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THE HAMILTONIAN CHROMATIC NUMBER OF A CONNECTED GRAPH WITHOUT LARGE HAMILTONIAN-CONNECTED SUBGRAPHS

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Abstract. If G is a connected graph of order $n \geqslant 1$, then by a hamiltonian coloring of G we mean a mapping c of $V(G)$ into the set of all positive integers such that $|c(x) - c(y)| \geqslant n - 1 - D_G(x, y)$ (where $D_G(x, y)$ denotes the length of a longest $x - y$ path in G) for all distinct $x, y \in V(G)$. Let G be a connected graph. By the hamiltonian chromatic number of G we mean

$$\min(\max(c(z);\ z \in V(G))),$$

where the minimum is taken over all hamiltonian colorings c of G.

The main result of this paper can be formulated as follows: Let G be a connected graph of order $n \geqslant 3$. Assume that there exists a subgraph F of G such that F is a hamiltonian-connected graph of order i, where $2 \leqslant i \leqslant \frac{1}{2}(n+1)$. Then $\mathrm{hc}(G) \leqslant (n-2)^2+1-2(i-1)(i-2)$.

Keywords: connected graphs, hamiltonian-connected subgraphs, hamiltonian colorings, hamiltonian chromatic number

MSC 2000: 05C15, 05C38, 05C45, 05C78

By a graph we mean a finite undirected graph with no loop or multiple edge, i.e. a graph in the sense of [1], for example. The letters f–n will be reserved for denoting non-negative integers. The set of all positive integers will be denoted by $\mathbb{N}$.

0.

If G_0 is a connected graph and $u, v \in V(G_0)$, then we denote by $D_{G_0}(u, v)$ the length of a longest $u - v$ path in G_0. If G is a connected graph of order $n \geqslant 1$ and

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$x, y \in V(G)$, then, following [5], we denote

$$D'_G(x, y) = n - 1 - D_G(x, y).$$

Consider a connected graph G. By a *hamiltonian coloring* of G we mean a mapping c of $V(G)$ into $\mathbb{N}$ such that

$$|c(u) - c(v)| \geqslant D'_G(u, v)$$

for all distinct $u, v \in V(G)$. If c is a hamiltonian coloring of G, then by $\mathrm{hc}(c)$ we mean

$$\max(c(w);\ w \in V(G)).$$

By the *hamiltonian chromatic number* $\mathrm{hc}(G)$ of G we mean

$$\min(\mathrm{hc}(c);\ c \text{ is a hamiltonian coloring of } G).$$

The notions of a hamiltonian coloring and the hamiltonian chromatic number of a connected graph were introduced by Chartrand, Nebeský and Zhang in [2]. The adjective "hamiltonian" in these terms has a transparent motivation: if G is a connected graph, then $\mathrm{hc}(G) = 1$ if and only if G is hamiltonian-connected. Note that if G is a connected graph with no hamiltonian path and c is a hamiltonian coloring of G, then $c(u) \neq c(v)$ for any distinct $u, v \in V(G)$.

Let $n \geqslant 3$. The connected graph of order n which is, in a very natural sense, the most different from the hamiltonian-connected graphs of order n is the star $K_{1,n-1}$. It was proved in [2] that $\mathrm{hc}(K_{1,n-1}) = (n-2)^2 + 1$. As was proved in [3], if G is a connected graph of order $n \geqslant 5$ which is not a star, then $\mathrm{hc}(G) \leqslant \mathrm{hc}(K_{1,n-1}) - 2$. As follows from another result proved in [2],

$$\mathrm{hc}(C_n) = \sqrt{\mathrm{hc}(K_{1,n-1}) - 1} = n - 2.$$

Let G be a connected graph. We will say that a hamiltonian coloring c of G is *normal*, if there exists $u \in V(G)$ such that $c(u) = 1$. Clearly, if c_0 is a hamiltonian coloring of G such that $\mathrm{hc}(c_0) = \mathrm{hc}(G)$, then c_0 is normal.

Observation 1. Let G_1 be a connected factor of a graph G_0. As immediately follows from Lemma 4.5 in [2], $\mathrm{hc}(G_0) \leqslant \mathrm{hc}(G_1)$. This result is easy but very useful. It implies, for instance, that if G is a hamiltonian graph of order $n \geqslant 3$, then $\mathrm{hc}(G) \leqslant n - 2$.

Further results concerning hamiltonian colorings were proved in [2], [3], [4], and [5].

Let G be a connected graph of order $n \geqslant 3$. Then G contains a nontrivial hamiltonian-connected graph as a subgraph. The main result of the present paper can be formulated as follows. If there exists a subgraph F of G such that F is a hamiltonian-connected graph of order i, where $2 \leqslant i \leqslant \frac{1}{2}(n+1)$, then

$$\mathrm{hc}(G) \leqslant (n-2)^2 + 1 - 2(i-1)(i-2)$$

(Theorem 4).

1.

We first introduce a special type of graphs. (Graphs of that type could be called pseudostars.) Let $n \geqslant 3$, let H be a connected graph of order k, $1 \leqslant k < n$, let $u_1, \ldots, u_j$, where $1 \leqslant j \leqslant k$, be pairwise distinct vertices of H, and let $b_1, \ldots, b_j$ be positive integers such that $b_1 + \ldots + b_j = n - k$. Consider pairwise distinct vertices

(1) $$v_{1,1}, \ldots, v_{1,b_1}, \ldots, v_{j,1}, \ldots, v_{j,b_j}$$

not belonging to H. We denote by

$$S(H; u_1\colon\ v_{1,1}, \ldots, v_{1,b_1}; \ldots; u_j\colon\ v_{j,1}, \ldots, v_{j,b_j})$$

the graph G_0 such that

$$V(G_0) = V(H) \cup \{v_{1,1}, \ldots, v_{1,b_1}, \ldots, v_{j,1}, \ldots, v_{j,b_j}\}$$

and

$$E(G_0) = E(H) \cup \{u_1v_{1,1}, \ldots, u_1v_{1,b_1}, \ldots, u_jv_{j,1}, \ldots, u_jv_{j,b_j}\}.$$

Moreover, we say that a graph G is

$$S(H; u_1, b_1; \ldots; u_j, b_j)$$

if there exist pairwise distinct vertices (1) not belonging to H such that

$$G = S(H; u_1\colon\ v_{1,1}, \ldots, v_{1,b_1}; \ldots; u_j\colon\ v_{j,1}, \ldots, v_{j,b_j}).$$

Lemma 1. *Let $n \geqslant 4$, let H be a connected graph of order k, where $2 \leqslant k \leqslant n-2$, let $u \in V(H)$, and let $v_1, \ldots, v_{n-k}$ be pairwise distinct vertices not belonging to H. Consider a normal hamiltonian coloring c of $S(H; u\colon v_1, \ldots, v_{n-k})$ such that*

$$1 = c(v_1) \leqslant \ldots \leqslant c(v_{n-k}) = \operatorname{hc}(c).$$

Then there exists j, $1 \leqslant j < n-k$, such that

$$c(v_{j+1}) - c(v_j) \geqslant n.$$

Proof. Put

$$G = S(H; u\colon v_1, \ldots, v_{n-k}).$$

For each i, $1 \leqslant i < n-k$, we denote by W_i the set of all $w \in V(H)$ such that $c(v_i) \leqslant w \leqslant c(v_{i+1})$. We distinguish two cases.

