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CYCLES WITH A GIVEN NUMBER OF VERTICES FROM EACH PARTITE SET IN REGULAR MULTIPARTITE TOURNAMENTS

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Abstract. If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. A digraph D is called regular, if there is a number $p \in \mathbb{N}$ such that $d^+(x) = d^-(x) = p$ for all vertices x of D.

A c-partite tournament is an orientation of a complete c-partite graph. There are many results about directed cycles of a given length or of directed cycles with vertices from a given number of partite sets. The idea is now to combine the two properties. In this article, we examine in particular, whether c-partite tournaments with r vertices in each partite set contain a cycle with exactly $r-1$ vertices of every partite set. In 1982, Beineke and Little [2] solved this problem for the regular case if $c = 2$. If $c \geqslant 3$, then we will show that a regular c-partite tournament with $r \geqslant 2$ vertices in each partite set contains a cycle with exactly $r-1$ vertices from each partite set, with the exception of the case that $c = 4$ and $r = 2$.

Keywords: multipartite tournaments, regular multipartite tournaments, cycles

MSC 2000: 05C40, 05C20

1. Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \rightsquigarrow Y$ denotes the fact that there is no arc leading from Y to X. For the number of arcs from X to Y we write $d(X, Y)$.

If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if the arc $xy \in E(D)$ exists, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and the *indegree* of x, respectively. Furthermore, the numbers $\delta_D^+ = \delta^+ = \min\{d^+(x)\colon\ x \in V(D)\}$ and $\delta_D^- = \delta^- = \min\{d^-(x)\colon\ x \in V(D)\}$ are the *minimum outdegree* and the *minimum indegree*, respectively.

For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

If we speak of a *cycle*, then we mean a directed cycle, and a cycle of length n is called an *n-cycle*. The length of a cycle C is denoted by $L(C)$. A cycle in a digraph D is *Hamiltonian* if $L(C) = |V(D)|$. A *cycle-factor* of a digraph D is a spanning subdigraph consisting of disjoint cycles. A digraph D is called *pancyclic* if it contains cycles of length n for all $n \in \{3, 4, \ldots, |V(D)|\}$, and *even pancyclic* if it contains cycles of all even lengths. If $x \in V(C)$ ($x \in V(P)$, respectively) for a cycle C (a path P), then we denote the successor of x in the given cycle (path) by x^+ and the predecessor by x^-. A digraph D is *cycle complementary* if there exist two vertex-disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$.

A digraph D is *strongly connected* or *strong* if for each pair of vertices u and v, there is a path from u to v in D. A digraph D with at least $k+1$ vertices is *k-connected* if for any set A of at most $k-1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The *connectivity*, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected. If $\kappa(D) = 1$ and x is a vertex of D such that $D - x$ is not strong, then we say that x is a *cut-vertex* of D.

There are several measures of how much a digraph differs from being regular. In [18], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)} \{d^+(x), d^-(x)\} - \min_{y \in V(D)} \{d^+(y), d^-(y)\}.$$

If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leqslant 1$, then D is called *almost regular*.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leqslant |V_2| \leqslant \ldots \leqslant |V_c|$, then $|V_c| = \alpha(D)$ is the independence number of D.

Let $B = B(r_1, r_2, r_3, r_4)$ be the following bipartite tournament, which will be useful later. Let R_1, R_2, R_3, R_4 be pairwise disjoint independent sets of vertices with

$|R_i| = r_i$ for $1 \leqslant i \leqslant 4$. Define $V(B) = R_1 \cup R_2 \cup R_3 \cup R_4$ such that $R_i \to R_{i+1}$ for $i = 1, 2, 3$ and $R_4 \to R_1$.

There is extensive literature on cycles in multipartite tournaments, see e.g., Bang-Jensen and Gutin [1], Guo [6], Gutin [7], Volkmann [13] and Yeo [17]. Many results are about the existence of cycles of a given length as e.g. the following result of Bondy [3].

Theorem 1.1 (Bondy [3]). *Each strongly connected c-partite tournament contains a cycle of order m for each $m \in \{3, 4, \ldots, c\}$.*

Other articles treat the existence of cycles containing vertices of a given number of partite sets. A good example is the following theorem of Goddard and Oellermann [5].

Theorem 1.2 (Goddard, Oellermann [5]). *If x is an arbitrary vertex of a strongly connected c-partite tournament D, then x belongs to a cycle that contains vertices from exactly q partite sets for each $q \in \{3, 4, \ldots, c\}$.*

An interesting question is now to find sufficient conditions for a multipartite tournament such that we are able to combine these two categories of results, that means to solve the following problem.

Problem 1.3. Which conditions have to be fulfilled in order that a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ contains a cycle with exactly r_i vertices of V_i for all $1 \leqslant i \leqslant c$ and given integers $0 \leqslant r_i \leqslant |V_i|$?

In 1997, A. Yeo [16] gave a solution of this problem for regular c-partite tournaments in the case that $r_i = |V_i|$ for all $1 \leqslant i \leqslant c$.

Theorem 1.4 (Yeo [16]). *Every regular multipartite tournament D is Hamiltonian.*

Since, according to the well known result of Moon [9] that every strongly connected tournament is vertex-pancyclic, a strongly connected tournament is Hamiltonian, we note that the next theorem also treats Problem 1.3.

Theorem 1.5 (Volkmann [12]). *Let D be an almost regular c-partite tournament with $c \geqslant 4$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c-1\}$.*

In a recent article, the authors [15] settled a conjecture of Volkmann in affirmative by proving that Theorem 1.5 remains valid for $p = c$ when $c \geqslant 5$. Thus, we arrive at the following result.

Theorem 1.6 (Volkmann, Winzen [15]). *Let D be an almost regular c-partite tournament with $c \geqslant 5$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c\}$.*

Hence, Theorem 1.6 presents a solution of Problem 1.3 for almost regular c-partite tournaments and the case that $r_i = 1$ for all $1 \leqslant i \leqslant c$. In this article, we will treat the case that D is a regular c-tournament and $r_i = |V_i| - 1$ for all $1 \leqslant i \leqslant c$. Since the vertices of a cycle in a bipartite tournament D alternate between the two partite sets of D, Beineke and Little [2] (for a stronger form, see also Zhang [19]) gave a solution to this problem if $c = 2$.

Theorem 1.7 (Beineke, Little [2]). *A bipartite tournament is even pancyclic if it is Hamiltonian and is not isomorphic to the bipartite tournament $B(r, r, r, r)$ with $r \geqslant 2$.*

If we remove one vertex of each partite set in the bipartite tournament $B(r, r, r, r)$, then obviously the remaining bipartite tournament is not Hamiltonian. The case that $c = 3$ is also solved if we pay attention to the next result.

