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ON k-SPACES AND k_R-SPACES

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Abstract. In this note we study the relation between k_R-spaces and k-spaces and prove that a k_R-space with a σ-hereditarily closure-preserving k-network consisting of compact subsets is a k-space, and that a k_R-space with a point-countable k-network consisting of compact subsets need not be a k-space.

Keywords: k_R-spaces, k-spaces, k-networks, σ-hereditarily closure-preserving collections, point-countable collections

MSC 2000: 54D50, 54C30

1. Introduction

Suppose X is a topological space and $\mathscr{P}$ is a collection of subsets of X. A space X is determined by $\mathscr{P}$ if $U \subset X$ is open (closed) in X if and only if $U \cap P$ is open (closed) in P for every $P \in \mathscr{P}$. A space X is a k-space, if it is determined by the cover consisting of all compact subsets of X. A space X is called a k_R-space, if X is completely regular and the necessary and sufficient condition for a real valued function f on X to be continuous is that the restriction of f on each compact subset is continuous. Obviously, every completely regular k-space is a k_R-space. The converse is false, as was first shown by an example of Katětov which appeared in a paper by V. Pták (see [1]). What additional properties of a k_R-space ensure it is a k-space has attracted considerable attention, and some noticeable results have been obtained in [2]–[5]. Such conditions were formulated using the notion of a k-network. $\mathscr{P}$ is a k-network for X if whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathscr{P}' \subset U$ for some finite $\mathscr{P}' \subset \mathscr{P}$ (see [6]). Suppose $\mathscr{P}$ is a k-network for X, then $\mathscr{P}$ is a compact k-network if P is compact in X for every $P \in \mathscr{P}$. A family $\mathscr{P}$

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of subsets in X is called locally countable (locally finite), if for each point x there is a neighborhood of x which meets at most countably many (finite many) members of $\mathscr{P}$. A family $\mathscr{P}$ of subsets in X is called star countable, if each member of $\mathscr{P}$ meets at most countably many other members of $\mathscr{P}$. A family $\mathscr{P}$ of subsets in X is called point-countable, if each single point meets at most countably many members of $\mathscr{P}$. A family $\{A_\alpha\colon \alpha \in I\}$ of subsets of a space X is said to be hereditarily closure-preserving (briefly, HCP) if $\bigcup_{\alpha\in J} \overline{B_\alpha} = \overline{\bigcup_{\alpha\in J} B_\alpha}$ whenever $J \subset I$ and $B_\alpha \subset A_\alpha$ for each $\alpha \in J$. A collection $\mathscr{P}$ in X is σ-locally countable (locally finite, HCP) if it is a collection that is the union of countably many locally countable (locally finite, HCP) families. Let $\mathscr{P}$ be a k-network consisting of compact subsets in a regular space X. Then $\mathscr{P}$ is locally countable $\Rightarrow$ $\mathscr{P}$ is σ-locally countable $\Rightarrow$ $\mathscr{P}$ is star countable $\Rightarrow$ $\mathscr{P}$ is point-countable. But the inverse implications are not true. In 1973, Michael constructed an example of a k_R-space which is not a k-space, but has a countable k-network (see [2]). In 1991, S. Lin showed that a k_R-space with a star countable compact k-network is a k-space (see [3]), which answered affirmatively a question posed in [4]. In 2000, Z. Yun proved in [5] that the following statements are equivalent for a k_R-space with a k-network $\mathscr{P}$ of compact subsets, and each of them implies that X is a k-space:

(a) $\mathscr{P}$ is star countable.

(b) $\mathscr{P}$ is locally countable.

(c) $\mathscr{P}$ is σ-locally countable.

Therefore, the following question is raised naturally:

(1) If a k_R-space X has a point-countable compact k-network, then is X a k-space?

It is known that locally finite families are HCP. Hence σ-locally finite families are σ-HCP. Further, σ-locally finite families of compact sets are easily seen to be star countable. Thus σ-HCP is a generalization of σ-locally finite in another direction than star countable.

Therefore, the following question seems to be of some interest.

(2) If a k_R-space X has a σ-HCP compact k-network, then is X a k-space?

In this paper, we show that question 1 has negative answer by the example below, and question 2 has affirmative answer. In fact, a stronger result is proved—with a k-cover instead of a k-network. A family $\mathscr{P}$ of subsets in X is a k-cover if for any compact subset K, $K \subset \bigcup \mathscr{P}'$ for some finite $\mathscr{P}' \subset \mathscr{P}$ (see [7]).

In this paper, all spaces are Hausdorff spaces, and $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{Q}$ denote the set of natural numbers, real numbers and rational numbers, respectively.

2. Results

The following Lemma 1 is easy to show.