1. Assume that $k \leqslant \frac{2}{3}(n-1)$. Clearly, there exists j, $1 \leqslant j < n-k$, such that $u \in W_j$. If $|W_j| = 1$, then $c(u) - c(v_j) \geqslant D'_G(u, v_j) = n-2$ and $c(v_{j+1}) - c(u) \geqslant n-2$, thus $c(v_{j+1}) - c(v_j) \geqslant 2n-4 \geqslant n$. Let now $|W_j| = 2$, and let w be the vertex in W_j different from u. Without loss of generality we may assume that $c(w) \leqslant c(u)$. Then $c(w) - c(v_j) \geqslant D'_G(w, v_j) \geqslant n-k-1$, $c(u) - c(w) \geqslant D'_G(u, w) \geqslant n-k$ and $c(v_{j+1}) - c(u) \geqslant n-2$. Thus

$$c(v_{j+1}) - c(v_j) \geqslant 3n - 2k - 3 \geqslant 3n - 4\,\frac{n-1}{3} - 3 = 5\,\frac{n-1}{3} > n.$$

Finally, let $|W_j| \geqslant 3$. Since $2 \leqslant k \leqslant \frac{2}{3}(n-1)$, we get

$$c(v_{j+1}) - c(v_j) \geqslant 4(n-k) - 2 \geqslant 4\Big(n - 2\frac{n-1}{3}\Big) - 2 > n.$$

2. Assume that $k > \frac{2}{3}(n-1)$. Put

$$m = \frac{n-1}{n-k-1}(n-k) - 2.$$

If $m \leqslant n$, then $k \leqslant \frac{2}{3}(n-1)$; a contradiction. Thus $m > n$. Since $k > \frac{2}{3}(n-1)$, we have

$$\frac{k}{n-k-1} > 2.$$

Clearly, there exists j, $1 \leqslant j < n-k$, such that

$$|W_j| \geqslant \frac{k}{n-k-1}.$$

This implies that

$$\begin{aligned}
c(v_{j+1}) - c(v_j) &\geqslant (|W_j| + 1)(n-k) - 2\\
&\geqslant \Big(\frac{k}{n-k-1} + 1\Big)(n-k) - 2\\
&= \frac{n-1}{n-k-1}(n-k) - 2 = m > n,
\end{aligned}$$

which completes the proof. □

Observation 2. Obviously, the complement of a path of order four is a path. On the other hand, the complement of $K_{1,n-1}$, where $n \geqslant 2$, has no hamiltonian path. As was shown in Lemma 4.9 of [2], if T is a tree different from a star, then the complement of T has a hamiltonian path. This result can be extended as follows: if F is a forest different from a star, then the complement of F has a hamiltonian path. The proof is easy and will be left to the reader.

Lemma 2. *Let G_0 be a connected graph of order $n \geqslant 3$, let H be a connected graph of order k, where $2 \leqslant k < n$, and let $u \in V(H)$. Assume that H is an induced subgraph of G_0, and that $G_0 - (V(H-u))$ is a tree. Then for every normal hamiltonian coloring c_1 of $S(H; u, n-k)$ there exists a hamiltonian coloring c_0 of G_0 such that*

$$\mathrm{hc}(c_0) = \mathrm{hc}(c_1).$$

P r o o f. The case when $n-k = 1$ is obvious. Let $n-k \geqslant 2$. Then $n \geqslant 4$. Consider pairwise distinct vertices $v_1, \ldots, v_{n-k}$ not belonging to H and put

$$G_1 = S(H; u\colon\ v_1, \ldots, v_{n-k}).$$

Denote $J_0 = G_0 - V(H)$. Obviously, J_0 is a forest.

Let c_1 be an arbitrary normal hamiltonian coloring of G_1. Without loss of generality we may assume that

$$c_1(v_1) \leqslant \ldots \leqslant c_1(v_{n-k}).$$

Since $D'_{G_1}(v_f, v_g) = n-3$ for all f and g such that $1 \leqslant f < g \leqslant n-k$, we get $c_1(v_{h+1}) - c_1(v_h) \geqslant n-3$ for each h, $1 \leqslant h < n-k$.

We will construct a mapping c_0 of $V(G_0)$ into $\mathbb{N}$ such that

$$c_0(v) = c_1(v) \quad \text{for each } v \in V(H). \tag{2}$$

We will show that

$$(3)\qquad c_0 \text{ is a hamiltonian coloring of } G_0 \quad\text{and}\quad \mathrm{hc}(c_0) = \mathrm{hc}(c_1).$$

The construction of c_0 will be divided into several cases and subcases.

1. Assume that J_0 is not a star. Observation 2 implies that there exists a linear ordering

$$u_1, \dots, u_{n-k}$$

of all the vertices of J_0 such that u_f and u_{f+1} are non-adjacent in G_0 for each f, $1 \leqslant f < n-k$. We define

$$c_0(u_f) = c_1(v_f) \quad \text{for each } f, \quad 1 \leqslant f \leqslant n-k.$$

Consider an arbitrary $w \in V(H)$. Using (2), we get

$$|c_0(u_f) - c_0(w)| = |c_1(v_f) - c_1(w)| \geqslant D'_{G_1}(v_f, w) \geqslant D'_{G_0}(u_f, w)$$

for each f, $1 \leqslant f \leqslant n-k$. Moreover, we have

$$c_0(u_{f+1}) - c_0(u_f) = c_1(v_{f+1}) - c_1(v_f) \geqslant D'_{G_1}(v_{f+1}, v_f) = n-3 \geqslant D'_{G_0}(u_{f+1}, u_f)$$

for each f, $1 \leqslant f < n-k$. Since $n \geqslant 4$, we see that $c_0(u_h) - c_0(u_g) \geqslant n-2$, for all g and h such that $1 \leqslant g$ and $g+2 \leqslant h \leqslant n$. It is clear that (3) holds.

2. Assume that J_0 is a star. We denote by y the vertex of J_0 adjacent to u in G_0. Recall that $n-k \geqslant 2$. Let first $n-k \geqslant 3$; we denote by x the central vertex of J_0; clearly, either $y = x$ or x and y are adjacent in J_0. If $n-k = 2$, then we put $x = y$.

2.1. Assume that $c_1(v_1) > 1$ or $c_1(v_{n-k}) < \mathrm{hc}(c_1)$. Without loss of generality, let $c_1(v_{n-k}) < \mathrm{hc}(c_1)$.

2.1.1. Assume that $y = x$. Let $u_2, \dots, u_{n-k}$ be the vertices of J_0 adjacent to x. We define $c_0(x) = c_1(v_1)$ and

$$c_0(u_f) = c_1(v_f) + 1 \quad \text{for each } f, \quad 2 \leqslant f \leqslant n-k.$$

Consider an arbitrary $w \in V(H)$. Using (2), we get

$$|c_0(u_f) - c_0(w)| = |c_1(v_f) + 1 - c_1(w)| \geqslant D'_{G_1}(v_f, w) - 1 = D'_{G_0}(u_f, w)$$

for each f, $2 \leqslant f \leqslant n-k$, and

$$|c_0(x) - c_0(w)| = |c_1(v_1) - c_1(w)| \geqslant D'_{G_1}(v_1, w) = D'_{G_0}(x, w).$$

Obviously, $c_0(x) < c_0(u_2) \leqslant \ldots \leqslant c_0(u_{n-k})$. We have

$$c_0(u_2) - c_0(x) = c_1(v_2) + 1 - c_1(v_1) \geqslant D'_{G_1}(v_2, v_1) + 1 = n - 2 = D'_{G_0}(u_2, x)$$

and

$$\begin{aligned} c_0(u_{f+1}) - c_0(u_f) &= (c_1(v_{f+1}) + 1) - (c_1(v_f) + 1) \\ &\geqslant D'_{G_1}(v_{f+1}, v_f) = n - 3 = D'_{G_0}(u_{f+1}, u_f) \end{aligned}$$

for each f, $2 \leqslant f < n - k$. Recall that $c_0(u_{n-k}) = c_1(v_{n-k}) + 1 \leqslant \mathrm{hc}(c_1)$. We see that (3) holds.