Theorem 1.8 (Volkmann [14]). *Let D be a regular 3-partite tournament with $|V(D)| \geqslant 6$. Then D contains two complementary cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to the digraph $D_{3,2}$ of Fig. 1.*

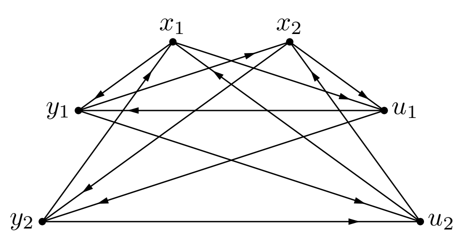

Figure 1. The 2-regular 3-partite tournament $D_{3,2}$

Since a 3-cycle contains vertices of exactly 3-partite sets and the digraph $D_{3,2}$ contains the cycle $x_2 y_2 u_2 x_2$, we see that a regular 3-partite tournament with r vertices of each partite set always contains a cycle with exactly $r - 1$ vertices of every partite set.

In the following, we will show that all regular c-partite tournaments with r vertices in every partite set contain a cycle with exactly $r - 1$ vertices of each partite set provided $c \geqslant 5$ or $c = 4$ and $r \geqslant 3$.

2. Preliminary results

The following results play an important role in our investigations.

Theorem 2.1 (Rédei [10]). *Every tournament contains a Hamiltonian path.*

Theorem 2.2 (Yeo [16]). *If D is a multipartite tournament with $\kappa(D) \geqslant \alpha(D)$, then D is Hamiltonian.*

Theorem 2.3 (Camion [4]). *A tournament is strongly connected if and only if it is Hamiltonian.*

Theorem 2.4 (Yeo [16]). *Let D be a $(\lfloor q/2 \rfloor + 1)$-connected c-partite tournament such that $\alpha(D) \leqslant q$. If D has a cycle-factor, then D is Hamiltonian.*

Theorem 2.5 (Yeo [18]). *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $|V_1| \leqslant |V_2| \leqslant \ldots \leqslant |V_c|$. If*

$$i_g(D) \leqslant \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 2}{2},$$

then D is Hamiltonian.

Lemma 2.6 (Yeo [17], Gutin, Yeo [8]). *A digraph D has no cycle-factor if and only if its vertex set $V(D)$ can be partitioned into four subsets Y, Z, R_1 and R_2 such that*

$$R_1 \rightsquigarrow Y \quad \text{and} \quad (R_1 \cup Y) \rightsquigarrow R_2,$$

where Y is an independent set and $|Y| > |Z|$.

Lemma 2.7 (Tewes, Volkmann, Yeo [11]). *If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D, then $\big||V_i| - |V_j|\big| \leqslant 2i_g(D)$ for $1 \leqslant i, j \leqslant c$.*

Since we consider only the case that $i_g(D) = 0$ in this article, we can note the following.

Remark 2.8. Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament. Then Lemma 2.7 implies that $r = |V_1| = |V_2| = \ldots = |V_c|$ and

$$d^+(x), d^-(x) = \frac{(c-1)r}{2}$$

for all $x \in V(D)$. That means especially that c is odd, if r is odd.

3. Main results

Theorem 3.1. *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geqslant 4$ and $|V_1| = |V_2| = \ldots = |V_c| = r \geqslant 2$. Furthermore, let X be an arbitrary subset of $V(D)$ consisting of m partite sets with exactly k vertices and $c-m$ partite sets with exactly $k-1$ vertices for $0 < m \leqslant c$ and $1 \leqslant k \leqslant r-1$. If*

$$r \geqslant \begin{cases} \left\lceil \dfrac{2k(c-1)-2}{c-3} \right\rceil + k & \text{and } m = c, \\ \left\lceil \dfrac{2k(c-1)-1}{c-3} \right\rceil + k & \text{and } m = c-1, \\ \left\lceil \dfrac{(2k-3)c+3m-2k+3}{c-3} \right\rceil + k & \text{and } m \leqslant c-2 \end{cases}$$

then D contains a cycle C such that $V(C) = V(D) - X$.

Proof. Let $D' = D - X$ with the partite sets $V_1', V_2', \ldots, V_c'$ such that $|V_1'| \leqslant |V_2'| \leqslant \ldots \leqslant |V_c'| \leqslant |V_1'| + 1$. Since D is regular, it follows that

$$i_g(D') \leqslant \begin{cases} k(c-1) & \text{if } c-1 \leqslant m \leqslant c, \\ (k-1)(c-1)+m & \text{if } m \leqslant c-2. \end{cases}$$

If

$$\begin{cases} k(c-1) \\ (k-1)(c-1)+m \end{cases} \leqslant \frac{|V(D')| - |V_{c-1}'| - 2|V_c'| + 2}{2} \quad \begin{array}{l} \text{if } c-1 \leqslant m \leqslant c, \\ \text{if } m \leqslant c-2, \end{array}$$

then Theorem 2.5 implies that D' is Hamiltonian, and hence the desired result. To show this, let us note that

$$\frac{|V(D')| - |V_{c-1}'| - 2|V_c'| + 2}{2} = \begin{cases} \dfrac{(c-3)(r-k)+2}{2} & \text{if } m = c, \\ \dfrac{(c-3)(r-k)+1}{2} & \text{if } m = c-1, \\ \dfrac{(c-3)(r-k)+c-m-1}{2} & \text{if } m \leqslant c-2. \end{cases}$$

If we distinguish the cases $m = c$, $m = c-1$ and $m \leqslant c-2$, then, noticing that $r \in \mathbb{N}$, equivalent transformations yield the bounds for r as in the assumptions of this theorem. This completes the proof of the theorem. □

In the following, we will treat only the case that $m = c$ and $k = 1$. In this case Theorem 3.1 leads to the next corollary.

Corollary 3.2. *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D such that $|V_1| = |V_2| = \ldots = |V_c| = r$. Furthermore, let $x_i \in V_i$ be arbitrary for all $1 \leqslant i \leqslant c$. If $c \geqslant 5$ and $r \geqslant 4$ or $c = 4$ and $r \geqslant 6$, then there exists a cycle C in D such that $V(C) = \bigcup\limits_{i=1}^{c} (V_i - x_i)$.*

The next example shows that the condition of Corollary 3.2 that $r \geqslant 4$, if $c \geqslant 5$, is the best possible.

