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ON MONOTONE PERMUTATIONS OF ℓ-CYCLICALLY ORDERED SETS

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Abstract. For an ℓ-cyclically ordered set M with the ℓ-cyclic order C let $P(M)$ be the set of all monotone permutations on M. We define a ternary relation $\overline{C}$ on the set $P(M)$. Further, we define in a natural way a group operation (denoted by $\cdot$) on $P(M)$. We prove that if the ℓ-cyclic order C is complete and $\overline{C} \neq \emptyset$, then $(P(M), \cdot, \overline{C})$ is a half cyclically ordered group.

Keywords: ℓ-cyclically ordered set, completeness, monotone permutation, half cyclically ordered group

MSC 2000: 06F15

0. Introduction

The notions of cyclic order and of partial cyclic order were studied by several authors; we mention here Novák [12], Novák and Novotný [13], [14], Quilliot [15], Fishburn and Woodall [5].

In this paper we apply the terminology and notation as in [10]. Some definitions are recalled in Section 1 below.

Let M be an ℓ-cyclically ordered set; the relation of cyclic order on M will be denoted by C. Further, we denote by $P(M)$ the set of all monotone permutations on M.

We remark that the investigation of $P(M)$ goes back to Droste, Giraudet and Macpherson [4].

We define in a natural way the group operation on the set $P(M)$. Next, let $\overline{C}$ be the set of all triples $(\varphi_1, \varphi_2, \varphi_3)$ of elements of $P(M)$ such that for each $t \in M$ the

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relation

$$(\varphi_1(t), \varphi_2(t), \varphi_3(t)) \in C$$

is valid.

The notion of a half cyclically ordered group was introduced in [10] generalizing the notion of a half partially ordered group which had been studied by Giraudet and Lucas [6] (cf. also Giraudet and Rachůnek [7], Černák [1], [2], [3], Ton [16], Černák and the author [11], the author [8], [9]).

In [10] the following result was proved.

Theorem (A). *Let M be a finite ℓ-cyclically ordered set with* $\operatorname{card} M \geqslant 3$. *Then the structure $(P(M), \cdot, \overline{C})$ is a half cyclically ordered group.*

If the ℓ-cyclically ordered set is infinite, then the set $\overline{C}$ can be empty and thus in such case $(P(M), \cdot, \overline{C})$ fails to be a half cyclically ordered group.

The following question has been left open in [10]:

Assume that M is an infinite ℓ-cyclically ordered set such that $\overline{C} \neq \emptyset$. Is $(P(M), \cdot, \overline{C})$ a half cyclically ordered group?

In this paper we show that the answer is 'No'. Further, we prove

Theorem (B). *Let M be an infinite ℓ-cyclically ordered set such that*
(i) *the relation $\overline{C}$ on $P(M)$ is nonempty,*
(ii) *the cyclic order C on M is complete.*
Then $(P(M), \cdot, \overline{C})$ is a half cyclically ordered group.

1. Preliminaries

For the following definition cf. Novák and Novotný [13], [14].

1.1. Definition. A nonempty set M endowed with a ternary relation C is said to be cyclically ordered if the following conditions (I), (II) and (III) are satisfied:

(I) If $(x, y, z) \in C$, then (y, x, z) does not belong to C.
(II) If $(x, y, z) \in C$, then $(y, z, x) \in C$.
(III) If $(x, y, z) \in C$ and $(x, z, u) \in C$, then $(x, y, u) \in C$.

The relation C is called a cyclic order on M.

1.1.1. Definition. Let $(M; C)$ be a cyclically ordered set. Suppose that the following condition is satisfied:

(IV) Whenever x, y and z are mutually distinct elements of M, then either $(x, y, z) \in C$ or $(z, y, x) \in C$.

Then M is said to be ℓ-cyclically ordered and C is called an ℓ-cyclic order on M.

Each nonempty subset of a cyclically ordered set M is considered cyclically ordered (under the induced cyclic order).