2.1.2 Assume that $y \neq x$. Then $n - k \geqslant 3$. We denote by $u_2, \ldots, u_{n-k-1}$ the vertices of J_0 adjacent to x and different from y. We define $c_0(y) = c_1(v_1)$,

$$c_0(u_f) = c_1(v_f) \quad \text{for each } f, \quad 2 \leqslant f < n - k,$$

and $c_0(x) = c_1(v_{n-k}) + 1$. Consider an arbitrary $w \in V(H)$. Using (2), we get

$$\begin{aligned} |c_0(y) - c_0(w)| &= |c_1(v_1) - c_1(w)| \geqslant D'_{G_1}(v_f, w) = D'_{G_0}(y, w), \\ |c_0(u_f) - c_0(w)| &= |c_1(v_f) - c_1(w)| \geqslant D'_{G_1}(v_f, w) = D'_{G_0}(u_f, w) + 2 \end{aligned}$$

for each f, $2 \leqslant f < n - k$, and

$$|c_0(x) - c_0(w)| = |c_1(v_{n-k}) + 1 - c_1(w)| \geqslant D'_{G_1}(v_{n-k}, w) - 1 = D'_{G_0}(x, w).$$

Obviously, $c_0(y) < c_0(u_2) \leqslant \ldots \leqslant c_0(u_{n-k-1}) < c_0(x)$. We have

$$\begin{aligned} c_0(u_2) - c_0(y) &= c_1(v_2) - c_1(v_1) \geqslant D'_{G_1}(v_2, v_1) = n - 3 = D'_{G_0}(u_2, y), \\ c_0(u_{f+1}) - c_0(u_f) &= c_1(v_{f+1}) - c_1(v_f) \geqslant D'_{G_1}(v_{f+1}, v_f) = n - 3 = D'_{G_0}(u_{f+1}, u_f) \end{aligned}$$

for each f, $2 \leqslant f \leqslant n - k - 2$, and

$$\begin{aligned} c_0(x) - c_0(u_{n-k-1}) &= c_1(v_{n-k}) + 1 - c_1(v_{n-k-1}) \geqslant D'_{G_1}(v_{n-k}, v_{n-k-1}) + 1 \\ &= n - 2 = D'_{G_0}(x, u_{n-k-1}). \end{aligned}$$

We see that $c_0(x) - c_0(y) > n - 2 = D'_G(x, y)$. Recall that $c_0(x) = c_1(v_{n-k}) + 1 \leqslant \mathrm{hc}(c_1)$. It is clear that (3) holds.

2.2. Assume that $c_1(v_1) = 1$ and $c_1(v_{n-k}) = \mathrm{hc}(c_1)$. By Lemma 1, there exists j, $1 \leqslant j < n - k$ such that $c_1(v_{j+1}) - c_1(v_j) \geqslant n$.

2.2.1. Assume that $1 < j < n - k - 1$. Then $n - k \geqslant 4$.

2.2.1.1. Assume that $y = x$. Similarly as 2.2.1, let $u_2, \ldots, u_{n-k}$ be the vertices of J_0 adjacent to x. We define $c_0(x) = c_1(v_1)$,

$$c_0(u_f) = c_1(v_f) + 1 \quad \text{for each } f, \quad 2 \leqslant f \leqslant j$$

and

$$c_0(u_f) = c_1(v_f) \quad \text{for each } f, \quad j + 1 \leqslant g \leqslant n - k.$$

Consider an arbitrary $w \in V(H)$. Using (2), we get

$$|c_0(x) - c_0(w)| = |c_1(v_1) - c_1(w)| \geqslant D'_{G_1}(v_1, w) = D'_{G_0}(x, w),$$
$$|c_0(u_f) - c_0(w)| = |c_1(v_f) + 1 - c_1(w)| \geqslant D'_{G_1}(v_f, w) - 1 = D'_{G_0}(u_f, w)$$

for each f, $2 \leqslant f \leqslant j$ and

$$|c_0(u_f) - c_0(w)| = |c_1(v_f) - c_1(w)| \geqslant D'_{G_1}(v_f, w) = D'_{G_0}(u_f, w) + 1$$

for each f, $j + 1 \leqslant f \leqslant n - k$. Obviously, $c_0(x) < c_0(u_2) \leqslant \ldots \leqslant c_0(u_{n-k})$. We see that

$$c_0(u_2) - c_0(x) = c_1(v_2) + 1 - c_1(v_1) \geqslant D'_{G_1}(v_2, v_1) + 1 = n - 2 = D'_{G_0}(u_2, x),$$
$$c_0(u_{f+1}) - c_0(u_f) = c_1(v_{f+1}) - c_1(v_f) \geqslant D'_{G_1}(v_{f+1}, v_f) = n - 3 = D'_{G_0}(u_{f+1}, u_f)$$

for each f, $2 \leqslant f \leqslant j - 1$,

$$c_0(u_{j+1}) - c_0(u_j) = c_1(v_{j+1}) - (c_1(v_j) + 1) \geqslant n - 1 > D'_{G_0}(u_{j+1}, u_j),$$

and

$$c_0(u_{f+1}) - c_0(u_f) = c_1(v_{f+1}) - c_1(v_f) \geqslant D'_{G_1}(v_{f+1}, v_f) = n - 3 = D'_{G_0}(u_{f+1}, u_f)$$

for each f, $j + 1 \leqslant f \leqslant n - k - 1$. Recall that $c_0(u_{n-k}) = c_1(v_{n-k})$. We see that (3) holds.

2.2.1.2. Assume that $y \neq x$. Let u_f, where $2 \leqslant f \leqslant j$ or $j + 2 \leqslant f \leqslant n - k$, be the vertices of J_0 adjacent to x and different from y. We define $c_0(y) = c_1(v_1)$,

$$c_0(u_f) = c_1(v_f) \quad \text{for each } f, \quad 2 \leqslant f \leqslant j \ \text{ or } \ j + 2 \leqslant f \leqslant n - k$$

and $c_0(x) = c_1(v_{j+1}) - 1$. Consider an arbitrary $w \in V(H)$. Using (2), we get

$$|c_0(y) - c_0(w)| = |c_1(v_1) - c_1(w)| \geqslant D'_{G_1}(v_f, w) = D'_{G_0}(y, w),$$
$$|c_0(u_f) - c_0(w)| = |c_1(v_f) - c_1(w)| \geqslant D'_{G_1}(v_f, w) = D'_{G_0}(u_f, w) + 2$$

for each f, $2 \leqslant f \leqslant j$ or $j+2 \leqslant f \leqslant n-k$, and

$$|c_0(x) - c_0(w)| = |c_1(v_{j+1}) - 1 - c_1(w)| \geqslant D'_{G_1}(v_{j+1}, w) - 1 = D_{G_0}(x, w).$$

Moreover, we get

$$c_0(u_2) - c_0(y) = c_1(v_2) - c_1(v_1) \geqslant D'_{G_1}(v_2, v_1) = n - 3 = D'_{G_0}(u_2, y),$$
$$c_0(u_{f+1}) - c_0(u_f) = c_1(v_{f+1}) - c_1(v_f) \geqslant D'_{G_1}(v_{f+1}, v_f) = n - 3 = D'_{G_0}(u_{f+1}, u_f)$$

for each f, $2 \leqslant f \leqslant j$ or $j+2 \leqslant f < n-k$,

$$c_0(x) - c_0(u_j) = c_1(v_{j+1}) - 1 - c_1(v_j) \geqslant n - 1 > D'_{G_0}(x, u_j),$$

and

$$c_0(u_{j+2}) - c_0(x) = c_1(v_{j+2}) - (c_1(v_{j+1}) - 1) \geqslant D'_{G_1}(v_{j+2}, v_{j+1}) + 1$$
$$= n - 2 = D'_{G_0}(u_{j+2}, x).$$

Clearly, $c_0(x) - c_0(y) \geqslant 2n - 4 \geqslant n > D'_G(x, y)$. This implies that (3) holds.

