$ *for some* $T \geqslant t_0$ *such that*

$$c(t) + r(t)f^2(t) - (r(t)f(t))' \leqslant 0 \quad \text{on } [T,\infty).$$

(b) *There is a function* $v \in C^1([T,\infty),\mathbb{R})$ *for some* $T \geqslant t_0$ *such that*

$$\varphi(t) + a(t)r(t)v^2(t) - (a(t)r(t)v(t))' \leqslant 0, \quad t \geqslant T,$$

where $a(t) \in C^2([t_0,\infty),(0,\infty))$ *is a given function and* $\varphi(t) = a(t)[c(t)+r(t)f^2(t) - (r(t)f(t))']$.

Clearly, condition (b) is condition (a) if $a(t) = 1$. We also have the following observation:

If $c(t) \leqslant 0$ for t large enough, then equation (1) is nonoscillatory. Suppose that "$c(t) \leqslant 0$ for t large enough" does not hold. If we can find $a, a_1 \in C^2([t_0,\infty),(0,\infty))$ such that the coefficient at $w(t)$ and $v(t)$ in (3) or (4) is nonpositive, then equation (1) is nonoscillatory.

Using Theorem A, Li and Yeh [11] obtained many nonoscillatory criteria for equation (1). In this section we use another method to derive Theorem A. Using this result, we establish some nonoscillatory criteria which generalize some results of [5], [6] and [11]. An alternative proof of the Hille-Wintner Comparison Theorem [14], [15] is also given.

Throughout this section, we assume that $a(t) \in C^2([t_0,\infty),(0,\infty))$ is a given function,

$$\varphi(t) := a(t)[c(t) + r(t)f^2(t) - (r(t)f(t))'] := a(t)\Big(c(t) + \frac{v^2(t)}{r(t)} + v'(t)\Big).$$

Here $f(t) := -a'(t)/2a(t)$ and $v(t) := -r(t)f(t)$.

As stated above, for a given function $a_1 \in C^2([t_0,\infty),(0,\infty))$, the second order differential equation

$$(r_1(t)u'(t))' + c_1(t)u(t) = 0 \tag{5}$$

is equivalent to the second order linear differential equation

$$(a_1(t)r_1(t)w'(t))' + \varphi_1(t)w(t) = 0, \tag{6}$$

where $r_1, c_1 \in C([t_0,\infty),\mathbb{R})$ with $r_1(t) > 0$ on $[t_0,\infty)$,

$$\varphi_1(t) := a_1(t)[c_1(t) + r_1(t)f_1^2(t) - (r_1(t)f_1(t))'].$$

Here $f_1(t) := -a_1'(t)/2a_1(t)$.

In order to prove our main result, we need the following Sturm Comparison Theorem:

Theorem B (Sturm Comparison Theorem). *Let $a(t)r(t) \geqslant a_1(t)r_1(t)$ and $\varphi(t) \leqslant \varphi_1(t)$. If equation (5) is nonoscillatory, then equation (1) is nonoscillatory. That is, if equation (1) is oscillatory, then equation (5) is oscillatory.*

Now, we can state and prove our main result as follows:

Theorem 1. *The following three statements are equivalent:*

(a) *Equation (1) is nonoscillatory.*

(b) *There is a function $v(t) \in C^1([T,\infty),\mathbb{R})$ such that*

$$v'(t) + \varphi(t) + \frac{v^2(t)}{a(t)r(t)} = 0, \quad t \geqslant T$$

for some $T \geqslant t_0$.

(c) *There is a function $v(t) \in C^1([T,\infty),\mathbb{R})$ such that*

$$v'(t) + \varphi(t) + \frac{v^2(t)}{a(t)r(t)} \leqslant 0, \quad t \geqslant T \tag{7}$$

for some $T \geqslant t_0$.