1.2. Definition. Let G be a group. Suppose that G is, at the same time, a cyclically ordered set satisfying the condition

(V) if $(x_1, x_2, x_3) \in C$, $a \in G$, $y_i = ax_i$, $z_i = x_i a$ $(i = 1, 2, 3)$, then $(y_1, y_2, y_3) \in C$ and $(z_1, z_2, z_3) \in C$.

Then G is called a cyclically ordered group. In particular, if C is an ℓ-cyclic order, then G is called an ℓc-group.

Now suppose that $(G; \cdot)$ is a group and $(G; C)$ is a cyclically ordered set. We denote by $G{\uparrow}$ (and $G{\downarrow}$) the set of all $x \in G$ such that, whenever $(y_1, y_2, y_3) \in C$, then $(xy_1, xy_2, xy_3) \in C$ (or $(xy_3, xy_2, xy_1) \in C$, respectively).

1.3. Definition. Let $(G; \cdot, C)$ be as above. G is called a half cyclically ordered group if the following conditions are satisfied:

1) the system C is nonempty;
2) if $x \in G$ and $(y_1, y_2, y_3) \in C$, then $(y_1 x, y_2 x, y_3 x) \in C$;
3) $G = G{\uparrow} \cup G{\downarrow}$;
4) if $(x, y, z) \in C$, then either $\{x, y, z\} \subseteq G{\uparrow}$ or $\{x, y, z\} \subseteq G{\downarrow}$.

1.4. Definition. Let $(M; C)$ be an ℓ-cyclically ordered set. We denote by $P(M)(+)$ (and $(P(M)(-))$ the set of all permutations p on M such that, whenever $(x, y, z) \in C$, then $(p(x), p(y), p(z)) \in C$ (or $(p(z), p(y), p(x)) \in C$, respectively). The elements of the set $P(M) = P(M)(+) \cup P(M)(-)$ are called monotone permutations on M.

For $\varphi_1, \varphi_2 \in P(M)$ we put $\varphi_1\varphi_2 = \varphi$, where $\varphi(t) = \varphi_1(\varphi_2(t))$ for each $t \in T$. Then $P(M)$ turns out to be a group.

Further, let $\overline{C}$ be the set of all triples $(\varphi_1, \varphi_2, \varphi_3)$ of elements of $P(M)$ such that $(\varphi_1(t), \varphi_2(t), \varphi_3(t)) \in C$ for each $t \in M$. The structure $(P(M); \overline{C})$ is a cyclically ordered set and we have

$$P(M){\uparrow} = P(M)(+), \quad P(M){\downarrow} = P(M)(-).$$

Let $(M; C)$ be as in 1.4, $C \neq \emptyset$ and let $a \in M$. For $x, y \in M$ we put $x \leqslant_a y$ if either $x = a$ or $(a, x, y) \in C$. Then $(M; \leqslant_a)$ is a linearly ordered set with the least element a. (Cf. Novák [12], Theorem 3.1 and Lemma 3.4.) If $x_1, x_2, x_3 \in M$, then $(x_1, x_2, x_3) \in C$ if and only if some of the following relations

$$x_1 <_a x_2 <_a x_3, \quad x_2 <_a x_3 <_a x_1, \quad x_3 <_a x_1 <_a x_2$$

is valid.

1.5. Lemma. *Let $(M;C)$ be as above and let $a,b \in M$. The following conditions are equivalent:*
(i) *Each nonempty upper-bounded subset of $(M;\leqslant_a)$ has a supremum in $(M;\leqslant_a)$.*
(ii) *Each nonempty upper-bounded subset of $(M;\leqslant_b)$ has a supremum in $(M;\leqslant_b)$.*

The proof will be omitted. Also, if (i) holds, then each nonempty lower-bounded subset of $(M;\leqslant_a)$ has an infimum in $(M;\leqslant_a)$.

1.6. Definition. Let $(M;C)$ be as in 1.5. If the condition (i) from 1.5 is satisfied, then the cyclic order C on M is called complete.

Let X be a partially ordered set. We denote by C the set of all triples (x,y,z) of elements of X such that some of the following conditions is valid:

$$(*) \qquad x<y<z, \quad y<z<x, \quad z<x<y.$$

It is well-known that $(X;C)$ is a cyclically ordered set.