2.2.2. Assume that $j = 1$ or $j = n - k - 1$. Without loss of generality we assume that $j = 1$. Let $u_2, \ldots, u_{n-k}$ be the vertices of J_0 adjacent to x. We define $c_0(x) = 1 = c_1(v_1)$ and

$$c_0(u_f) = c_1(v_f) \quad \text{for each } f, \quad 2 \leqslant f \leqslant n-k.$$

Recall that $c_1(v_2) - c_1(v_1) \geqslant n$. Then $c_0(u_2) - c_0(x) \geqslant n > D'_{G_0}(x, u_2)$. Using (2), we can easily show that (3) holds.

Thus the lemma is proved. □

Corollary 1. *Let G be a connected graph of order $n \geqslant 3$, let H be a connected graph of order k, where $2 \leqslant k < n$, and let $u \in V(H)$. Assume that H is an induced subgraph of G and that $G - (V(H - u))$ is connected. Then*

$$\operatorname{hc}(G) \leqslant \operatorname{hc}(S(H; u, n-k)).$$

P r o o f. Obviously, there exists a connected factor G_0 of G such that H is an induced subgraph of G_0 and $G_0 - (V(H-u))$ is a tree. As follows from Observation 1, $\operatorname{hc}(G) \leqslant \operatorname{hc}(G_0)$. Combining this inequality with Lemma 2, we get the desired result. □

The next theorem is an important step towards the main result of this paper:

Theorem 1. *Let G be a connected graph of order $n \geqslant 3$ and let F be an induced subgraph of G. Assume that F is a connected graph of order i, where $2 \leqslant i < n$. Then there exist pairwise distinct $u_1, \ldots, u_j \in V(F)$, where $1 \leqslant j \leqslant i$, and positive integers $b_1, \ldots, b_j$ such that $b_1 + \ldots + b_j = n - i$ and*

$$\operatorname{hc}(G) \leqslant \operatorname{hc}(S(F; u_1, b_1; \ldots; u_j, b_j)). \tag{4}$$

Proof. Obviously, there exists a connected factor G^* of' G such that no edge of $G^* - E(F)$ belongs to a cycle in G^*. By Observation 1,

$$\operatorname{hc}(G) \leqslant \operatorname{hc}(G^*).$$

Since $i < n$, we see that there exist pairwise distinct vertices $u_1, \ldots, u_j$ of G^*, where $1 \leqslant j \leqslant i$, and pairwise vertex-disjoint subtrees $L_1, \ldots, L_j$ of G^* such that

$$V(L_f) \cap V(F) = \{u_f\} \quad \text{for each } f, \quad 1 \leqslant f \leqslant j,$$

and $V(L_1) \cup \ldots \cup V(L_j) \cup V(F) = V(G^*)$. Put $b_f = |V(L_f)| - 1$ for each f, $1 \leqslant f \leqslant j$. Moreover, we put $G_0^* = G^*$ and

$$G_f^* = S(G_{f-1}^* - V(L_f - \{u_f\}); u_f, b_f) \quad \text{for each } f, \quad 1 \leqslant f \leqslant j.$$

It is clear that

$$G_j^* = S(F; u_1, b_1; \ldots; u_j, b_j).$$

It follows from Lemma 2 that

$$\operatorname{hc}(G_0^*) \leqslant \operatorname{hc}(G_1^*) \leqslant \ldots \leqslant \operatorname{hc}(G_j^*),$$

which completes the proof. □

2.

As we will see, Theorem 1 can be improved under the condition that $i \leqslant \frac{1}{2}(n+1)$ and F is hamiltonian-connected.

Recall that every complete graph is hamiltonian-connected. If f and i are positive integers, then by $S(K_i; f)$ we mean a graph $S(H; u, f)$, where H is a complete graph of order i and $u \in V(H)$.

Proposition 1. *Let F be a complete graph of order $i \geqslant 2$, let $u_1, \ldots, u_j \in V(F)$, where $1 \leqslant j \leqslant i$, be pairwise distinct vertices of F, and let $b_1, \ldots, b_j$ be positive integers. Put*

$$G = S(F; u_1, b_1; \ldots; u_j, b_j).$$

Consider an arbitrary $A \subseteq E(F)$ such that $F - A$ is hamiltonian-connected. Then every hamiltonian coloring of G is a hamiltonian coloring of $G - A$.

P r o o f. The proposition immediately follows from the definition of a hamiltonian coloring. □

Observation 3. Put $G = S(K_i; n-i)$, where $n \geqslant 4$ and $2 \leqslant i \leqslant n-2$. Consider arbitrary distinct $v, w \in V(G)$ such that $\deg_G v \leqslant \deg_G w$. Then

if $\deg_G v = \deg_G w = 1$, then $D'_G(v, w) = n - 3$,
if $\deg_G v = 1$ and $\deg_G w = i - 1$, then $D'_G(v, w) = n - i - 1$,
if $\deg_G v = 1$ and $\deg_G w = n - 1$, then $D'_G(v, w) = n - 2$,
if $\deg_G v = i - 1$ and $\deg_G w = i - 1$ or $n - 1$, then $D'_G(v, w) = n - i$.

Lemma 3. *Let F be a complete graph of order $i \geqslant 2$, let $u_1, \ldots, u_j$, where $1 \leqslant j \leqslant i$, be pairwise distinct vertices of F, and let $b_1, \ldots, b_j$ be positive integers such that $i \leqslant b_1 + \ldots + b_j + 1$, and*

$$j \geqslant 3 \quad \text{or} \quad b_j \geqslant 2.$$

Then for every hamiltonian coloring c^* of $S(F; u_j, b_1 + \ldots + b_j)$ there exists a hamiltonian coloring c of $S(F; u_1, b_1; \ldots; u_j, b_j)$ such that $\mathrm{hc}(c) = \mathrm{hc}(c^*)$.

P r o o f. The case when $j = 1$ is obvious. Let $j \geqslant 2$. Put

$$n = i + b_1 + \ldots + b_j, \ \ G = S(F; u_1, b_1; \ldots; u_j, b_j) \ \text{and} \ G^* = S(F; u_j, n - i).$$

Obviously, $i \leqslant \frac{1}{2}(n+1)$. Since $j \geqslant 2$, we have $n - i \geqslant 2$. Put $W = V(G) \setminus V(F)$ and $W^* = V(G^*) \setminus V(F)$. For every f, $1 \leqslant f \leqslant j$, we denote by W_f the set of all vertices in W adjacent to u_f in G. Thus $|W| = n - i = |W^*|$ and $|W_f| = b_f$ for each f, $1 \leqslant f \leqslant j$.