Proof. (a) $\Rightarrow$ (b) If (1) is nonoscillatory and $u(x)$ is a solution of (1) on $[t_0,\infty)$, then there is a number $T \geqslant t_0$ such that $u(x) \neq 0$ on $[T,\infty)$. Let

$$v(t) = a(t)r(t)\Big(\frac{u'(t)}{u(t)} + f(t)\Big), \quad t \geqslant T.$$

Then

$$\begin{aligned} v'(t) &= \Big[a(t)\frac{r(t)u'(t)}{u(t)} + a(t)r(t)f(t)\Big]' \\ &= a(t)\frac{(r(t)u'(t))'}{u(t)} + a'(t)\frac{r(t)u'(t)}{u(t)} - \frac{a(t)r(t)(u'(t))^2}{u^2(t)} \\ &\quad + a'(t)r(t)f(t) + a(t)(r(t)f(t))' \end{aligned}$$

$$
\begin{aligned}
&= -a(t)c(t) - 2a(t)f(t)\frac{r(t)u'(t)}{u(t)} - \frac{1}{a(t)r(t)}\Big(\frac{a(t)r(t)u'(t)}{u(t)}\Big)^2 \\
&\quad - 2a(t)r(t)f^2(t) + a(t)(r(t)f(t))' \\
&= -a(t)[c(t) - (r(t)f(t))' + r(t)f^2(t)] \\
&\quad - \frac{1}{a(t)r(t)}\Big[\Big(\frac{a(t)r(t)u'(t)}{u(t)}\Big)^2 + 2a^2(t)r^2(t)\frac{u'(t)}{u(t)} + (a(t)r(t)f(t))^2\Big] \\
&= -a(t)[c(t) - (r(t)f(t))' + r(t)f^2(t)] \\
&\quad - \frac{1}{a(t)r(t)}\Big[\frac{a(t)r(t)u'(t)}{u(t)} + a(t)r(t)f(t)\Big]^2 \\
&= -a(t)[c(t) - (r(t)f(t))' + r(t)f^2(t)] - \frac{1}{a(t)r(t)}v^2(t),
\end{aligned}
$$

which implies

$$v'(t) + a(t)[c(t) + r(t)f^2(t) - (r(t)f(t))'] + \frac{1}{a(t)r(t)}v^2(t) = 0$$

for $t \geqslant T$. Hence

$$v'(t) + \varphi(t) + \frac{v^2(t)}{a(t)r(t)} = 0, \quad t \geqslant T.$$

(b) $\Rightarrow$ (c) It is clear.

(c) $\Rightarrow$ (a) If there exists a function $v(t)$ satisfying

$$-\varphi_1(t) := v'(t) + \frac{v^2(t)}{a(t)r(t)} \leqslant -\varphi(t) \quad \text{for } t \geqslant T, \tag{8}$$

then

$$w(t) = \exp\Big(\int_T^t \frac{v(s)}{a(s)r(s)}\,\mathrm{d}s\Big) \tag{9}$$

satisfies

$$(a(t)r(t)w'(t))' + \varphi_1(t)w(t) = 0, \quad t \geqslant T.$$

In fact,

$$w'(t) = w(t)\frac{v(t)}{a(t)r(t)},$$

which implies

$$
\begin{aligned}
(a(t)r(t)w'(t))' &= (w(t)v(t))' \\
&= w'(t)v(t) + w(t)v'(t) \\
&= \frac{w(t)v^2(t)}{a(t)r(t)} + w(t)\Big(-\varphi_1(t) - \frac{v^2(t)}{a(t)r(t)}\Big).
\end{aligned}
$$

Thus, (9) is a nonoscillatory solution of

$$(a(t)r(t)w'(t))' + \varphi_1(t)w(t) = 0, \quad t \geqslant T. \tag{10}$$

It follows from (8), (10) and the Sturm Comparison Theorem that equation (3) is nonoscillatory and hence, equation (1) is nonoscillatory. This completes our proof. □

Taking $v(t) = -a(t)r(t)w(t)$, our Theorem 1 reduces to condition (b) of Theorem A.