2. Auxiliary results

In this section we assume that $(M;C)$ is an ℓ-cyclically ordered set. Let $a \in M$ and $C \neq \emptyset$. Our aim is to characterize the elements of $P(M)\uparrow$ (and, similarly, the elements of $P(M)\downarrow$) by applying the linear order $\leqslant_a$ on M.

2.1. Lemma. *Let φ be a permutation on M such that $\varphi(a)=a$. Then the following conditions are equivalent:*
(i) *$\varphi \in P(M)\uparrow$;*
(ii) *φ is increasing with respect to the linear order $\leqslant_a$.*

P r o o f. Let (i) be valid. Suppose that $x,y \in M$, $x <_a y$. First assume that $x=a$. Then $a=\varphi(x) \neq \varphi(y)$, whence $\varphi(x) <_a \varphi(y)$. Next, suppose that $x \neq a$. Then we have $a <_a x <_a y$, thus $(a,x,y) \in C$. This yields that $(\varphi(a),\varphi(x),\varphi(y)) \in C$. Hence some of the relations

$$(1) \qquad \varphi(a) <_a \varphi(x) <_a \varphi(y), \quad \varphi(x) <_a \varphi(y) <_a \varphi(a), \quad \varphi(y) <_a \varphi(a) <_a \varphi(x)$$

is valid. Since $\varphi(a)=a$, we get that $\varphi(y) <_a \varphi(a)$ cannot hold; thus

$$\varphi(a) <_a \varphi(x) <_a \varphi(y).$$

Therefore (ii) is satisfied.

Conversely, suppose that (ii) is valid. Let $(x, y, z) \in C$. Thus some of the relations

$$x <_a y <_a z, \quad y <_a z <_a x, \quad z <_a x <_a y$$

is valid. Hence, according to (ii), some of the conditions

$$\varphi(x) <_a \varphi(y) <_a \varphi(z), \quad \varphi(y) <_a \varphi(z) <_a \varphi(x), \quad \varphi(z) <_a \varphi(x) <_a \varphi(y)$$

is satisfied. Therefore $(\varphi(x), \varphi(y), \varphi(z)) \in C$. $\square$

By analogous steps we obtain

2.2. Lemma. *Let φ be a permutation on M such that $\varphi(a)$ is the greatest element of M under the linear order $\leqslant_a$. The following conditions are equivalent:*

(i) $\varphi \in P(M)\downarrow$;

(ii) *φ is decreasing with respect to the linear order $\leqslant_a$.*

Now suppose that φ is a permutation on M such that $\varphi(a) \neq a$. We denote $\varphi(a) = q$, $\varphi^{-1}(a) = u$. We also put

$$\begin{aligned} M_1 &= \{t \in M\colon\ t <_a u\}, \\ M_2 &= \{t \in M\colon\ t \geqslant_a u\}. \end{aligned}$$

2.3. Lemma. *Let φ be a permutation on M such that $\varphi(a) \neq a$. The following conditions are equivalent:*

(i) $\varphi \in P(M)\uparrow$;

(ii) *with respect to the linear order $\leqslant_a$, φ is increasing on both the sets M_1 and M_2; moreover, $\varphi(x) <_a q$ for each $x \in M_2$.*

P r o o f. Let (i) be valid.

a) Assume that $x_1 \in M$, $a <_a x_1 <_a u$. Then $(a, x_1, u) \in C$, hence in view of (i) we obtain $(\varphi(a), \varphi(x_1), \varphi(u)) \in C$ and therefore $(q, \varphi(x_1), a) \in C$. This yields that some of the following relations is valid:

$$q <_a \varphi(x_1) <_a a, \quad a <_a q <_a \varphi(x_1), \quad \varphi(x_1) <_a q <_a a.$$

Since $q \not<_a a$, we must have $q <_a \varphi(x_1)$.

b) Assume that $x_1, x_2 \in M_1$, $x_1 <_a x_2$. Then by applying the result of a) and using the method as in a) (we take x_1, x_2 instead of a, x) we obtain the relation $\varphi(x_1) <_a \varphi(x_2)$. Hence φ is increasing on M_1.