Lemma 1. *Let X be a topological space, $\mathscr{P}$ an* HCP*-cover of X by closed sets.*
(1) *If P is a k-space for each $P \in \mathscr{P}$, then so is X.*
(2) *If P is normal for each $P \in \mathscr{P}$, then so is X.*

Lemma 2. *Suppose that X is a k_R-space and $X = \bigcup \mathscr{P}$, where $\mathscr{P} = \{X_n \colon n \in \mathbb{N}\}$ and X_n is a closed normal k-space. If $\mathscr{P}$ is a k-cover for X, then X is a k-space.*

Proof. First we shall show that X is determined by $\mathscr{P}$. Suppose not. There is a set A which is not closed in X such that for any $n \in \mathbb{N}$, $A \cap X_n$ is closed in X. Taking $a \in \overline{A} \setminus A$, we have $a \in X_m$ for some $m \in \mathbb{N}$. For $i \in \mathbb{N}$, let $Y_i = \bigcup\{X_n \colon n \leqslant m+i-1\}$, then $a \in Y_1 \subset Y_i \subset Y_{i+1}$. Y_i is a normal k-subspace by Lemma 1, and $A \cap Y_i$ is closed in X. We can assume that $A \cap Y_1 \neq \emptyset$. Since $a \notin A \cap Y_1$, there is a continuous function f_1 on Y_1 such that $f_1(a) = 1$, and $f_1(A \cap Y_1) = \{0\}$. We define $g_1 \colon A \cap Y_2 \to \mathbb{R}$ such that $g_1(A \cap Y_2) = \{0\}$. Since Y_1 and $A \cap Y_2$ are closed in X, f_1 is continuous on Y_1, g_1 is continuous on $A \cap Y_2$ and $f_1 = g_1$ on $Y_1 \cap (A \cap Y_2) = A \cap Y_1$, we can define a real valued function $h_1 \colon Y_1 \cup (A \cap Y_2) \to \mathbb{R}$ such that $h_1(x) = f_1(x)$ if $x \in Y_1$; $h_1(x) = g_1(x)$ if $x \in A \cap Y_2$. So h_1 is continuous on $Y_1 \cup (A \cap Y_2)$. Since Y_2 is a normal space and $Y_1 \cup (A \cap Y_2)$ is closed in Y_2, h_1 can be expanded continuously to Y_2, that is, we can define $f_2 \colon Y_2 \to \mathbb{R}$ such that f_2 is continuous on Y_2 with the restriction of f_2 on Y_1 being f_1, i.e. $f_2 | Y_1 = f_1$, and $f_2(A \cap Y_2) = \{0\}$. By induction, we can define a sequence of real valued continuous functions $f_n \colon Y_n \to \mathbb{R}$ such that $f_n(A \cap Y_n) = \{0\}$ and $f_n | Y_{n-1} = f_{n-1}$. Define $f \colon X \to \mathbb{R}$ by $f | Y_n = f_n$, then $f(A) = \{0\}$ and $f(a) = 1$. From the fact $a \in \overline{A}$ we know that $f(\overline{A}) \not\subset \overline{f(A)}$, and hence f is not continuous on X. On the other hand, for any compact subset $K \subset X$ there exists $n \in \mathbb{N}$ such that $K \subset Y_n$. f is continuous on K because f is continuous on Y_n. Since X is a k_R-space, f is continuous on X. This is a contradiction. Hence X is determined by $\mathscr{P}$. Next, let $F \subset X$ be such that $F \cap K$ is closed in K for each compact set $K \subset X$. As each X_n is a k-space, $(F \cap X_n) \cap K = (F \cap K) \cap X_n = F \cap K$ is closed in K for each compact set $K \subset X_n$, so $F \cap X_n$ is closed in X_n for each $n \in \mathbb{N}$. Since X is determined by $\mathscr{P}$, F is closed in X. Hence X is a k-space. □

Theorem 3. *A k_R-space with a σ-*HCP *k-cover consisting of compact subsets is a k-space.*

Proof. Suppose X is a k_R-space and has a σ-HCP k-cover consisting of compact subsets. Let $\mathscr{P} = \bigcup\{\mathscr{P}_n \colon n \in \mathbb{N}\}$ be a σ-HCP k-cover, where each $\mathscr{P}_n$ is a

HCP collection consisting of compact subsets. For any $n \in \mathbb{N}$, put $X_n = \bigcup \mathscr{P}_n$. Clearly each X_n is closed in X. By Lemma 1, each X_n is a normal k-space. By Lemma 2, X is a k-space. □

Corollary 4. *A k_R-space with a σ-HCP compact k-network is a k-space.*

As for point-countable compact k-networks, we have

Example 5. There exists a k_R-space X with a point-countable compact k-network, such that X is not a k-space.