Corollary 2. *If* $(a(t)r(t))' \leqslant 0$ *for* t *large enough and*

$$\limsup_{t\to\infty} \frac{t^2\varphi(t)}{a(t)r(t)} < \frac{1}{4}, \tag{11}$$

then equation (1) *is nonoscillatory.*

Proof. It follows from (11) that there exist two numbers $T \geqslant t_0$ and $\lambda < \frac{1}{4}$ such that

$$\varphi(t) \leqslant \frac{\lambda r(t)a(t)}{t^2} \quad \text{for } t \geqslant T.$$

Let

$$v(t) = a(t)r(t)h(t),$$

where $h(t) = 1/2t$. Then, for $t \geqslant T$,

$$\begin{aligned}
v'(t) + \varphi(t) + \frac{v^2(t)}{a(t)r(t)} &= (a(t)r(t))'h(t) + a(t)r(t)\Big(\frac{-1}{2t^2}\Big) + \varphi(t) + a(t)r(t)h^2(t) \\
&\leqslant a(t)r(t)\Big(\frac{-1}{2t^2} + \frac{\lambda}{t^2} + \frac{1}{4t^2}\Big) + (a(t)r(t))'h(t) \\
&\leqslant a(t)r(t)\frac{4\lambda - 1}{4t^2} \leqslant 0.
\end{aligned}$$

This and Theorem 1 imply (1) is nonoscillatory. □

Remark 1.
(a) Let $a(t) \equiv 1$, then $\varphi(t) = c(t)$. Thus our Corollary 2 reduces to Theorem 3.5 in [11].
(b) Let $a(t) = r(t) = 1$. Then Corollary 2 reduces to the result of [5], [6].

Corollary 3. *If* $(a(t)r(t))' \leqslant 0$ *for* t *large enough and*

$$\limsup_{t\to\infty} t^2 \log^2 t\Big(\frac{\varphi(t)}{a(t)r(t)} - \frac{1}{4t^2}\Big) < \frac{1}{4}, \tag{12}$$

then equation (1) *is nonoscillatory.*

Proof. It follows from (12) that there exist two numbers $T \geqslant t_0$ and $\lambda < \frac{1}{4}$ such that

$$\varphi(t) < a(t)r(t)\Big(\frac{1}{4t^2} + \frac{\lambda}{t^2\log^2 t}\Big) \quad \text{for } t \geqslant T.$$

Let

$$v(t) = a(t)r(t)h(t),$$

where

$$h(t) = \frac{1}{2}\Big(\frac{1}{t} + \frac{1}{t\log t}\Big).$$

Then, for $t \geqslant T$,

$$h'(t) = -\frac{1}{2}\Big(\frac{1}{t^2} + \frac{1}{t^2\log t} + \frac{1}{t^2\log^2 t}\Big).$$

So, for $t \geqslant T$,

$$\begin{aligned}
&v'(t) + \varphi(t) + \frac{v^2(t)}{a(t)r(t)}\\
&\quad= (a(t)r(t))'h(t) + a(t)r(t)h'(t) + \varphi(t) + a(t)r(t)h^2(t)\\
&\quad\leqslant a(t)r(t)\Big[\Big(-\frac{1}{2}\Big)\Big(\frac{1}{t^2} + \frac{1}{t^2\log t} + \frac{1}{t^2\log^2 t}\Big)\Big] + a(t)r(t)\Big(\frac{1}{4t^2} + \frac{\lambda}{t^2\log^2 t}\Big)\\
&\qquad+ a(t)r(t)\Big(\frac{1}{4}\Big)\Big(\frac{1}{t^2} + \frac{2}{t^2\log t} + \frac{1}{t^2\log^2 t}\Big)\\
&\quad= a(t)r(t)\frac{4\lambda - 1}{4t^2\log^2 t} \leqslant 0.
\end{aligned}$$

Thus, by Theorem 1, equation (1) is nonoscillatory. $\square$

Remark 2.

(a) Let $a(t) \equiv 1$, then $\varphi(t) = c(t)$. Thus our Corollary 3 reduces to Theorem 3.6 in [11].

(b) Let $a(t) = r(t) = 1$. Then Corollary 3 reduces to the result of [5], [6].