c) Assume that $x_3 \in M$, $u <_a x_3$. Thus $(a, u, x_3) \in C$ and then (i) yields $(\varphi(a), \varphi(u), \varphi(x_3)) \in C$. Therefore $(q, a, \varphi(x_3)) \in C$. Hence some of the following relations is valid:

$$q <_a a <_a \varphi(x_3), \quad \varphi(x_3) <_a q <_a a, \quad a <_a \varphi(x_3) <_a q.$$

Since $q \not<_a a$, we must have $\varphi(x_3) <_a q$.

d) Let $x_3, x_4 \in M$, $u <_a x_3 <_a x_4$. Hence $(u, x_3, x_4) \in C$ and in view of (i), $(a, \varphi(x_3), \varphi(x_4)) \in C$. Thus some of the relations

$$a <_a \varphi(x_3) <_a \varphi(x_4), \quad \varphi(x_4) <_a a <_a \varphi(x_3), \quad \varphi(x_3) <_a \varphi(x_4) <_a a$$

is valid. Since $\varphi(x_4) \not<_a a$, we must have $\varphi(x_3) <_a \varphi(x_4)$. This yields that φ is increasing on M_2.

e) Conversely, let (ii) be satisfied. Let $(x, y, z) \in C$. Thus without loss of generality we can assume that $x <_a y <_a z$.

If either $\{x, y, z\} \subseteq M_1$ or $\{x, y, z\} \subseteq M_2$, then in view of (ii) we have

$$\varphi(x) <_a \varphi(y) <_a \varphi(z),$$

hence $((\varphi(x), \varphi(y), \varphi(z)) \in C$.

Suppose that $x, y \in M_1$ and $z \in M_2$. Hence $a \leqslant_a x <_a y$. Since φ is increasing on M_1 and $\varphi(a) = q$ we get $q \leqslant_a \varphi(a) <_a \varphi(y)$. Further, since $z \in M_2$, in view of (ii) we have $\varphi(z) <_a q$. Thus

$$\varphi(z) <_a \varphi(x) <_a \varphi(y),$$

hence $(\varphi(z), \varphi(x), \varphi(y)) \in C$ and so $(\varphi(x), \varphi(y), \varphi(z)) \in C$.

Finally, suppose that $x \in M_1$ and $y, z \in M_2$. From $a \leqslant_a x$ and from the fact that φ is increasing on M_1 we obtain $q \leqslant_a \varphi(x)$. Since φ is increasing on M_2 we have $\varphi(y) <_a \varphi(z)$. Thus

$$\varphi(y) <_a \varphi(z) <_a \varphi(x).$$

Hence $((\varphi(y), \varphi(z), \varphi(x)) \in C$ and therefore $((\varphi(x), \varphi(y), \varphi(z)) \in C$.

Summarizing, we conclude that φ belongs to $P(M)\!\uparrow$. □

2.4. Corollary. *Let φ be as in 2.3 and let the condition* (i) *from 2.3 be satisfied. Then*

$$\varphi(M_1) = \{x \in M\colon\ x \geqslant_a q\},$$
$$\varphi(M_2) = \{x \in M\colon\ x <_a q\}.$$

Again, let φ be a permutation on M such that $\varphi(a) \neq a$, and let q, u be as above. Denote

$$M_1' = \{t \in M\colon\ a \leqslant_a t \leqslant_a u\}, \quad M_2' = \{t \in M\colon\ t >_a u\}.$$

By a method analogous to the proof of 2.3 we obtain

2.5. Lemma. *Let φ be a permutation on M such that $\varphi(a) \neq a$. Then φ belongs to $P(M)\!\downarrow$ if and only if the following conditions are satisfied:*

(i) *φ is decreasing on the set M_1';*

(ii) *if $M_2' \neq \emptyset$, then φ is decreasing on M_2 and $q <_a \varphi(x)$ for each $x \in M_2'$.*

3. The completeness condition

The aim of the present section is to prove the following result:

Theorem 3.0. *Let M be an ℓ-cyclically ordered set with the ℓ-cyclic order such that*

(i) *the relation $\overline{C}$ on $(P(M))$ is nonempty;*

(ii) *the ℓ-cyclic order C on M is complete.*

Then $(P(M), \cdot, \overline{C})$ is a half cyclically ordered group.