Let X be the plane and τ_0 its usual topology. Let $A \subset X$ be the x-axis. For each $x \in A$, let $U(x)$ be the vertical line through x; also let $\mathscr{V}(x)$ be the collection of all $V \subset X$ of the form $V = B(x, \delta) - H(x)$, where $B(x, \delta)$ is an open disc centered at x with radius δ and $H(x)$ is a τ_0-closed subspace of $X - \{x\}$ which is disjoint from $U(x)$. Let $\wedge$ be the topology on X with the following open neighborhood system: an open neighborhood of a point $p \in X - A$ is an open disc centered at p; an open neighborhood of a point $q \in A$ is a set which results from picking a $V(x) \in \mathscr{V}(x)$, for each $q_1 - \varepsilon < x_1 < q_1 + \varepsilon$, and forming the union of these $V(x)$; it will be denoted by $B(q, \varepsilon, \{V(x)\})$. In [8], R. Borges proved that $(X, \wedge)$ is homeomorphic to the space (X, τ) of Example 1.1 in [2]. Recall that τ is the coarsest topology on X which makes every function $f\colon X \to \mathbb{R}$ (the real line) τ_0-continuous on $X - A$ and τ_0-separately continuous at each $x \in A$, (i.e., for each $x \in A$, $f|U(x)$ and $f|x$-axis are continuous). In [2], Michael showed that (X, τ) is a σ-space and a cosmic k_R-space which is not a k-space. By the construction of the topological space (X, τ), the subspaces A and $X \setminus A$ of X have their usual topology, and so they have a countable k-network consisting of compact subsets in A and $X \setminus A$, which are denoted by α, β, respectively. For every $x = (x_1, 0) \in A$ and every $p, q \in \mathbb{Q}$, we denote $F(x, p, q) = \{(x_1, y_2) \in X\colon p \leqslant y_2 \leqslant q\}$. Since the space $\{x_1\} \times \mathbb{R}$ has its usual topology, $F(x, p, q)$ is compact in X. Let $\mathscr{P} = \alpha \cup \beta \cup \{F(x, p, q)\colon x = (x_1, 0) \in A,\ p, q \in \mathbb{Q}\}$. Clearly $\mathscr{P}$ is a point-countable cover consisting of compact subsets in X. We shall show $\mathscr{P}$ is a k-network for X. Assume that C and U are respectively compact and open in X and such that $C \subset U$. Since $C \cap A \subset U \cap A$ and $C \cap A$ is compact and $U \cap A$ open in A, there exists a finite $\alpha' \subset \alpha$ such that $C \cap A \subset \bigcup \alpha' \subset U \cap A$. By Lemma 3.4 in [2], a compact subset of X has the following property:

If C is compact in X, then there are $\varepsilon > 0$ and a finite $A' \subset A$ such that for $y = (y_1, y_2) \in C$ and $0 < |y_2| < \varepsilon$, there is $x = (x_1, x_2) \in A'$ with $y_1 = x_1$. Take $m \in \mathbb{N}$ with $1/m < \varepsilon$. Let $L = \{(x_1, x_2) \in X\colon x_1 \in \mathbb{R}$ and $|x_2| \leqslant 1/m\}$. Then L is closed in X, $C \setminus \operatorname{int}(L) \subset U \setminus A$ with $C \setminus \operatorname{int}(L)$ compact in $X \setminus A$ and $U \setminus A$ open in $X \setminus A$, thus $C \setminus \operatorname{int}(L) \subset \bigcup \beta' \subset U \setminus A$ for some finite $\beta' \subset \beta$. For every $x = (x_1, 0) \in A'$, since $F(x_1, -1/m, 1/m) \cap C \subset (\{x_1\} \times \mathbb{R}) \cap U$ and

$F(x_1, -/m, 1/m) \cap C$ are compact and $(\{x_1\} \times \mathbb{R}) \cap U$ is open in $\{x_1\} \times \mathbb{R}$, and $\{F(x,p,q)\colon\ p,q \in \mathbb{Q}\}$ is a k-network for $\{x_1\} \times \mathbb{R}$, there is a finite $\gamma_x \subset \{F(x,p,q)\colon$ $p,q \in \mathbb{Q}\}$ such that $F(x,-1/m,1/m) \cap C \subset \bigcup \gamma_x \subset (\{x_1\} \times \mathbb{R}) \cap U$. Clearly $C \subset \bigcup(\alpha' \cup \beta' \cup \{\gamma_x\colon\ x \in A'\}) \subset U$, and $\alpha' \cup \beta' \cup \{\gamma_x\colon\ x \in A'\}$ is a finite subfamily of $\mathscr{P}$. Thus $\mathscr{P}$ is a k-network for X.

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