Consider an arbitrary hamiltonian coloring c^* of G^*. Since $i \geqslant 2$ and $n - i \geqslant 2$, we see that G^* has no hamiltonian path; therefore $c^*(v) \neq c^*(w)$ for all distinct $v, w \in V(G^*)$. If $j \geqslant 3$, then, without loss of generality, we assume that

$$c^*(u_1) < \ldots < c^*(u_{j-1}).$$

Consider an arbitrary f, $1 \leqslant f \leqslant j - 1$. If there exists $x \in W^*$ such that $c^*(x) < c^*(u_f)$ and there exists no $r \in V(G^*)$ such that $c^*(x) < c^*(r) < c^*(u_f)$, then we put $u_f^- = x$. If there exists $x \in W^*$ such that $c^*(u_f) < c^*(x)$ and there exists no $s \in V(G^*)$ such that $c^*(u_f) < c^*(s) < c^*(x)$, then we put $u_f^+ = x$.

Moreover, we put

$X_f = \{u_f^-, u_f^+\}$ if both u_f^- and u_f^+ are defined,
$X_f = \{u_f^-\}$ if u_f^- is defined and u_f^+ is not,
$X_f = \{u_f^+\}$ if u_f^+ is defined and u_f^- is not, and
$X_f = \emptyset$ if neither u_f^- nor u_f^+ are defined.

Recall that if $j \geqslant 3$, then $c^*(u_1) < c^*(u_{j-1})$. This means that if $j \geqslant 3$ and u_{j-1}^+ is defined, then $u_{j-1}^+ \notin X_1$.

We introduce the following notation. Consider arbitrary vertices $z_1, \ldots, z_f$ of G^* such that $c^*(z_1) < \ldots < c^*(z_f)$, where $f \geqslant 1$. Put $Z = \{z_1, \ldots, z_f\}$. If $1 \leqslant g \leqslant f$, then we write

$$Z_{\langle g \rangle} = \{z_1, \ldots, z_g\}.$$

We now define the sets W_f^*, where $1 \leqslant f \leqslant j$, as follows:

$$W_1^* = (W^* \setminus X_1)_{\langle b_1 - 1 \rangle} \cup \{u_{j-1}^+\}$$
$$\text{if } j \geqslant 3,\ u_{j-1}^+ \text{ is defined and } u_{j-1}^+ \notin (W^* \setminus X_1)_{\langle b_1 - 1 \rangle},$$
$$W_1^* = (W^* \setminus X_1)_{\langle b_1 \rangle} \text{ otherwise;}$$

if $j \geqslant 3$ and $2 \leqslant f < j$, then

$$W_f^* = ((W^* \setminus (W_1^* \cup \ldots \cup W_{f-1}^*)) \setminus X_f)_{\langle b_f \rangle};$$

finally

$$W_j^* = W^* \setminus (W_1^* \cup \ldots \cup W_{j-1}^*).$$

Clearly, if $j \geqslant 3$, then

$$|(W^* \setminus (W_1^* \cup \ldots \cup W_{j-2}^*)) \cap \{X_{j-1})| \leqslant 1.$$

It is easy to see that the sets $W_1^*, \ldots, W_{j-1}^*, W_j^*$ are well-defined.

Let c be a mapping of $V(G)$ into $\mathbb{N}$ such that

$$c(v) = c^*(v) \quad \text{for every } v \in V(F)$$

and

$$c(w_f) = c^*(w_f^*) \quad \text{for each } f, \quad 1 \leqslant f \leqslant j.$$

Consider distinct $w_1, w_2 \in W$. Then there exist distinct $w_1^*, w_2^* \in W^*$ such that $c(w_1) = c^*(w_1^*)$ and $c(w_2) = c^*(w_2^*)$. Thus

$$|c(w_1) - c(w_2)| = |c^*(w_1^*) - c^*(w_2^*)| \geqslant D'_{G^*}(w_1^*, w_2^*) = n - 3 \geqslant D'_G(w_1, w_2).$$

Consider an arbitrary f, $1 \leqslant f \leqslant j$, and an arbitrary $w \in W_f$. There exists $w^* \in W_f^*$ such that $c(w) = c^*(w^*)$. Clearly,

$$|c(w) - c(u_j)| = |c^*(w^*) - c^*(u_j)| \geqslant D'_{G^*}(w^*, u_j) = n - 2 \geqslant D'_G(w, u_j).$$

Let $v \in V(F)$ and $u_f \neq v \neq u_j$. Then

$$|c(w) - c(v)| = |c^*(w^*) - c^*(v)| \geqslant D'_{G^*}(w^*, v) = n - i - 1 = D'_G(w, v).$$

Without loss of generality we assume that $c^*(w^*) < c^*(u_f)$. As follows from the definition of W_f^*, there exists $r \in V(G^*)$ such that $c^*(w^*) < c^*(r) < c^*(u_f)$. Clearly,

$$|c(u_f) - c(w)| = c^*(w^*) - c^*(u_f) \geqslant (c^*(u_f) - c^*(r)) + (c^*(r) - c^*(w^*)).$$

Obviously, if $r \in V(F - u_j)$, then $c^*(u_f) - c^*(r) \geqslant n - i$ and $c^*(r) - c^*(w^*) \geqslant n - i - 1$; if $r = u_j$, then $c^*(u_f) - c^*(r) \geqslant n - i$ and $c^*(r) - c^*(w^*) \geqslant n - 2$; and if $r \in W^*$, then $c^*(u_f) - c^*(r) \geqslant n - i - 1$ and $c^*(r) - c^*(w^*) \geqslant n - 3$. Hence

$$|c(u_f) - c(w)| \geqslant \min(2n - 2i - 1,\ 2n - i - 4).$$

Recall that $i \leqslant \frac{1}{2}(n + 1)$. We see that

$$2n - 2i - 1 \geqslant n - 2 = D'_G(u_f, w).$$

Since $n \geqslant 4$ and i is an integer, we see that

$$2n - i - 4 \geqslant n - 2 = D'_G(u_f, w)$$

again.

This implies that c is a hamiltonian coloring of G and $\mathrm{hc}(c) = \mathrm{hc}(c^*)$, which completes the proof. □

Lemma 4. *Let F be a complete graph of order $i \geqslant 2$, and let u_1 and u be distinct vertices of F. Then* $\operatorname{hc}(S(F; u_1, 1; u, 1) \leqslant \operatorname{hc}(S(F; u, 2)$.

P r o o f. Put $G = S(F; u_1, 1; u_2, 1)$ and $G^* = S(F; u_2, 2)$. If $i = 2$, then it is easy to show that $\operatorname{hc}(G) = 4 < 5 = \operatorname{hc}(G^*)$.