Proof. By looking at Definition 1.3 we see that it suffices to show that the condition 4) from 1.3 is valid in our case; thus we have to verify the validity of

$$(**)\qquad \text{if } (x,y,z) \in \overline{C}, \text{ then either } \{x,y,z\} \subseteq P(M)\!\uparrow \text{ or } \{x,y,z\} \subseteq P(M)\!\downarrow.$$

By way of contradiction, assume that the condition $(**)$ does not hold. Then it is easy to verify that without loss of generality we can suppose that there exist $x \in P(M)\!\downarrow$ and $y \in P(M)\!\uparrow$ such that

$$(1)\qquad (e, x, y) \in \overline{C},$$

where e is the neutral element of the group $P(M)$.

Let a and $\leqslant_a$ be as in the previous section. To simplify the notation, we write in the present section $\leqslant$ and $<$ instead of $\leqslant_a$ or $<_a$, respectively. Since a is the least element of $(M; \leqslant)$, the relation (1) yields

$$(2)\qquad a < x(a) < y(a).$$

Denote $x(a) = p$, $u = x^{-1}(a)$. Hence $a < u$ and according to 2.4 we have

3.1. *The partial mapping* $x|[a,u]$ *is a dual isomorphism of the interval* $[a,u]$ *onto the interval* $[a,p]$.

We put

$$A=\{t\in[a,u]\colon\ e(t)<x(t)\}.$$

Then $A\neq\emptyset$, since $a\in A$. Moreover, $u\notin A$, thus A is an upper-bounded subset of M. Therefore, in view of the completeness condition, there exists

$$t_0=\sup A$$

in M; clearly $a<t_0\leqslant u$.

3.2. *A is an ideal of the lattice* $[a,u]$.

Proof. Let $t_1\in A$, $t_2\in[a,u]$, $t_2<t_1$. Then

$$e(t_2)<e(t_1)<x(t_1)<x(t_2), \tag{3}$$

whence $t_2\in A$. □

3.3. *For each* $t_1,t_2\in A$, $x(t_1)>t_2$.

Proof. The case $t_2=t_1$ is obvious. If $t_2<t_1$, then it suffices to apply (3). Let $t_2>t_1$. We get $x(t_1)>x(t_2)>t_2$. □

The completeness condition also yields that there exists

$$p_1=\inf\{x(t)\colon\ t\in A\}$$

in the interval $[a,p]$. Then from 3.1 we conclude

3.4. $p_1=x(t_0)$.

3.5. $t_0\in A$.

Proof. In view of 3.3 we have

$$\inf\{x(t)\colon\ t\in A\}\geqslant\sup\{t_1\colon\ t_1\in A\},$$

whence $x(t_0)\geqslant t_0$. If $x(t_0)>t_0$, then $t_0\in A$. Otherwise we would have $x(t_0)=t_0=e(t_0)$ and this contradicts the relation (1). □

We denote

$$A' = \{t \in [u, a]\colon\ x(t) < e(t)\}.$$

We have $u \in A'$ and $a \notin A'$. Hence A' is nonempty and lower-bounded. In view of the completeness condition there exists $t_0' \in [a, u]$ such that

$$t_0' = \inf A'.$$

By analogous steps as above we obtain

3.6. $t_0' \in A'$.

If x, y are elements of a lattice L such that $x < y$ and there is no $z \in L$ with $x < z < y$, then $[x, y]$ is a *prime interval* in L.

From 3.5 and 3.6 we conclude

3.7. $t_0 < t_0'$.

3.8. $[t_0, t_0']$ *is a prime interval of the lattice* $[a, u]$.

Proof. If $[t_0, t_0']$ fails to be a prime interval, then there exists $t \in [t_0, t_0']$ with $t_0 \neq t \neq t_0'$. Suppose that t has this property.