Example 3.3. Let D be a regular $(2p+1)$-partite tournament with $r = 3$ vertices in each partite set. If D consists of three regular disjoint subtournaments H_1, H_2, H_3 of order $2p+1$ such that $H_1 \leadsto H_2 \leadsto H_3 \leadsto H_1$, then $D' = D - V(H_1)$ contains no Hamiltonian cycle.

Nevertheless, if $r = 3$ and thus, according to Remark 2.8, $c = 2p + 1$, then there exist vertices $x_1, x_2, \ldots, x_c$ with $x_i \in V_i$ such that D contains a cycle C with $V(C) = \bigcup\limits_{i=1}^{c} (V_i - x_i)$, as the following theorem demonstrates.

Theorem 3.4. *Let $V_1, V_2, \ldots, V_{2p+1}$ be the partite sets of a regular $(2p+1)$-partite tournament with $p \geqslant 2$ such that $|V_1| = |V_2| = \ldots = |V_{2p+1}| = 3$. Then D contains a cycle with exactly 2 vertices of each partite set.*

Proof. Suppose that D contains no cycle with exactly 2 vertices of each partite set. Let T_1 be a subtournament of D with $|V(T_1)| = 2p+1$. Then we define $D' = D - V(T_1)$. Since D is regular, Remark 2.8 with $r = 3$ implies $d^+(x), d^-(x) = 3p$ and thus $d^+_{D'}(x), d^-_{D'}(x) \geqslant p$.

First, let D' be 2-connected. Because of $\alpha(D') = 2$, Theorem 2.2 yields that D' is Hamiltonian, a contradiction.

Secondly, let D' be not strong. Then D' can be partitioned into strong components $D_1, D_2, \ldots, D_t$ such that $D_i \leadsto D_j$ for $i < j$. The fact that $d^-_{D_1}(x) \geqslant p$ for all $x \in V(D_1)$ implies $|V(D_1)| \geqslant 2\delta^-_{D_1} + 1 \geqslant 2p + 1$. Analogously, we observe that $|V(D_t)| \geqslant 2p+1$. Since $|V(D_1)| + |V(D_2)| + \ldots + |V(D_t)| = 4p+2$, we deduce that $t = 2$ and $|D_1| = |D_2| = 2p+1$. This is possible only if $D_2 \leadsto T_1 \leadsto D_1$ and D_1, D_2, T_1 are regular tournaments. Hence, D is the multipartite tournament from Example 3.3. If $a_1 a_2 \ldots a_{2p+1} a_1$ is a Hamiltonian cycle of T_1, $v_1 \in V(D_1) \cap V(a_1)$ and $b_1 b_2 \ldots b_{2p+1} b_1$ is a Hamiltonian cycle of D_2 such that $b_1 \in V(a_1)$, then $a_1 a_2 \ldots a_{2p+1} v_1 b_2 b_3 \ldots b_{2p+1} a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Thirdly, let D' be exactly 1-connected. This yields that D' contains a cut-vertex u such that $D' - \{u\}$ consists of strong components $D_1, D_2, \ldots, D_t$ with the property that $D_i \leadsto D_j$ for $i < j$. Furthermore, there are vertices $v_1 \in V(D_1)$ and $v_t \in V(D_t)$

such that $v_t \to u \to v_1$. Since $d^-_{D_1}(x) \geqslant p-1$ for all $x \in V(D_1)$, we conclude that $|V(D_1)| \geqslant 2\delta^-_{D_1} + 1 \geqslant 2p-1$. Analogously, we see that $|V(D_t)| \geqslant 2p-1$. Without loss of generality, let $|V(D_1)| \leqslant |V(D_t)|$, since otherwise we use the converse D^{-1} of D. Now we distinguish the two possible cases $|V(D_1)| = 2p-1$ and $|V(D_1)| = 2p$.

Case 1. Suppose that $|V(D_1)| = 2p-1$. This is possible only if D_1 is a $(p-1)$-regular tournament with $u \to V(D_1)$, $V(T_1) \rightsquigarrow V(D_1)$ and $2p-1 \leqslant |V(D_t)| \leqslant 2p+2$. Let $C = a_1a_2\dots a_{2p-1}a_1$ be a Hamiltonian cycle of D_1.

Subcase 1.1. Let $|V(D_t)| = 2p-1$. As above, we deduce that D_t is a regular tournament with a Hamiltonian cycle $\tilde{C} = b_1b_2\dots b_{2p-1}b_1$ such that $V(D_t) \rightsquigarrow V(T_1)$ and $V(D_t) \to u$. The fact that $|V(D_2)| + |V(D_3)| + \ldots + |V(D_{t-1})| = 3$ implies that $t = 3$ or $t = 5$.

First, let $t = 3$. In this case, D_2 is a 3-cycle $c_1c_2c_3c_1$. Without loss of generality, we may suppose that $a_{2p-1} \notin V(c_1)$ and $b_1 \notin V(c_3)$. Now, $a_1a_2\dots a_{2p-1}c_1c_2c_3b_1b_2\dots b_{2p-1}ua_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Secondly, let $t = 5$. This yields that $|D_2| = |D_3| = |D_4| = 1$ so that $D_2 = \{v_2\}$, $D_3 = \{v_3\}$ and $D_4 = \{v_4\}$. If $v_2 \notin V(v_3)$ and $v_3 \notin V(v_4)$, then the vertices of $V(C)$ and $V(\tilde{C})$ can be chosen so that $a_{2p-1} \notin V(v_2)$ and $b_1 \notin V(v_4)$. Now, $a_1a_2\dots a_{2p-1}v_2v_3v_4b_1b_2\dots b_{2p-1}ua_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. If $v_2 \in V(v_3)$ and $v_2' \in V(T_1) \cap V(v_3)$, then, without loss of generality, the numbering of the cycles C and $\tilde{C}$ can be chosen so that $v_4 \notin V(b_2)$ and $a_{2p-1} \notin V(b_1)$. In this case we see that $b_1ua_1v_3v_4b_2b_3\dots b_{2p-1}v_2'a_2a_3\dots a_{2p-1}b_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. Analogously, we arrive at a contradiction if $v_3 \in V(v_4)$.

Subcase 1.2. Assume that $|V(D_t)| = 2p$ and thus $t = 4$ and $|V(D_2)| = |V(D_3)| = 1$. Let $D_2 = \{v_2\}$ and $D_3 = \{v_3\}$.

Subcase 1.2.1. Suppose that D_4 is Hamiltonian with a Hamiltonian cycle $C' = b_1b_2\dots b_{2p}b_1$.