Let $i \geqslant 3$. The definition of a hamiltonian coloring implies that $\operatorname{hc}(G^*) \geqslant 2i - 1$. Let $u_2, \ldots, u_{i-1}$ be the vertices of F different from u_1 and u, and let v_1 and v be the vertices of degree one in G such that $u_1v_1, uv \in E(G)$. We denote by c the mapping of $V(G)$ into $\mathbb{N}$ defined as follows:

$$c(u_1) = 1,\ \ c(u_2) = 3,\ \ \ldots,\ \ c(u_{i-1}) = 2i - 3,\ \ c(u) = 2i - 1,\ \ c(v) = 2,$$

and

$$c(v_1) = i + 1 \ \text{ if } \ i \ \text{ is odd,} \quad \text{and} \quad c(v_1) = i + 2 \ \text{ if } \ i \ \text{ is even.}$$

It is easy to see that c is a hamiltonian coloring of G. Thus $\operatorname{hc}(G) \leqslant \operatorname{hc}(G^*)$, which completes the proof. □

The next theorem is a further important step towards the main result of this paper:

Theorem 2. *Let G be a connected graph of order $n \geqslant 3$ and let F be an induced subgraph of G. Assume that F is a hamiltonian-connected graph of order i, where $2 \leqslant i \leqslant \frac{1}{2}(n + 1)$. Then*

$$\operatorname{hc}(G) \leqslant \operatorname{hc}(S(K_i; n - i)).$$

P r o o f. By Theorem 1, there exist pairwise distinct $u_1, \ldots, u_j \in V(F)$, where $1 \leqslant j \leqslant i$, and positive integers $b_1, \ldots, b_j$ such that $b_1 + \ldots + b_j = n - i$ and (4) holds. Without loss of generality we assume that

$$\text{if } \ b_j = 1, \ \text{ then } \ b_f = 1 \quad \text{for each } f, \quad 1 \leqslant f \leqslant j - 1.$$

If $j \geqslant 3$ or $b_j \geqslant 2$, the result follows from Proposition 1 and Lemma 3. Let now $j = 2$ and $b_j = 1$. Then $n - i = 2$. The result immediately follows from Proposition 1 and Lemma 4. □

3.

Let $n \geqslant 3$. Then $S(K_2; n-2) = K_{1,n-1}$ and thus, by Theorem 3.2 of [2], $\operatorname{hc}(S(K_2; n-2) = (n-2)^2 + 1$. Moreover, as follows from Lemma 2.3 of [2], $\operatorname{hc}(S(K_{n-1}; 1) = n-1$.

We will prove that if $2 \leqslant i \leqslant \frac{1}{2}(n+1)$, then $\operatorname{hc}(S(K_i, n-i)) = (n-2)^2 + 1 - 2(i-1)(i-2)$.

Let G be a connected graph of order $n \geqslant 1$, and let c be a mapping of $V(G)$ into $\mathbb{N}$. We will say that c is a *pseudohamiltonian* coloring of G if there exists an ordering

$$u_1, \ldots, u_n$$

of $V(G)$ such that

$$c(u_1) \leqslant \ldots \leqslant c(u_n)$$

and

$$c(u_{f+1}) - c(u_f) \geqslant D'_G(u_{f+1}, u_f) \quad \text{for each } f, \quad 1 \leqslant f < n.$$

Obviously, every hamiltonian coloring of G is pseudohamiltonian. On the other hand, we will prove that if $G = S(K_i; n-i)$, where $n \geqslant 4$ and $3 \leqslant i \leqslant \frac{1}{2}(n+1)$, then every pseudohamiltonian coloring of G is hamiltonian.

In the rest of this paper we will study $S(K_i; n-i)$.

We now introduce several useful conventions. Let $G = S(K_i; n-i)$, where $n \geqslant 4$ and $3 \leqslant i \leqslant n-2$. We denote by u the only vertex of degree $n-1$ in G, by V_1 the set of all vertices of degree one in G, and by V_{i-1} the set of all vertices of degree $i-1$ in G. Clearly, $|V_1| = n-i$ and $|V_{i-1}| = i-1$. Put $R = V_{i-1} \cup \{u\}$.

Consider an arbitrary pseudohamiltonian coloring c of G. There exists an ordering

$$v_1^c, \ldots, v_{n-i}^c$$

of V_1 such that

$$c(v_1^c) < \ldots < c(v_{n-i}^c).$$

We denote

$$R_0^c = \{r \in R;\ c(r) < c(v_1^c)\},$$
$$R_f^c = \{r \in R;\ c(v_f^c) < c(r) < c(v_{f+1}^c)\} \quad \text{for each } f, \quad 1 \leqslant f < n-i,$$

and

$$R_{n-i}^c = \{r \in R;\ c(v_{n-i}^c) < c(r)\}.$$

Moreover, we denote

$$a_f^c = |R_f^c| \quad \text{for each } f, \quad 0 \leqslant f \leqslant n-i.$$

Consider an arbitrary f, $0 \leqslant f \leqslant n-i$ such that $a_f^c \geqslant 1$. Then there exists an ordering

$$r_{f,1}^c, \ldots, r_{f,a_f}^c$$

of R_f^c such that

$$c(r_{f,1}^c) < \ldots < c(r_{f,a_f}^c).$$

Obviously, there exist integers $j(c)$ and $m(c)$ such that

$$0 \leqslant j(c) \leqslant n-i, \;\; a_{j(c)}^c \geqslant 1, \;\; 1 \leqslant m(c) \leqslant a_{j(c)}^c, \;\; \text{and } r_{j(c),m(c)}^c = u.$$

Let $a_1, \ldots, a_{n-i}, j$ and m be non-negative integers such that

$$a_1 + \ldots + a_{n-i} = i, \quad j \leqslant n-i \quad \text{and} \quad 1 \leqslant m \leqslant a_j. \tag{5}$$

Consider a pseudohamiltonian coloring c of G. If

$$a_f^c = a_f \quad \text{for each } f, \quad 0 \leqslant f \leqslant n-i,$$

$j(c) = j$ and $m(c) = m$, then we say that c has the type

$$(a_0, \ldots, a_{n-i}; j, m). \tag{6}$$

Let c be a pseudohamiltonian coloring of $G = S(K_i; n-i)$, where $n \geqslant 5$ and $3 \leqslant i \leqslant n-2$. Then there exist non-negative integers $a_0, \ldots, a_{n-i}$ such that (5) holds and (6) is the type of c. Clearly, there exists an ordering

$$u_1, \ldots u_n$$

of $V(G)$ such that

$$|c(u_{f+1}) - c(u_f)| \geqslant D'_G(u_{f+1}, u_f) \quad \text{for each } f, \quad 1 \leqslant f < n.$$

If $c(u_1) = 1$ and

$$|c(u_{f+1}) - c(u_f)| = D'_G(u_{f+1}, u_f) \quad \text{for each } f, \quad 1 \leqslant f < n,$$

then we will say that c is the *minimum* pseudohamiltonian coloring of the type (6) and we will write

$$c = M(a_0, \ldots, a_{n-i}; j, m).$$

Lemma 5. *Let $G = S(K_i; n-i)$, where $n \geqslant 5$ and $3 \leqslant i \leqslant n-2$, and let $a_0, \dots, a_{n-i}, j$ and m be non-negative integers such that (5) holds, and let $c = M(a_0, \dots, a_{n-i}; j, m)$. Put $k = \max(c(u);\ u \in V(G))$. Then*

if $a_0 = 0$, then $c(v_1^c) = 1$;

if $a_0 \geqslant 1$ and ($j \geqslant 1$ or ($j = 0$ and $m < a_0$)), then $c(v_1^c) = a_0(n-i)$;

if $a_0 \geqslant 1$, $j = 0$ and $m = a_0$, then $c(v_1^c) = (a_0 - 1)(n-i) + n - 1$;

if $1 \leqslant f < n-i$ and $a_f = 0$, then $c(v_{f+1}^c) = c(v_f^c) + n - 3$;

if $1 \leqslant f < n-i$, $a_f \geqslant 1$, and ($j \neq f$ or ($j = f$ and $1 < m < a_f$)), then $c(v_{f+1}^c) = c(v_f^c) + (a_f + 1)(n-i) - 2$;

if $1 \leqslant f < n-i$ and $a_f \geqslant 2$ and ($m = 1$ or a_f), then $c(v_{f+1}^c) = c(v_f^c) + a_f(n-i) + n - 3$;

if $1 \leqslant f < n-i$, $a_f = 1$ and $j = f$, then $c(v_{f+1}^c) = c(v_f^c) + 2(n-2)$;

if $a_{n-i} = 0$, then $k = c(v_{n-i}^c)$;

if $a_{n-i} \geqslant 1$ and ($j < n-i$ or ($j = n-i$ and $m \geqslant 2$)), then $k = c(v_{n-i}^c) + a_{n-i}(n-i) - 1$; and

if $a_{n-i} \geqslant 1$, $j = n-i$ and $m = 1$, then $k = c(v_{n-i}^c) + (a_{n-i} - 1)(n-i) + n - 2$.