If $x(t) > t$, then $t \leqslant t_0$, which is impossible. Similarly, if $x(t) < t$, then $t \geqslant t_0'$, which cannot hold. Hence $x(t) = t = e(t)$; in view of (1) we arrive at a contradiction. □

Let z_1 and z_2 be elements of a lattice such that $[z_1, z_2]$ is a prime interval; we express this fact by writing $z_1 \prec z_2$.

In view of 3.8 we have $t_0 \prec t_0'$. Then according to 3.1, $x(t_0') \prec x(t_0)$. Since M is linearly ordered, 3.5 yields $t_0 \leqslant x(t_0')$. Further, according to 3.6, $x(t_0') < t_0'$. Then we must have $x(t_0') = t_0$. Therefore $x(t_0') \prec t_0'$. Hence we obtain

3.9. $x(t_0') \prec e(t_0')$.

The relation (1) yields

$$(e(t_0'), x(t_0'), y(t_0')) \in C.$$

Since $x(t_0') < e(t_0')$, we get

$$x(t_0') < y(t_0') < e(t_0').$$

In view of 3.9 we arrive at a contradiction. Thus the relation $(**)$ must hold. □

As a corollary, we obtain that Theorem (B) is valid.

Now let $(M; C)$ be an ℓ-cyclically ordered set such that M is finite. Since each nonempty subset of a finite linearly ordered set has a supremum, in view of 1.5 we conclude that the ℓ-cyclic order C is complete. Further, suppose that $\operatorname{card} M \geqslant 3$, $\operatorname{card} M = n$. Then without loss of generality we can assume that $M = \{0, 1, 2, \ldots, n-1\}$ with the natural linear order $<$ and that C is the set of all triples (x, y, z) such that one of the relations $(*)$ in Section 1 is valid (cf. the quotation from [12] in Section 1). For $t \in M$ we define $\varphi_1(t)$, $\varphi_2(t)$ and $\varphi_3(t)$ as follows: $\varphi_1(t) = t$; $\varphi_2(t) = x$, where $x \in M$ and $x \equiv t + 1 \pmod{n}$; $\varphi_3(t) = y$, where $y \in M$ and $y \equiv t + 2 \pmod{n}$. Then $(\varphi_1(t), \varphi_2(t), \varphi_3(t)) \in C$ for each $t \in M$, whence $(\varphi_1, \varphi_2, \varphi_3) \in \overline{C}$ and thus $\overline{C} \neq \emptyset$.

Therefore Theorem 3.0 includes also Theorem (A).

4. An example

We denote by $\mathbb{R}$ the set of all reals with the natural linear order. Further, let $\mathbb{Q}$ be the set of all rationals.

Let us apply the following notation. Suppose that $u, v \in \mathbb{R}$, $u < v$ and that g is a real function defined on the set

$$\mathbb{Q}_1 = \{t \in \mathbb{Q}\colon\ u \leqslant t \leqslant v\}.$$

If for each sequence (t_n) such that $t_n \in \mathbb{Q}_1$ and the sequence (t_n) converges to u (in the usual sense), the corresponding sequence $(g(t_n))$ converges to a real r, then we write

$$\lim_{t \to u+} g(t) = r.$$

The notation

$$\lim_{t \to v-} g(t) = r_1$$

has an analogous meaning.

For $x, y \in \mathbb{R}$ with $x < y$ we put

$$[x, y]_{\mathbb{Q}} = \{t \in \mathbb{Q}\colon\ x \leqslant t \leqslant y\}, \quad (x, y)_{\mathbb{Q}} = \{t \in \mathbb{Q}\colon\ x < t < y\};$$

the meaning of the symbols $[x, y)_{\mathbb{Q}}$ and $(x, y]_{\mathbb{Q}}$ is analogous.

We choose reals p, q, u, v, u_1, v_1 such that

$$0 < u < v < p < u_1 < v_1 < q < 1, \quad p, q, u, u_1 \in \mathbb{Q} \quad \text{and} \quad v, v_1 \in \mathbb{R} \setminus \mathbb{Q}.$$

We denote

$$A_1 = [0, u)_{\mathbb{Q}}\,, \quad A_2 = [u, v)_{\mathbb{Q}}\,, \quad A_3 = (v, u_1]_{\mathbb{Q}}\,,$$
$$A_4 = (u_1, v_1)_{\mathbb{Q}}\,, \quad A_5 = (v_1, 1)_{\mathbb{Q}}\,.$$

We put $e(t) = t$ for each $t \in [0, 1)_{\mathbb{Q}}$.