Theorem 4. *Theorems* B *and* 1 *are equivalent.*

P r o o f. It follows from the proof of Theorem 1 that Theorem B implies Theorem 1. Now, we prove that Theorem 1 implies Theorem B. Since equation (5) is nonoscillatory, it follows from Theorem 1 that there is a function $v \in C^1([T,\infty),\mathbb{R})$ for some $T \geqslant t_0$ such that

$$v'(t) + \varphi_1(t) + \frac{v^2(t)}{a_1(t)r_1(t)} \leqslant 0, \quad t \geqslant T.$$

This and $\varphi(t) \leqslant \varphi_1(t)$, $a(t)r(t) \geqslant a_1(t)r_1(t)$ imply

$$v'(t) + \varphi(t) + \frac{v^2(t)}{a(t)r(t)} \leqslant 0.$$

Thus, by Theorem 1, equation (1) is nonoscillatory. □

Using Theorem 1 and Corollary 1 in [13], we can answer an open question in Theorem 2 of [16], which is a generalization of the Hille-Wintner Comparison Theorem ([5], [16], [18]).

Theorem 5 (Hille-Wintner Comparison Theorem). *Let* $a(t)r(t) \geqslant a_1(t)r_1(t)$ *and*

$$\left|\int_t^\infty \varphi(s)\,\mathrm{d}s\right| \leqslant \int_t^\infty \varphi_1(s)\,\mathrm{d}s < \infty \tag{13}$$

for $t \geqslant t_0$. *If equation* (5) *is nonoscillatory, then equation* (1) *is nonoscillatory. That is, if equation* (1) *is oscillatory, then equation* (5) *is oscillatory.*

P r o o f. It will be convenient to divide the proof into two cases:

$$\int^\infty \frac{1}{a_1(s)r_1(s)}\,\mathrm{d}s = \infty, \tag{i}$$

$$\int^\infty \frac{1}{a_1(s)r_1(s)}\,\mathrm{d}s < \infty. \tag{ii}$$

C a s e (i). If equation (5) is nonoscillatory, then equation (6) is nonoscillatory. Thus, as in the proof of Theorem 1, there exists a function $v \in C^1([T,\infty),\mathbb{R})$ for some $T \geqslant t_0$ such that

$$v'(t) + \frac{v^2(t)}{a_1(t)r_1(t)} + \varphi_1(t) = 0, \quad t \geqslant T.$$

Integrating it from t to ξ $(t < \xi)$, we obtain

$$v(\xi) - v(t) + \int_t^{\xi} \varphi_1(s)\,\mathrm{d}s + \int_t^{\xi} \frac{v^2(s)}{a_1(s)r_1(s)}\,\mathrm{d}s = 0. \tag{14}$$

We can prove that

$$\int_t^{\infty} \frac{v^2(s)}{a_1(s)r_1(s)}\,\mathrm{d}s < \infty, \quad t \geqslant T$$

and $\lim\limits_{\xi\to\infty} v(\xi) = 0$. Letting $\xi \to \infty$ in (14), we conclude

$$v(t) = \int_t^{\infty} \varphi_1(s)\,\mathrm{d}s + \int_t^{\infty} \frac{v^2(s)}{a_1(s)r_1(s)}\,\mathrm{d}s, \quad t \geqslant T.$$

Let

$$y(t) := v(t) - \left(\int_t^{\infty} \varphi_1(s)\,\mathrm{d}s - \left|\int_t^{\infty} \varphi(s)\,\mathrm{d}s\right|\right), \quad t \geqslant T.$$

This and (13) imply $v(t) \geqslant y(t) > 0$. Moreover,

$$y'(t) = v'(t) + \varphi_1(t) - \varphi(t) = -\frac{v^2(t)}{a_1(t)r_1(t)} - \varphi_1(t) + \varphi_1(t) - \varphi(t)$$

for $t \geqslant T$. Thus

$$y'(t) + \frac{v^2(t)}{a_1(t)r_1(t)} + \varphi(t) = 0, \quad t \geqslant T.$$

It follows from $v(t) \geqslant y(t) > 0$ and $1/a(t)r(t) \leqslant 1/a_1(t)r_1(t)$ that

$$y'(t) + \frac{y^2(t)}{a(t)r(t)} + \varphi(t) \leqslant 0, \quad t \geqslant T.$$

Hence, by Theorem 1, equation (1) is nonoscillatory.