First, let $v_2 \notin V(v_3)$. Because of

$$2p^2 \leqslant \sum_{x \in V(D_4)} d^+_{D'}(x) \leqslant \frac{2p(2p-1)}{2} + d(D_4, u) = 2p^2 - p + d(D_4, u),$$

we deduce that $d(D_4, u) \geqslant p \geqslant 2$. Hence, there exists a vertex $b_i \in V(C')$ such that $b_i \to u$ and $b_i^+ \notin V(v_3)$. Now the vertices of C can be numbered so that $a_{2p-1} \notin V(v_2)$ and $a_1a_2\dots a_{2p-1}v_2v_3b_i^+\dots b_i^-b_iua_1$ is a Hamiltonian cycle of D', a contradiction.

Secondly, let $v_2 \in V(v_3)$. This implies that D_4 is a tournament. Let $v_2' \in V(T_1) \cap V(v_3)$. If $v_2' \to D_4$, then we observe that $d^+_{D_4}(y) \geqslant 3p - (|V(T_1)| - 2) - |\{u\}| = p$ for all $y \in V(D_4)$, and thus

$$2p^2 - p = |E(D_4)| \geqslant 2p^2,$$

a contradiction. Let $\{v_2', u\} = \{x, y\}$ and $x \to y$. If $y' \in N^-(x) \cap V(D_4)$, then let $y'b_2b_3 \ldots b_{2p}y'$ be a Hamiltonian cycle of D_4. Summarizing our results, we see that $a_1a_2 \ldots a_{2p-1}v_2b_2b_3 \ldots b_{2p}y'xya_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Subcase 1.2.2. Let D_4 be not Hamiltonian. Since D_4 is strongly connected, Theorem 2.3 implies that D_4 is no tournament. The fact that D_1 is a tournament and $u \to D_1$ yields that D_4 consists of vertices of exactly $2p-1$ partite sets, and thus $v_2 \notin V(v_3)$.

Let $x \in V(T_1)$ be arbitrary. Then we observe that

$$\begin{aligned} 6p^2 = \sum_{y \in V(D_4)} d^+(y) &\leqslant \sum_{y \in V(D_4)} d^+_{D_4}(y) + d(D_4, u) + d(D_4, T_1) \\ &\leqslant 2p^2 - p - 1 + 2p - |V(u) \cap V(D_4)| - |N^+(u) \cap V(D_4)| \\ &\quad + 4p^2 - |N^+(x) \cap V(D_4)|, \end{aligned}$$

and it follows that

$$(1) \qquad |V(u) \cap V(D_4)| + |N^+(x) \cap V(D_4)| + |N^+(u) \cap V(D_4)| \leqslant p - 1.$$

Theorem 1.2 implies that D_4 contains a cycle C' with vertices of all the $2p-1$ partite sets of D_4, and thus $L(C') = 2p - 1$. Let $\{v_4\} = V(D_4) - V(C')$ and $v_4' \in V(T_1) \cap V(v_4)$. If $C' = b_1b_2 \ldots b_{2p-1}b_1$ then, according to (1), there are at least $|V(C')| - (p-1) = p \geqslant 2$ vertices $b_i, b_j \in V(C') - (V(u) \cup N^-(v_4') \cup N^-(u))$ such that $\{b_i, b_j\} \to u$ and $\{b_i, b_j\} \rightsquigarrow v_4'$. Let $b_j \to v_4'$. If $v_3 \notin V(b_i^+)$, then the vertices of C can be numbered so that $a_{2p-1} \notin V(b_j^+)$ and $a_{2p-2} \notin V(v_2)$, and we see that $a_1a_2 \ldots a_{2p-2}v_2v_3b_i^+ \ldots b_jv_4'a_{2p-1}b_j^+ \ldots b_i^-b_iua_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. If $v_3 \in V(b_i^+)$ and thus $v_2 \notin V(b_i^+)$, $v_3 \notin V(b_j^+)$ and $V(C) \to v_3$, then the vertices of C can be numbered so that $a_{2p-2} \notin V(v_2)$. This implies that $a_1a_2 \ldots a_{2p-2}v_2b_i^+ \ldots b_jv_4'a_{2p-1}v_3b_j^+ \ldots b_i^-b_iua_1$ is a cycle with exactly two vertices from every partite set, again a contradiction.

Subcase 1.3. Assume that $|V(D_t)| = 2p + 1$. This implies $t = 3$ and $|V(D_2)| = 1$. Let $V(D_2) = \{v_2\}$.

Subcase 1.3.1. Suppose that D_3 is Hamiltonian with a Hamiltonian cycle $C' = b_1b_2 \ldots b_{2p+1}b_1$. Let $u' \in V(T_1) \cap V(u)$. If $|N^-(u) \cap V(D_3)| = 1$ and $|N^-(u') \cap V(D_3)| \leqslant 1$, then we conclude that $d(D_3, u) \leqslant 1$ and $|N^+(u') \cap V(D_3)| \geqslant 2p - 1$, and thus

$$\begin{aligned} (2p+1)3p = \sum_{y \in V(D_3)} d^+(y) &= \sum_{y \in V(D_3)} d^+_{D_3}(y) + d(D_3, u) + d(D_3, T_1) \\ &\leqslant 2p^2 + p + 1 + (2p+1)2p - (2p-1) = 6p^2 + p + 2, \end{aligned}$$

a contradiction to $p \geqslant 2$. Hence, it follows that $|N^-(u) \cap V(D_3)| \geqslant 2$ or $|N^-(u') \cap V(D_3)| \geqslant 2$. If $|N^-(u) \cap V(D_3)| \geqslant 2$, then the vertices of C' can be numbered so that $b_{2p+1} \to u$ and $b_1 \notin V(v_2)$. Let $a_{2p-1} \notin V(v_2)$. Then $a_1a_2 \ldots a_{2p-1}v_2b_1b_2 \ldots b_{2p+1}ua_1$ is a cycle with exactly 2 vertices of every partite set, a contradiction. Analogously, the case that $|N^-(u') \cap V(D_3)| \geqslant 2$ leads to a contradiction.

Subcase 1.3.2. Let D_3 be not Hamiltonian. Since D_3 is strongly connected, Theorem 2.3 implies that D_3 is not a tournament. Since $\{u\} \cup V(D_1)$ consists of $2p$ partite sets, it follows that D_3 consists of the vertices of exactly $2p$ partite sets and $v_2 \to D_3$. Analogously to Subcase 1.2.2, we see that

$$(2) \qquad |V(u) \cap V(D_3)| + |N^+(x) \cap V(D_3)| + |N^+(u) \cap V(D_3)| \leqslant 2p$$

for an arbitrary vertex $x \in V(T_1)$. Theorem 1.2 implies that D_3 contains a cycle C' with the vertices of all the $2p$ partite sets of D_3.