P r o o f is easy and will be left to the reader. □

Remark. Let c and k be the same as in Lemma 5. If c is hamiltonian, then $\mathrm{hc}(c) = k$.

Proposition 2. *Let $n \geqslant 5$, and let $3 \leqslant i \leqslant n-2$. Then every pseudohamiltonian coloring c of $S(K_i; n-i)$ is hamiltonian if and only if $i \leqslant \frac{1}{2}(n+1)$.*

P r o o f. Put $G = S(K_i; n-i)$.

Let first $i \leqslant \frac{1}{2}(n+1)$. Consider an arbitrary pseudohamiltonian coloring c of G. Then there exist non-negative integers $a_1, \dots, a_{n-i}, j$ and m such that (5) holds and that (6) is the type of c.

Consider an arbitrary f, $0 < f < n-i-1$; assume that $a_f \geqslant 1$. Then

$$\begin{aligned} c(r_{f+1,1}^c) - c(r_{f,a_f}^c) &= (c(r_{f+1,1}^c) - c(v_{f+1}^c)) + (c(v_{f+1}^c) - c(r_{f,a_f}^c)) \\ &\geqslant D'_G(r_{f+1,1}^c, v_{f+1}^c) + D'_G(v_{f+1}^c, r_{f,a_f}^c) \\ &\geqslant 2(n-i-1) \geqslant n-i = D'(r_{f+1,1}^c, r_{f,a_f}^c). \end{aligned}$$

Consider an arbitrary f, $0 < f < n-i$ such that $a_f \geqslant 1$; if $f \neq j$ or ($f = j$ and $1 < m < a_f$), then

$$c(v_{f+1}^c) - c(v_f^c) \geqslant (a_f + 1)(n-i) - 2 \geqslant 2(n-i) - 2 \geqslant n - 3 = D'_G(v_{f+1}^c, v_f^c);$$

if $f = j$ and ($m = 1$ or a_f), then

$$c(v^c_{f+1}) - c(v^c_f) > \max(c(v^c_{f+1}) - c(u), c(u) - c(v^c_f)) \geqslant n - 2 > D'(v^c_{f+1}, v^c_f).$$

If $j < n - i$ and $m < a_j$, then

$$c(v^c_{j+1}) - c(u) \geqslant (a_j - m + 1)(n - i) - 1 \geqslant 2(n - i) - 1 \geqslant n - 2 = D'_G(v^c_{j+1}, u).$$

If $j > 0$ and $m > 1$, then

$$c(u) - c(v^c_j) \geqslant m(n - i) - 1 \geqslant 2(n - i) - 1 \geqslant n - 2 \geqslant D'_G(u, v^c_j).$$

As easily follows from these observations, c is a hamiltonian coloring of G.

Let now $i > \frac{1}{2}(n + 1)$. Consider an arbitrary pseudohamiltonian coloring of G such that (6) is the type of c,

$$a_0 = 2, \quad a_1 = 1, \quad a_f = 0 \quad \text{for each } f,$$
$$1 < f < n - i, \quad a_{n-i} = n - i - 3, \quad j = 0 \quad \text{and} \quad m = 1,$$

and the following holds

$$c(r^c_{0,1}) = 1, \quad c(r^c_{0,2}) = 1 + (n - i), \quad c(v^c_1) = c(r^c_{0,2}) + n - i - 1,$$
$$c(r^c_{1,1}) = c(v^c_1) + n - i - 1 \quad \text{and} \quad c(v^c_2) = c(v^r_{1,1}) + n - i - 1.$$

Recall that $r^c_{0,1} = u$. Since $i > \frac{1}{2}(n + 1)$, we get

$$c(v^c_1) - c(u) = 2n - 2i - 1 < n - 2$$

and

$$c(v^c_2) - c(v^c_1) = 2n - 2i - 2 < n - 3.$$

Thus c is not a hamiltonian coloring of G. □

Remark. Using the technique of the proof of Proposition 1, it is easy to show that every pseudohamiltonian coloring of $K_{1,n-1}$, where $n \geqslant 3$, is hamiltonian.

Lemma 6. *Let $G = S(K_i; n-i)$, where $n \geqslant 5$, and let $3 \leqslant i \leqslant \frac{1}{2}(n+1)$. Consider non-negative integers $a_0, \ldots, a_{n-i}$ such that*

$$a_0 + \ldots + a_{n-i} = i.$$

Assume that there exist f and g, $1 < f < n-i$ and $0 \leqslant g \leqslant n-i$, such that

$$\begin{gathered} a_f = 0, \\ a_g \geqslant 3 \quad \textit{if} \quad g = 0, \\ a_g \geqslant 2 \quad \textit{if} \quad 1 \leqslant g < n-i, \ \textit{and} \\ a_g \geqslant 1 \quad \textit{if} \quad g = n-i. \end{gathered}$$

Put

$$a_f^+ = 1,\ \ a_g^+ = a_g - 1 \ \ \textit{and} \ \ a_h^+ = a_h \quad \textit{for each } h, \quad 0 \leqslant h \leqslant n-i,\ \ f \neq h \neq g.$$

Then

$$\mathrm{hc}(M(a_0^+, \ldots, a_{n-i}^+; 0, 1)) < \mathrm{hc}(M(a_0, \ldots, a_{n-i}; 0, 1)).$$

P r o o f. Put $c = M(a_0, \ldots, a_{n-i}; 0, 1)$ and $c^+ = M(a_0^+, \ldots, a_{n-i}^+; 0, 1)$. By Lemma 5, $c(v_{f+1}^c) - c(v_f^c) = n-3$. If $g < n-i$ or ($g = n-i$ and $a_g \geqslant 2$), then

$$\mathrm{hc}(c^+) = \mathrm{hc}(c) - ((n-i) + (n-3)) + 2(n-i-1) = \mathrm{hc}(c) + 1 - i.$$

If $g = n-i$ and $a_g = 1$, then $\mathrm{hc}(c^+) = \mathrm{hc}(c) + 2 - i$. Since $i \geqslant 3$, the lemma is proved. □

The next theorem is the last important step to the main result of this paper:

Theorem 3. *Let $n \geqslant 3$ and $2 \leqslant i \leqslant \frac{1}{2}(n+1)$. Then*

$$\mathrm{hc}(S(K_i; n-i) = (n-2)^2 + 1 - 2(i-1)(i-2).$$

P r o o f. If $i = 2$, then the result immediately follows from Theorem 3.2 in [2]. We assume that $i \geqslant 3$. Then $n \geqslant 5$.