Case (ii) Let condition (ii) hold. It follows from (13) and (ii) that

$$\int^{\infty} \frac{1}{a(s)r(s)}\,\mathrm{d}s < \infty \quad \text{and} \quad \int_t^{\infty} \varphi(s)\,\mathrm{d}s < \infty.$$

Thus, by Corollary 1 of [13], equation (1) is nonoscillatory. $\square$

Letting $a(t) = a_1(t) = 1$ in Theorem 5, we have

Corollary 6. *Let* $r(t) \geqslant r_1(t)$ *and*

$$\left|\int_t^\infty c(s)\,\mathrm{d}s\right| \leqslant \int_t^\infty c_1(s)\,\mathrm{d}s < \infty$$

on $[t_0, \infty)$*. If equation* (5) *is nonoscillatory, then equation* (1) *is nonoscillatory.*

2. More results

In 1950, Leighton [8] showed the following oscillation criterion:

Leighton Oscillatory Theorem. *If*

$$\int^\infty \frac{1}{r(t)}\,\mathrm{d}t = \int^\infty c(t)\,\mathrm{d}t = \infty,$$

then equation (1) *is oscillatory.*

In this section, we will extend the Leighton Oscillatory Theorem to the second order half-linear ordinary differential equation (2) by using Coles' technique [1].

Theorem 7 (Leighton Oscillatory Theorem)**.** *If*

$$\int^\infty c(t)\,\mathrm{d}t = \int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty,$$

where $1/p + 1/q = 1$*, then equation* (2) *is oscillatory.*

Proof. Suppose this is not the case. Then (2) has a nonoscillatory solution $u(t) \neq 0$ on $[T, \infty)$ for some $T \geqslant t_0$. Without loss of generality, we may assume that $u(t) > 0$ on $[T, \infty)$. Define

$$v(t) = \frac{r(t)\Phi(u'(t))}{\Phi(u(t))}, \quad t \geqslant T.$$

Then, for $t \geqslant T$,

$$\begin{aligned} v'(t) &= -c(t) - \frac{r(t)\Phi(u'(t))\Phi'(u(t))u'(t)}{\Phi^2(u(t))} \\ &= -c(t) - \frac{r(t)u'(t)|u'(t)|^{p-2}(p-1)u^{p-2}(t)u'(t)}{u^{2p-2}(t)} \\ &= -c(t) - \frac{(p-1)r(t)|u'(t)|^p}{u^p(t)} \\ &= -c(t) - (p-1)r^{1-\frac{p}{(p-1)}}(t)\left[\frac{r(t)|u'(t)|^{p-1}}{u^{p-1}(t)}\right]^{\frac{p}{p-1}} \\ &= -c(t) - (p-1)r^{1-q}(t)|v(t)|^q. \end{aligned}$$

Thus, for $t \geqslant T$,

$$v'(t) + c(t) + (p-1)r^{1-q}(t)|v(t)|^q = 0. \tag{15}$$

It follows from (15) that, for $t \geqslant T$,

$$v(t) = v(t_0) - \int_{t_0}^{t} c(s)\,\mathrm{d}s - \int_{t_0}^{t} (p-1)r^{1-q}(s)|v(s)|^q\,\mathrm{d}s.$$

Since $\int^{\infty} c(t)\,\mathrm{d}t = \infty$, we can always find $t_1 \geqslant t_0$ such that

$$v(t_0) - \int_{t_0}^{t} c(s)\,\mathrm{d}s < 0$$

for all $t \in [t_1, \infty)$. Thus

$$v(t) < -\int_{t_0}^{t} (p-1)r^{1-q}(s)|v(s)|^q\,\mathrm{d}s$$

for all $t \geqslant t_1$. Let

$$R(t) := \int_{t_0}^{t} (p-1)r^{1-q}(s)|v(s)|^q\,\mathrm{d}s,$$

then $R(t) > 0$, $|v(t)|^q > R^q(t)$ and

$$R'(t) = (p-1)r^{1-q}(t)|v(t)|^q > (p-1)r^{1-q}(t)R^q(t)$$

for $t \geqslant t_1 \geqslant t_0$. Thus

$$\frac{R'(t)}{R^q(t)} > (p-1)r^{1-q}(t).$$

Integrating it from t_1 to t, we have

$$\begin{aligned}\frac{-R^{1-q}(t_1)}{1-q} &> \frac{1}{1-q}(R^{1-q}(t) - R^{1-q}(t_1)) = \int_{t_1}^{t} \frac{\mathrm{d}R(s)}{R^q(s)} \\ &> \int_{t_1}^{t} (p-1)r^{1-q}(s)R^q(s)\,\mathrm{d}s.\end{aligned}$$

Letting $t \to \infty$, we obtain

$$\infty > \frac{-R^{1-q}(t_1)}{1-q} > (p-1)\int_{t_1}^{\infty} r^{1-q}(s)\,\mathrm{d}s = \infty,$$

which is a contradiction. Thus (2) is oscillatory. $\square$

Remark 3. Let $p = 2$. Then Theorem 7 reduces to the Leighton Oscillatory Theorem.