Hence, let $L(C') = 2p$ for $C' = b_1b_2 \ldots b_{2p}b_1$. Let us define $\{v_3\} = V(D_3) - V(C')$ and $\{v_3'\} = V(T_1) \cap V(v_3)$.

Assume that $u \rightsquigarrow C'$. Since $N^-(u) \cap V(D_3) \neq \emptyset$, it follows that $v_3 \to u$. Furthermore, (2) yields that $C' \rightsquigarrow v_3'$. If $\{\tilde{v}_3\} = V(C') \cap V(v_3)$, then let the vertices of C be numbered so that $a_{2p-1} \notin V(v_2)$. In this case $a_1v_3ua_2a_3 \ldots a_{2p-1}v_2\tilde{v}_3^+ \ldots \tilde{v}_3^- v_3'a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. Hence, let $N^-(u) \cap V(C') \neq \emptyset$ and $b_i \to u$.

Suppose now that $v_3' \rightsquigarrow C'$. This yields that $v_2 \to v_3'$, since otherwise we observe that

$$3p = d^+(v_3') \geqslant |V(D_1)| + |V(C')| - 1 + |\{v_2\}| = 4p - 1,$$

a contradiction to $p \geqslant 2$. If the vertices of C are numbered so that $a_{2p-2} \notin V(v_2)$ and $a_{2p-1} \notin V(b_i^+)$, then $a_1a_2 \ldots a_{2p-2}v_2v_3'a_{2p-1}b_i^+ \ldots b_i^- b_iua_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. Consequently, let $N^-(v_3') \cap V(C') \neq \emptyset$.

Let $\{x, y\} = \{u, v_3'\}$ such that $x \to y$ and $b \in V(C')$ with $b \to x$. If $a_{2p-1} \notin V(v_2)$, then we see that $a_1a_2 \ldots a_{2p-1}v_2b^+ \ldots b^- bxya_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Subcase 1.4. Assume that $|V(D_t)| = 2p + 2$. This implies that $t = 2$.

Subcase 1.4.1. Suppose that D_2 is Hamiltonian with a Hamiltonian cycle $C' = b_1b_2 \ldots b_{2p+2}b_1$. It is easy to see that the vertices of C and C' can be numbered so that $b_{2p+2} \to u$ and $a_{2p-1} \notin V(b_1)$. Now, we observe that $a_1a_2 \ldots a_{2p-1}b_1b_2 \ldots b_{2p+2}ua_1$ is a Hamiltonian cycle of D', a contradiction.

Subcase 1.4.2. Let D_2 be not Hamiltonian. Since D_1 is a tournament and $u \notin V(x)$ for all $x \in V(D_1)$, we conclude that D_2 contains the vertices of exactly $2p+1$ partite sets. Theorem 1.2 implies that D_2 contains a cycle C' with vertices of all the $2p+1$

partite sets of D_2. If $L(C') = 2p + 2$, then D_2 is Hamiltonian and Subcase 1.4.1 yields a contradiction.

Consequently, it remains to consider the case that $L(C') = 2p + 1$ with $C' = b_1 b_2 \ldots b_{2p+1} b_1$. Let us define $\{v_2\} = V(D_2) - V(C')$ and $\{v_2'\} = V(T_1) \cap V(v_2)$. If $N^-(u) \cap V(C') = \emptyset$, then we observe that

$$3p = d^+(u) \geqslant |V(D_1)| + |V(C')| - 1 = 2p - 1 + 2p = 4p - 1,$$

a contradiction to $p \geqslant 2$. Hence, there exists a vertex $b_i \in V(C')$ such that $b_i \to u$. Analogously, we see that there exists a vertex $b_j \in V(C')$ such that $b_j \to v_2'$.

Noticing that either $u \to v_2'$ or $v_2' \to u$ and that the vertices of C can be numbered so that $a_{2p-1} \notin V(b_i^+) \cup V(b_j^+)$, we observe that either $a_1 a_2 \ldots a_{2p-1} b_i^+ \ldots b_i^- b_i u v_2' a_1$ or $a_1 a_2 \ldots a_{2p-1} b_j^+ \ldots b_j^- b_j v_2' u a_1$ is a cycle of D with exactly 2 vertices of every partite set, a contradiction.

Case 2. Assume that $|V(D_1)| = 2p$. This implies that $|V(D_t)| = 2p$ and $t = 3$ or $|V(D_t)| = 2p + 1$ and $t = 2$. Let D_1 consist of the vertices of exactly k partite sets with $p \leqslant k \leqslant 2p$. It follows that

$$\begin{aligned} 6p^2 &= \sum_{y \in V(D_1)} d^-(y) = \sum_{y \in V(D_1)} d^-_{D_1}(y) + d(u, D_1) + d(T_1, D_1) \\ &\leqslant \frac{2(2p-k)(2p-2) + (2k-2p)(2p-1)}{2} + 2p - |V(u) \cap V(D_1)| \\ &\quad - |N^-(u) \cap V(D_1)| + 4p^2 - \sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)| \\ &= 6p^2 + k - p - |V(u) \cap V(D_1)| - |N^-(u) \cap V(D_1)| \\ &\quad - \sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)|, \end{aligned}$$

and thus

$$|N^-(u) \cap V(D_1)| + \sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)| \leqslant k - p - |V(u) \cap V(D_1)|. \tag{3}$$

Let $y_1 \in V(D_2) \cup \ldots \cup V(D_t)$ be an arbitrary vertex. We will show that there exists a Hamiltonian path in $D_1 \cup \{u, y_1\}$ with the initial vertex u and the terminal vertex y_1. Suppose that this is not true.

First, let D_1 be Hamiltonian with a Hamiltonian cycle $C = a_1 a_2 \ldots a_{2p} a_1$. If $|N^+(u) \cap V(D_1)| \geqslant 2$, then, without loss of generality, let $u \to a_1$ and $a_{2p} \notin V(y_1)$. But now $u a_1 a_2 \ldots a_{2p} y_1$ is a Hamiltonian path in $D_1 \cup \{u, y_1\}$, a contradiction. Hence, let $|N^+(u) \cap V(D_1)| = 1$. Together with (3), this implies

$$p - |V(u) \cap V(D_1)| \geqslant k - p - |V(u) \cap V(D_1)| \\ \geqslant |N^-(u) \cap V(D_1)| \geqslant 2p - 1 - |V(u) \cap V(D_1)|,$$

a contradiction to $p \geqslant 2$.