Let c be an arbitrary hamiltonian coloring of G. It is easy to see that there exist non-negative integers $a_0, \ldots, a_{n-i}, j$ and m such that (5) holds and (6) is the type of c. Put

$$c_0 = M(a_0, \ldots, a_{n-i}; j, m).$$

By Proposition 2, c_0 is a hamiltonian coloring of G. Obviously, $\mathrm{hc}(c_0) \leqslant \mathrm{hc}(c)$.

Consider the hamiltonian coloring

$$c^* = M(a_0^*, \dots, a_{n-i}^*; 0, 1)$$

of G, where $a_0^*, \dots a_{n-i}^*$ will be defined in exactly one of the following Cases 1–6:

1. Assume that $a_0 \geqslant 2$ and $j = 0$. Put $a_0^* = a_0, \dots, a_{n-i}^* = a_{n-i}$.

   If $m < a_0$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0)$.
   If $m = a_0$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i-1)$.

2. Assume that $a_0 = 1$ and $j = 0$. Clearly, there exists k, $1 \leqslant k \leqslant n-i$, such that $a_k \geqslant 1$. Put $a_0^* = 2$, $a_k^* = a_k - 1$, and $a_f^* = a_f$ for each f, $1 \leqslant f \leqslant n-i$, $f \neq k$.

   If $k < n-i$ and $a_k \geqslant 2$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i-1)$.
   If $k < n-i$ and $a_k = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0)$.
   If $k = n-i$ and $a_k \geqslant 2$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i-1)$.
   If $k = n-i$ and $a_k = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i-2)$.

3. Assume that $a_0 \geqslant 2$ and $j \geqslant 1$. Put $a_0^* = a_0, \dots a_{n-i}^* = a_{n-i}$.

   If $j < n-i$ and $1 < m < a_j$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0)$.
   If $j < n-i$, $a_j \geqslant 2$, and ($m = 1$ or a_j), then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i-1)$.
   If $j < n-i$ and $a_j = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (2i-2)$.
   If $j = n-i$ and $m > 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0)$.
   If $j = n-i$ and $m = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i-1)$.

4. Assume that $a_0 = 1$ and $j \geqslant 1$. Put $a_0^* = 2$, $a_j^* = a_j - 1$, and $a_f^* = a_f$ for each f, $1 \leqslant f \leqslant n-i$, $f \neq j$.

   If $j < n-i$ and $1 < m < a_j$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0)$.
   If $j < n-i$, $a_j \geqslant 2$, and ($m = 1$ or a_j), then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i-1)$.
   If $j < n-i$ and $a_j = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i-1)$.
   If $j = n-i$ and $m > 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0)$.
   If $j = n-i$, $a_j \geqslant 2$, and $m = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i-1)$.
   If $j = n-i$ and $a_j = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i-2)$.

5. Assume that $a_0 = 0$ and $a_j \geqslant 2$. Put $a_0^* = 2$, $a_j^* = a_j - 2$ and $a_f^* = a_f$ for each f, $1 \leqslant f \leqslant n-i$, $f \neq j$.

   If $j < n-i$ and $1 < m < a_j$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - 1$.
   If $j < n-i$ and $a_j \geqslant 3$ and $m = 1$ or a_j, then $\mathrm{hc}(c_0) - i$.
   If $j < n-i$ and $a_j = 2$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - 1$.
   If $j = n-i$, $a_j \geqslant 3$, and $m > 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - 1$.
   If $j = n-i$, $a_j = 2$, and $m = 2$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0)$.
   If $j = n-i$, $a_j \geqslant 3$, and $m = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - i$.

If $j = n - i$, $a_j = 2$, and $m = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i - 1)$.

6. Assume that $a_0 = 0$ and $a_j = 1$. Clearly there exists k, $1 \leqslant k \leqslant n - i$, such that $k \neq j$ and $a_k \geqslant 1$. Put $a_0^* = 2$, $a_j^* = 0$, $a_k^* = a_k - 1$, and $a_f^* = a_f$ for each f, $1 \leqslant f \leqslant n - i$, $j \neq f \neq k$.

   If $j < n - i$, $k < n - i$ and $a_k \geqslant 2$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - i$.
   If $j < n - i$, $k < n - i$ and $a_k = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - 1$.
   If $j = n - i$ and $a_k \geqslant 2$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i - 1)$.
   If $j = n - i$ and $a_k = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0)$.
   If $k = n - i$ and $a_k \geqslant 2$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - i$.
   If $k = n - i$ and $a_k = 1$, then $\mathrm{hc}(c^*) = \mathrm{hc}(c_0) - (i - 1)$.

Since $i \geqslant 3$, we have $\mathrm{hc}(c^*) \leqslant \mathrm{hc}(c_0)$. Lemma 6 implies that there exist non-negative integers $a_1^+, \ldots, a_{n-i-1}^+$ such that

$$a_1^+ \leqslant 1, \ldots, a_{n-i-1}^+ \leqslant 1, \quad a_1^+ + \ldots + a_{n-i-1}^+ = i - 2$$

and

$$\mathrm{hc}(M(2, a_1^+, \ldots, a_{n-i-1}^+, 0; 0, 1)) \leqslant \mathrm{hc}(c^*).$$

There exists a permutation α of $(1, \ldots, n - i - 1)$ such that

$$a_{\alpha(1)}^+ \geqslant \ldots \geqslant a_{\alpha(n-i-1)}^+.$$

Put

$$c_{\mathrm{opt}} = M(2, a_{\alpha(1)}^+, \ldots, a_{\alpha(n-i-1)}^+, 0; 0, 1).$$

It is clear that $\mathrm{hc}(c_{\mathrm{opt}}) = \mathrm{hc}(M(2, a_{\alpha(1)}^+, \ldots, a_{\alpha(n-i-1)}^+, 0; 0, 1))$.

We have proved that $\mathrm{hc}(c_{\mathrm{opt}}) \leqslant \mathrm{hc}(c)$ for every hamiltonian coloring c of G. It follows from Lemma 5 that

$$\begin{aligned}\mathrm{hc}(c_{\mathrm{opt}}) &= 2(n-1) + (i-2)(2n-2i-2) + (n-2i+3)(n-3) \\ &= n^2 - 4n - 2i^2 + 6i + 1 \\ &= (n-2)^2 + 1 - 2(i-1)(i-2),\end{aligned}$$

which completes the proof of the theorem. □

4.

Let G be a connected graph of order $n \geqslant 3$, and let $2 \leqslant i \leqslant n$. It is obvious that G contains a hamiltonian-connected graph of order i as a subgraph if and only if G contain a hamiltonian-connected graph of order i as an induced subgraph.

Clearly, every nontrivial connected graph contains a nontrivial hamiltonian-connected graph as a subgraph.

The next theorem is the main result of the this paper:

Theorem 4. *Let G be a connected graph of order $n \geqslant 3$. If $2 \leqslant i \leqslant \frac{1}{2}(n+1)$ and there exists a hamiltonian-connected graph F of order i such that F is a subgraph of G, then*

$$\mathrm{hc}(G) \leqslant (n-2)^2 + 1 - 2(i-1)(i-2).$$

Proof. The result immediately follows from Theorems 2 and 3. □

Remark. Let G, i and F be the same as in Theorem 4. As immediately follows from Proposition 1 and Theorem 3, if $G = S(F; n-i)$, then

$$\mathrm{hc}(G) = (n-2)^2 + 1 - 2(i-1)(i-2).$$

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