Using Leighton's Oscillatory Theorem, we have

Corollary 8. *Let $a, a_1 \in C^2([t_0,\infty),(0,\infty))$. If either*

$$\int^{\infty} \frac{1}{a(t)r(t)}\,\mathrm{d}t = \int^{\infty} \varphi(t)\,\mathrm{d}t = \infty$$

or

$$\int^{\infty} \frac{1}{a_1(t)a(t)r_1(t)}\,\mathrm{d}t = \int^{\infty} a(t)[\varphi(t) + a(t)r(t)g^2(t) - (a(t)r(t)g(t))']\,\mathrm{d}t = \infty,$$

where $\varphi(t)$ and $g(t)$ are defined as in Section 1, *then equation* (1) *is oscillatory.*

Remark 4. In [4], Harris used a very complicated transformation which transformed equation (1) into a Riccati integral equation and then he proved that Corollary 8 (Theorem 1 in [4]) holds.

In 1995, Li and Yeh [10] obtained the following theorem for the half-linear differential equation (2).

Theorem 9 (see Theorem 3.2 of [10]). *The following three statements are equivalent:*

(a) *Equation* (2) *is nonoscillatory.*

(b) *There is a function $v \in C^1([T,\infty),\mathbb{R})$ such that*

$$v'(t) + c(t) + (p-1)r^{1-q}(t)|v(t)|^q = 0, \quad t \geqslant T$$

for some $T \geqslant t_0$.

(c) *There is a function $v \in C^1([T,\infty),\mathbb{R})$ such that*

$$v'(t) + c(t) + (p-1)r^{1-q}(t)|v(t)|^q \leqslant 0, \quad t \geqslant T$$

for some $T \geqslant t_0$.

Just as Theorems B and 1 are equivalent, so are Theorem 9 and the following generalized Sturm Comparison Theorem for the half-linear differential equation (2). The analogue of Theorem 5 for the half-linear differential equation (2) reads as follows:

Theorem 10 (Sturm Comparison Theorem). *Consider equation* (2) *and the differential equation*

$$(r_1(t)\Phi(u'(t)))' + c_1(t)\Phi(u(t)) = 0, \tag{16}$$

where

(i)* $r_1, c_1 \in C^1([t_0,\infty),\mathbb{R})$ *and* $r_1(t) > 0$ *on* $[t_0,\infty)$ *for some* $t_0 \geqslant 0$,

(ii)* $\Phi(u)$ *is defined as in* (ii).

Let $r(t) \geqslant r_1(t)$ *and* $c(t) \leqslant c_1(t)$. *If equation* (16) *is nonoscillatory, then equation* (2) *is nonoscillatory; that is, if equation* (2) *is oscillatory, then equation* (16) *is oscillatory.*

Theorem 11 (Hille-Wintner Comparison Theorem)**.** *Let* $r(t) \geqslant r_1(t)$ *and*

$$\left|\int_t^\infty c(s)\,\mathrm{d}s\right| \leqslant \int_t^\infty c_1(s)\,\mathrm{d}s, \quad t \geqslant t_0.$$

If equation (16) *is nonoscillatory, then equation* (2) *is nonoscillatory; that is, if equation* (2) *is oscillatory, then equation* (16) *is oscillatory.*

By Theorem 10, we have the following corollary which extends a result of Fink and Mary [3].

Corollary 12. *If the half-linear differential equation* (2) *is oscillatory, then the half-linear differential equation*

$$(r(t)\Phi(u'(t)))' + \lambda c(t)\Phi(u(t)) = 0$$

is also oscillatory for any $\lambda > 1$.