Secondly, let D_1 be not Hamiltonian, and thus, according to Theorem 2.3, $k \neq 2p$. Theorem 1.2 implies that D_1 contains a cycle with vertices of all the k partite sets. Let $C = a_1a_2 \dots a_la_1$ be a cycle which fulfils this condition and which has the maximal cardinality of all cycles that contain vertices of all the k partite sets of D_1. If $L(C) = 2p$, then D_1 is Hamiltonian and as above, we arrive at a contradiction. Hence, let $L(C) < 2p$ and $T_1' = D_1 - V(C)$. It is obvious that T_1' is a tournament and, according to Theorem 2.1, T_1' contains a Hamiltonian path $P = b_1b_2 \dots b_{2p-l}$.

If $|N^+(u) \cap V(C)| \leqslant p - 1$, then it follows that $|N^-(u) \cap V(C)| \geqslant k - p + 1 - |V(u) \cap V(C)|$, a contradiction to (3). Hence, we conclude that

$$(4) \qquad |N^+(u) \cap V(C)| \geqslant p \ \ (\geqslant 2).$$

Let $u \to a_i$. If $a_i^- \to b_1$ then, noticing that $P \to y_1$, $ua_ia_i^+ \dots a_i^-b_1b_2 \dots b_{2p-l}y_1$ is a Hamiltonian path in $D_1 \cup \{u, y_1\}$, a contradiction. Consequently, let $b_1 \rightsquigarrow a_i^-$. Suppose that $b_j \to b_1$ for some $j \geqslant 3$. Let $j_{\max} = \max\{j \geqslant 3\colon\ b_j \to b_1\}$.

First, let $a_i^- \in V(b_1)$. If $b_1 \to a_{i-2}$, then, because of the maximality of C, we deduce that $b_1 \rightsquigarrow C$, and thus

$$p - 1 \leqslant d^-_{D_1}(b_1) \leqslant |V(T_1')| - 2 \quad \Rightarrow \quad |V(T_1')| \geqslant p + 1,$$

a contradiction. Hence, let $a_{i-2} \to b_1$. Now, the maximality of C implies that $a_i^- \to \{b_2, b_3, \dots, b_{2p-l}\}$. If $j_{\max} \neq 2p-l$ then $ua_ia_i^+ \dots a_i^-b_2b_3 \dots b_{j_{\max}}b_1b_{j_{\max}+1} \dots b_{2p-l}y_1$ is a Hamiltonian path of $D_1 \cup \{u, y_1\}$ and if $j_{\max} = 2p - l$ then $ua_ia_i^+ \dots a_i^-b_2b_3 \dots b_{2p-l}b_1y_1$ is a Hamiltonian path of $D_1 \cup \{u, y_1\}$, in both cases a contradiction.

Consequently, it remains to consider the case that $b_1 \to a_i^-$. If $a_p \in V(b_1) \cap V(C)$, then the maximality of C implies that $b_1 \to \{a_{p+1}, a_{p+2}, \dots, a_i^-\}$ and thus $p \neq i$. If $b_1 \to a_{p-1}$, then analogously as above we see that $b_1 \rightsquigarrow C$, a contradiction. Again the maximality of C yields that $a_p \to \{b_2, b_3, \dots b_{2p-l}\}$. If $j_{\max} \neq 2p - l$, then $b_2b_3 \dots b_{j_{\max}}b_1b_{j_{\max}+1} \dots b_{2p-l}$ is a Hamiltonian path of T_1' and if $j_{\max} = 2p - l$, then $b_2b_3 \dots b_{2p-l}b_1$ is a Hamiltonian path of T_1'. Both the Hamiltonian paths have the initial vertex b_2. Analogously as above, we see that $b_2 \to a_i^-$, $b_2 \to \{a_{q+1}, a_{q+2}, \dots, a_i^-\}$ and $a_q \to \{b_1, b_3, b_4, \dots, b_{2p-l}\}$ if $a_q \in V(b_2) \cap V(C)$ $(q \neq i)$. Without loss of generality, we may suppose that $i > q > p$ (modulo l). But now, the fact that $a_q \to b_1$ and $b_1 \to \{a_{p+1}, a_{p+2}, \dots, a_{i-1}\}$ yields a contradiction.

Summarizing our results, we see that $b_1 \to \{b_2, b_3, \ldots, b_{2p-l}\}$. Now, suppose that $u \to b_1$. Let $a_w \in V(y_1) \cap V(C)$ (or $a_w \in V(C) - V(b_{2p-l})$ arbitrary if $V(C) \cap V(y_1) = \emptyset$). Then it follows that $a_w \to b_{2p-l}$, since otherwise $ub_1b_2 \ldots b_{2p-l}a_wa_w^+ \ldots a_w^- y_1$ is a Hamiltonian path in $D_1 \cup \{u, y_1\}$, a contradiction. The maximality of C implies that $a_{w+1} \rightsquigarrow b_{2p-l}$. If $m \notin \{1, 2, \ldots, l\} - \{w, w+1\}$ and $b_{2p-l} \to a_m$, then $ub_1b_2 \ldots b_{2p-l}a_ma_m^+ \ldots a_m^- y_1$ is a Hamiltonian path of $D_1 \cup \{u, y_1\}$, a contradiction. Altogether, we have $C \rightsquigarrow b_{2p-l}$. If $a_n \in V(b_{2p-l})$, then we conclude that $a_n \to b_{2p-l-1}$, since otherwise

$$ub_1b_2 \ldots b_{2p-l-1}a_na_n^+ \ldots a_n^- b_{2p-l}y_1 \tag{5}$$

is a Hamiltonian path in $D_1 \cup \{u, y_1\}$. The maximality of D_1 yields that $a_{n+1} \rightsquigarrow b_{2p-l-1}$. To get no contradiction as in (5), we deduce that $C \rightsquigarrow b_{2p-l-1}$. Successively, it follows that $C \rightsquigarrow \{b_1, b_2, \ldots, b_{2p-l}\}$, a contradiction to the strong connectivity of D_1.

Consequently, let $b_1 \to u$ and thus $d^-_{D_1}(b_1) \geqslant p$. Furthermore, using (4) and the results above, we conclude that

$$\begin{aligned} |N^+_{D_1}(b_1)| &\geqslant |N^+(u) \cap V(C)| - |V(b_1) \cap V(C)| + |V(T_1') - \{b_1\}| \\ &\geqslant |N^+(u) \cap V(C)| - 1 \geqslant p - 1. \end{aligned}$$

Altogether, we arrive at the contradiction

$$2p = |V(D_1)| = d^+_{D_1}(b_1) + d^-_{D_1}(b_1) + 2 \geqslant 2p + 1.$$

Hence, for an arbitrary vertex $y_1 \in V(D_1) \cup \ldots \cup V(D_t)$ there exists a Hamiltonian path of $D_1 \cup \{u, y_1\}$ with the initial vertex u and the terminal vertex y_1.