For each $i \in \{1, 2, 3, 4, 5\}$ there exist functions x_i and y_i from A_i to $[0, 1)_{\mathbb{Q}}$ such that

(i) the function x_i is decreasing on A_i;

(ii) the function y_i is increasing on A_i;

(iii) the following conditions are satisfied (cf. Fig. 1):

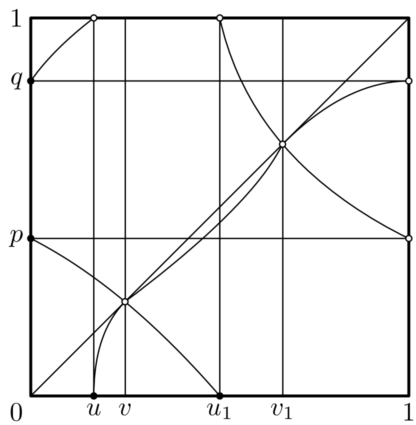

Fig. 1

1) $x_1(0) = p$, $x_1(u) > v$, $y_1(0) = q$,

$$\lim_{t \to u-} y_1(t) = 1;$$

2) $x_2(u) = x_1(u)$, $y_2(u) = 0$, $y_2(t) < t < x_2(t)$ for each $t \in A_2$,

$$\lim_{t \to v-} x_2(t) = \lim_{t \to v-} y_2(t) = v;$$

3) $x_3(u_1) = 0$, $x_3(t) < y_3(t) < t$ for each $t \in A_3$,

$$\lim_{t \to v+} x_3(t) = \lim_{t \to v+} y_3(t) = v;$$

4) $y_4(t) < t < x_4(t)$ for each $t \in A_4$,

$$\lim_{t \to u_1+} x_4(t) = 1, \quad \lim_{t \to v_1-} y_4(t) = v_1;$$

5) $\lim_{t\to v_1+} x_5(t) = v_1$, $\lim_{t\to 1-} x_5(t) = p$, $\lim_{t\to v_1+} y_5(t) = v_1$, $\lim_{t\to 1-} y_5(t) = q$,

$$p < x_5(t) < y_5(t) < t \quad \text{and} \quad y_5(t) < q \text{ for each } t \in A_5.$$

Put $M = [0,1)_{\mathbb{Q}}$. We define a mapping x of M into M by putting

$$x(t) = x_i(t) \quad \text{whenever} \quad t \in A_i \ (i = 1,2,3,4,5);$$

analogously we define a mapping y of M into M.

Then we have

1a) $e(t) < x(t) < y(t)$ for each $t \in A_1$,
2a) $y(t) < e(t) < x(t)$ for each $t \in A_2$,
3a) $x(t) < y(t) < e(t)$ for each $t \in A_3$,
4a) $y(t) < e(t) < x(t)$ for each $t \in A_4$,
5a) $x(t) < y(t) < e(t)$ for each $t \in A_5$.

Thus for each $i \in \{1,2,3,4,5\}$ and for each $t \in A_i$ the relation

$$(e(t), x(t), y(t)) \in C \tag{1}$$

is valid.

Moreover, x is decreasing on the sets

$$A_1 \cup A_2 \cup A_3 \quad \text{and} \quad A_4 \cup A_5;$$

y is increasing on the sets

$$A_1 \quad \text{and} \quad A_2 \cup A_3 \cup A_4 \cup A_5.$$

Consider the cyclic order C on M defined as in $(*)$ of Section 1. From the definitions of x, y and e, from the results of Section 2 and from (1) we obtain

4.1. Lemma. *The functions e and y belong to $P(M)\uparrow$, x belongs to $P(M)\downarrow$. Moreover, $(e,x,y) \in \overline{C}$.*

In view of 4.1 and according to the condition 4) of 1.3 we conclude that the structure $(P(M); \cdot, \overline{C})$ fails to be a half cyclically ordered group.

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