On the other hand, if we let

$$v(t) := t^{-p/q}\left(\int_t^\infty s^{p/q} g(s)\,\mathrm{d}s + \frac{1}{2}\right)$$

in Theorem 9, then we obtain the following corollary which is due to Li and Yeh [10]. We will use this corollary to give a nonoscillatory criterion for equation (2).

Corollary 13. *If*

$$\left|\int_t^\infty s^{p/q} g(s)\,\mathrm{d}s\right| < \infty, \quad t \geqslant t_0 \geqslant 0,$$

then $(\Phi(u'))' + g(t)\Phi(u) = 0$ *is nonoscillatory, where* $1/p + 1/q = 1$ *and* $g \in C[t_0,\infty)$.

If we make a change of variables

$$\tau = \int^t r^{1-q}(s)\,\mathrm{d}s \quad \text{and} \quad u(t) = x(\tau),$$

then equation (2) is equivalent to the half-linear equation

$$\Phi(x^{\cdot})^{\cdot} + r^{q-1}(t(\tau))c(t(\tau))\Phi(x) = 0, \tag{17}$$

where $x^{\cdot} = \mathrm{d}x/\mathrm{d}\tau$.

Corollary 14. *If*

$$\left|\int^{\infty} c(t)\Big(\int^{t} r^{1-q}(s)\,\mathrm{d}s\Big)^{p/q}\,\mathrm{d}t\right| < \infty, \tag{18}$$

then equation (2) *is nonoscillatory.*

Proof. It follows from

$$\begin{aligned}\infty &> \left|\int^{\infty} c(t)\Big(\int^{t} r^{1-q}(s)\,\mathrm{d}s\Big)^{p/q}\,\mathrm{d}t\right| \\ &= \left|\int^{\infty} c(t(\tau))r^{q-1}(t(\tau))\Big(\int^{t(\tau)} r^{1-q}(s)\,\mathrm{d}s\Big)^{p/q}\,\mathrm{d}\tau\right|\end{aligned}$$

and Corollary 13 that equation (17) is nonoscillatory. Thus, equation (2) is nonoscillatory. □

The following results generalize some results in [19].

Theorem 15. *Equation* (2) *is nonoscillatory if and only if there exist positive functions* $h(t) \in C(t_0,\infty)$ *and* $f(t) \in C^1(t_0,\infty)$ *such that*

$$h(t)\Phi(f'(t)) + \int^{t} \Phi(f(s))c(s)\,\mathrm{d}s = 0, \tag{19}$$

and $0 < h(t) \leqslant r(t)$ *for* $t_0 \leqslant t < \infty$.

Proof. It follows from (19) that

$$(h(t)\Phi(f'(t))' + c(t)\Phi(f(t)) = 0, \quad t_0 \leqslant t < \infty. \tag{20}$$

Clearly, equation (20) has a nonoscillatory solution $f = f(t)$ on $[t_0,\infty)$. Since $h(t) \leqslant r(t)$, the Sturm Comparison Theorem (Theorem 9) implies that equation (2) is nonoscillatory. The converse follows from the following Mirzov's result [12]: For $t_1 \in [t_0,\infty)$ and any given real constants y_0, y_1, equation (2) under the initial condition

$$u(t_1) = y_0, \quad u'(t_1) = y_1$$

has a unique continuous solution on $[t_0,\infty)$. Let $u = f(t)$ be a nonoscillatory solution of (2) and let it satisfy $f'(t_1) = 0$ $(t_1 > t_0)$. If we integrate (2) from t_1 to t $(t > t_1)$, we get

$$r(t)\Phi(f'(t)) + \int_{t_1}^{t} \Phi(f(s))c(s)\,\mathrm{d}s = 0.$$

Taking $r(t) = h(t)$, the above equation reduces to (19). □

Corollary 16. *If*

$$(r_i(t)\Phi(u'))' + c_i(t)\Phi(u) = 0, \quad i = 1, 2, \ldots, n,$$

are nonoscillatory, where $r_i, c_i \in C([t_0, \infty), \mathbb{R})$ *and* $r_i(t) > 0$ *on* $[t_0, \infty)$ *for some* $t_0 \geqslant 0$ *then*