Subcase 2.1. Assume that $|V(D_t)| = 2p$ and thus $t = 3$ and $D_2 = \{v_2\}$. Observing the converse D^{-1} of D, we see that for an arbitrary vertex $y_2 \in V(D_1) \cup V(D_2)$ there exists a Hamiltonian path of $D_2 \cup \{u, y_2\}$ with the initial vertex y_2 and the terminal vertex u. Choosing $y_1 = y_2 = v_2$, we get a Hamiltonian cycle of D', a contradiction.

Subcase 2.2. Suppose that $|V(D_t)| = 2p + 1$ and thus $t = 2$. According to (3), we have

$$\sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)| \leqslant k - p.$$

We conclude that there are at least $k - (k - p) = p \geqslant 2$ vertices in $V(T_1)$ belonging to the partite sets represented in $V(D_1)$ such that they (weakly) dominate D_1. Hence, let $w_1 \in V(T_1)$ with $w_1 \rightsquigarrow D_1$ and $x_1 \in V(D_1) \cap V(w_1)$. Let $D'' = [D' \cup \{w_1\}] - \{x_1\}$. Assume that there is a vertex $x \in V(D'')$ such that $d^+_{D''}(x) \leqslant p - 1$ or $d^-_{D''}(x) \leqslant p - 1$.

This yields the contradiction $3p = d^+_D(x), d^-_D(x) \leqslant p-1+|V(T_1)|-1 = 3p-1$. Hence, let $d^+_{D''}(x), d^-_{D''}(x) \geqslant p \geqslant 2$ for all $x \in V(D'')$, and thus

$$(6) \qquad d(D_2, w_1) \geqslant p-1 \geqslant 1.$$

If $D_1 - \{x_1\}$ is not strongly connected, then let $D'_1, D'_2, \dots, D'_{t'}$ be the strong components of $D_1 - \{x_1\}$ such that $D'_i \rightsquigarrow D'_j$ for $i < j$. If $D'_1 \rightsquigarrow u$, then it follows that $d^-_{D'_1}(y) \geqslant p-1$ for all $y \in V(D'_1)$, and thus $|V(D'_1)| \geqslant 2p-1$, a contradiction to $|V(D_1)| = 2p$. Consequently, we may assume that there is a vertex $y \in D'_1$ such that $u \to y$ provided $D_1 - \{x_1\}$ is not strongly connected. If $D_1 - \{x_1\}$ is strongly connected and $D_1 - \{x_1\} \rightsquigarrow u$, then we see that

$$2p-1 \leqslant |V(u) \cap V(D_1)| + |N^-(u) \cap V(D_1)| \leqslant k-p \Rightarrow 3p-1 \leqslant k \leqslant 2p,$$

a contradiction to $p \geqslant 2$. Consequently, we observe that there is a vertex $y \in V(D_1) - \{x_1\}$ such that $u \to y$ provided $D_1 - \{x_1\}$ is strong.

The above results guarantee that D'' is strong. If D'' is 2-connected, then Theorem 2.2 yields that D'' is Hamiltonian, a contradiction. Hence D'' is exactly 1-connected. Obviously, the vertices u and w_1 are no cut-vertices of D''. Since $D_1 - \{x_1\} \rightsquigarrow D_2$, $x_1 \in V(w_1)$, $N^-(w_1) \cap V(D_2) \neq \emptyset$, $N^-(u) \cap V(D_2) \neq \emptyset$, $w_1 \rightsquigarrow D_1 - \{x_1\}$ and $d^+_{D''}(x), d^-_{D''}(x) \geqslant 2$ for all $x \in V(D'')$ furthermore, there is also no cut-vertex of D'' in $D_1 - \{x_1\}$. Hence, let $x' \in V(D_2)$ be a cut-vertex of D''. Because of (6), $N^+(u) \cap V(D'_1) \neq \emptyset$, $w_1 \rightsquigarrow D_1$, $d^-_{D''}(u) \geqslant 2$ and $d^-_{D''}(w_1) \geqslant 2$, so if the vertex x' were no cut-vertex of D_2, then necessarily $x' \to \{w_1, u\} \rightsquigarrow D_2 - \{x'\} =: \hat{D}$ and $u \to w_1$. Since $d^-_{D''}(w_1) \geqslant p$ this implies that $p = 2$, and thus $|\hat{D}| = 4$. Let $y \in \hat{D} - V(w_1)$ be such that $d^+_{\hat{D}}(y) = 1$. Then we observe that $d^+(y) \leqslant 1 + |\{x'\}| + |V(T_1)| - 2 = 5$, a contradiction. Hence, x' is a cut-vertex of D_2. Let $D''_1, D''_2, \dots, D''_{t''}$ be the strong components of $D_2 - \{x'\}$ such that $D''_i \rightsquigarrow D''_j$ for $i < j$.

Suppose that there is a vertex $y \in V(D''_{t''})$ with $y \to u$. Since $N^+(u) \cap V(D'_1) \neq \emptyset$, $w_1 \rightsquigarrow D_1$ and (6) is valid, we conclude that $D'' - \{x'\}$ is strongly connected, a contradiction. Consequently, let $u \rightsquigarrow D''_{t''}$. This yields that $d^+_{D''_{t''}}(x) \geqslant 3p - (|V(T_1)|-1) - |\{x'\}| = p-1$ for all $x \in V(D''_{t''})$ and thus $|V(D''_{t''})| \geqslant 2\delta^+_{D''_{t''}} + 1 \geqslant 2p-1$. To get no contradiction, it follows that $t'' = 2$, $|V(D''_2)| = 2p-1$ and $D''_2 \rightsquigarrow T_1 \cup \{x'\}$. Since $D_1 - \{x_1\} \rightsquigarrow D_2$, $d^-_{D''}(u) \geqslant 2$ and $w_1 \rightsquigarrow D_1 - \{x_1\}$, we deduce that $D'' - \{x'\}$ is strongly connected, a contradiction.

This completes the proof of the theorem. $\square$

Combining Corollary 3.2 with Theorems 1.6 and 3.4 we can see that we have found a solution of Problem 1.3 for regular multipartite tournaments and $r_i = |V_i| - 1$ for all $1 \leqslant i \leqslant c$.