$$\left[\left(\sum_{i=1}^{n} k_i r_i(t)\right)\Phi(u')\right]' + \left(\sum_{i=1}^{n} k_i c_i(t)\right)\Phi(u) = 0 \tag{21}$$

is nonoscillatory, where k_i *are arbitrary nonnegative constants.*

Proof. It follows from Theorem 9 that there exist functions $v_i(t) \in C^1([T, \infty), \mathbb{R})$ $(i = 1, 2, \ldots, n)$ such that

$$v_i'(t) + c_i(t) + (p-1)r_i^{1-q}(t)|v_i(t)|^q \leqslant 0, \quad t \geqslant T. \tag{22}$$

Thus,

$$\sum_{i=1}^{n} k_i v_i'(t) + \sum_{i=1}^{n} k_i c_i(t) + (p-1)\sum_{i=1}^{n}(k_i r_i(t))^{1-q}|k_i v_i(t)|^q \leqslant 0, \quad t \geqslant T. \tag{23}$$

Using the Hölder inequality

$$\sum_{i=1}^{n} a_i b_i \geqslant \left(\sum_{i=1}^{n} a_i^{\alpha}\right)^{1/\alpha}\left(\sum_{i=1}^{n} b_i^{\beta}\right)^{1/\beta}, \quad \text{where } \alpha < 1 \text{ and } \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

we have

$$\begin{aligned}
\sum_{i=1}^{n}(k_i r_i(t))^{1-q}|k_i v_i(t)|g &\geqslant \left(\sum_{i=1}^{n}((k_i r_i(t))^{1-q})^{\frac{1}{1-q}}\right)^{1-q}\left(\sum_{i=1}^{n}(|k_i v_i(t)|^q)^{1/q}\right)^q \\
&= \left(\sum_{i=1}^{n} k_i r_i(t)\right)^{1-q}\left(\sum_{i=1}^{n}|k_i v_i(t)|\right)^q \\
&\geqslant \left(\sum_{i=1}^{n} k_i r_i(t)\right)^{1-q}\left|\sum_{i=1}^{n} k_i v_i(t)\right|^q.
\end{aligned}$$

Thus

$$v'(t) + \sum_{i=1}^{n} k_i c_i(t) + (p-1)\left(\sum_{i=1}^{n} k_i r_i(t)\right)^{1-q}|v|^q \leqslant 0, \quad t \geqslant T,$$

where $v(t) = \sum\limits_{i=1}^{n} k_i v_i(t)$. By Theorem 9, equation (2) is nonoscillatory. $\square$

Corollary 17. *Let $G(t) \in C^1[t_0,\infty)$ be any function such that $G'(t) = -g(t)$. If*

$$(\Phi(u'))' + (p-1)2^q|G(t)|^q\Phi(u) = 0 \tag{24}$$

is nonoscillatory, then $(\Phi(u'))' + g(t)\Phi(u) = 0$ is nonoscillatory, where $1/p + 1/q = 1$.

Proof. It follows from Theorem 9 that the nonoscillation of (24) implies that there exists a function $v(t) \in C^1[T,\infty)$ such that

$$v'(t) + (p-1)2^q|G(t)|^q + (p-1)|v(t)|^q \leqslant 0, \quad t \geqslant T, \tag{25}$$

for some $T \geqslant t_0$. Let $w(t) = G(t) + \frac{1}{2}v(t)$, then

$$\begin{aligned} w'(t) + g(t) + (p-1)|w(t)|^q &= G'(t) + \frac{v'(t)}{2} + g(t) + (p-1)\Big|G(t) + \frac{v(t)}{2}\Big|^q \\ &\leqslant \frac{v'(t)}{2} + (p-1)\Big[|G(t)| + \frac{|v(t)|}{2}\Big]^q \\ &\leqslant \frac{v'(t)}{2} + (p-1)2^{q-1}\Big[|G(t)|^q + \frac{|v(t)|^q}{2^q}\Big] \\ &= \frac{1}{2}[v'(t) + (p-1)2^q|G(t)|^q + (p-1)|v(t)|^q] \leqslant 0. \end{aligned}$$

It follows from Theorem 9 that $(\Phi(u'))' + g(t)\Phi(u) = 0$ is nonoscillatory. □

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