Corollary 3.5. *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geqslant 5$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \geqslant 2$. Then D contains a cycle with exactly $r-1$ vertices of each partite set.*

Now, we shall prove the main theorem of this article.

Theorem 3.6. *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $|V_1| = |V_2| = \ldots = |V_c| = r \geqslant 2$. If $c \geqslant 5$ or $c = 4$ and $r \geqslant 4$ or $c = 3$ or $c = 2$ and D is not isomorphic to $B\left(\frac{1}{2}r, \frac{1}{2}r, \frac{1}{2}r, \frac{1}{2}r\right)$, then D contains a cycle with exactly $r-1$ vertices from each partite set.*

Proof. If $c \geqslant 5$, then Corollary 3.5 yields the desired result. Since, according to Theorem 1.4, D is Hamiltonian, the result for $c = 2$ follows directly from Theorem 1.7 while for the case $c = 3$ we use Theorems 1.1 and 1.8. Hence, let $c = 4$. According to Remark 2.8, r has to be even. If $r \geqslant 6$, then Corollary 3.2 leads to the desired result.

Consequently, it remains to consider the case that $c = r = 4$. Suppose that D does not contain any cycle with exactly 3 vertices of every partite set. Let T_1 be a subtournament of D of order 4 and $D' = D - V(T_1)$. This implies that $\alpha(D') = 3$. With respect to Remark 2.8, we observe that $d^+(x), d^-(x) = 6$ for all $x \in V(D)$ and $d^+_{D'}(x), d^-_{D'}(x) \geqslant 3$ for all $x \in V(D')$. Now we distinguish four different cases.

Case 1. Let $\kappa(D') \geqslant 3$. In this case Theorem 2.2 yields that D' is Hamiltonian, a contradiction.

Case 2. Assume that $\kappa(D') = 0$. Let $D_1, D_2, \ldots, D_t$ be the strong components of D such that $D_i \rightsquigarrow D_j$ for $i < j$. Since $d^-_{D_1}(x) \geqslant 3$ for all $x \in V(D_1)$, we deduce that $|V(D_1)| \geqslant 7$. Analogously, we conclude that $|V(D_t)| \geqslant 7$. Hence, we arrive at the contradiction $12 = |V(D')| \geqslant |V(D_1)| + |V(D_t)| \geqslant 14$.

Case 3. Suppose that $\kappa(D') = 1$. Let u be a cut-vertex of D' such that $D' - u$ consists of the strong components $D_1, D_2, \ldots, D_t$ with $D_i \rightsquigarrow D_j$ for $i < j$. This implies that $d^-_{D_1}(x) \geqslant 2$ for all $x \in V(D_1)$ and thus, since $c = 4$, we conclude that $|V(D_1)| \geqslant 6$. Analogously, we observe that $|V(D_t)| \geqslant 6$, a contradiction to $|V(D')| = 12$.

Case 4. Assume that $\kappa(D') = 2$.

First, let D' contain a cycle-factor. In this case, because of $\alpha(D') = 3$, Theorem 2.4 yields that D' is Hamiltonian, a contradiction.

Secondly, let D' contain no cycle-factor. Now, Lemma 2.6 implies that $V(D')$ can be partitioned into four subsets Y, Z, R_1 and R_2 such that $R_1 \rightsquigarrow Y$ and $(R_1 \cup Y) \rightsquigarrow R_2$, where Y is an independent set and $|Y| > |Z|$.

If $|Z| \leqslant 1$, then we deduce that $\kappa(D') \leqslant 1$, a contradiction to $\kappa(D') = 2$. If $|Z| \geqslant 3$, then Y has to be an independent set with $|Y| \geqslant 4$, a contradiction to

$\alpha(D') = 3$. Hence, let $|Z| = 2$ and $|Y| = 3$, which means that Y is a partite set of D'. Without loss of generality, let $|R_1| \leqslant |R_2|$.

Assume that $|R_1| = 0$. This yields that $|R_2| = 7$ and thus $d^+_{D'}(y) \geqslant 7$ for all $y \in Y$, a contradiction to $d^+(x), d^-(x) = 6$ for all $x \in V(D)$.

Now, let $1 \leqslant |R_1| \leqslant 2$. In this case we see that there is a vertex $x \in R_1$ with $d^-_{D[R_1]}(x) = 0$ and thus $d^-_{D'}(x) \leqslant |Z| = 2$, a contradiction.

Finally, let $|R_1| = 3$. Because $d^-_{D'}(x) \geqslant 3$ for all $x \in R_1$, we conclude that $D[R_1]$ is a 3-cycle and $Z \to R_1$. Since $D' - Y$ and R_1 consist of vertices of 3 partite sets, this is impossible. This completes the proof of the theorem. $\square$

For the case that $c = 4$ and $r = 2$, Theorem 3.6 is not true in general as the following example (see also [12]) demonstrates.

Example 3.7. Let $V_i = \{v'_i, v''_i\}$ for $i = 1, 2, 3, 4$ be the partite sets of a 4-partite tournament such that $v'_1 \to v'_2 \to v'_3 \to v'_1$, $v''_1 \to v''_2 \to v''_3 \to v''_1$,

$$\{v'_1, v'_2, v'_3\} \to v'_4 \to \{v''_1, v''_2, v''_3\} \to v''_4 \to \{v'_1, v'_2, v'_3\},$$
$$v'_1 \to v''_3 \to v'_2 \to v''_1 \to v'_3 \to v''_2 \to v'_1$$

(see also Fig. 2). Now it is a simple matter to check that the resulting 4-partite tournament is 3-regular without a cycle containing exactly $r - 1 = 1$ vertices of every partite set.

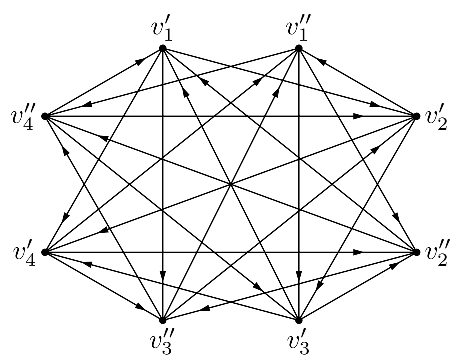

Figure 2. A regular 4-partite tournament without a strong subtournament of order 4

The results of the Theorems 1.4 and 1.6 and Corollary 3.5 lead us to the following conjecture.

Conjecture 3.8. *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geqslant 5$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \geqslant 2$. Then D contains a cycle with exactly m vertices of each partite set for every $m \in \{1, 2, \ldots, r\}$.*

Note that, according to Theorem 3.1, Conjecture 3.8 is valid for a given m if c and r are sufficiently large.

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