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SUBALGEBRA EXTENSIONS OF PARTIAL MONOUNARY ALGEBRAS

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Abstract. For a subalgebra $\mathscr{B}$ of a partial monounary algebra $\mathscr{A}$ we define the quotient partial monounary algebra $\mathscr{A}/\mathscr{B}$. Let $\mathscr{B}$, $\mathscr{C}$ be partial monounary algebras. In this paper we give a construction of all partial monounary algebras $\mathscr{A}$ such that $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ and $\mathscr{C} \cong \mathscr{A}/\mathscr{B}$.

Keywords: partial monounary algebra, subalgebra, congruence, quotient algebra, subalgebra extension, ideal, ideal extension

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0. Introduction

In the present paper we deal with subalgebra extensions of partial monounary algebras.

The extension problem for groups is as follows: Given two groups H and K, construct all groups G which have a normal subgroup N such that N is isomorphic to H and the quotient G/N of G by N is isomorphic to K. G is the well known Schreier's extension of H by K. Following the extension of groups, the ideal extension of semigroups has been considered by A. H. Clifford [1]. Related investigations dealing with extensions by ideals were performed for lattice ordered groups (in connection with the product of torsion classes, cf. Martinez [6]), for ordered and totally ordered semigroups (Kehayopulu, Tsingelis [5], Hulin [2]) and for lattices (Kehayopulu, Kiriakuli [4]).

Let $\mathscr{U}$ be the class of all partial monounary algebras, $\mathscr{A} \in \mathscr{U}$. If $\mathscr{B}$ is a subalgebra of $\mathscr{A}$, then the quotient partial algebra $\mathscr{A}/\mathscr{B}$ is defined. Similarly, the notion of an ideal of $\mathscr{A}$ is introduced and if $\mathscr{X}$ is an ideal of $\mathscr{A}$, then $\mathscr{A}/\mathscr{X}$ is defined.

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Let us consider the following two problems:

(α) Let $\mathscr{B},\mathscr{C} \in \mathscr{U}$. Find all $\mathscr{A} \in \mathscr{U}$ such that $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ and $\mathscr{A}/\mathscr{B} \cong \mathscr{C}$.

(β) Let $\mathscr{X},\mathscr{C} \in \mathscr{U}$. Find all $\mathscr{A} \in \mathscr{U}$ such that $\mathscr{X}$ is an ideal of $\mathscr{A}$ and $\mathscr{A}/\mathscr{X} \cong \mathscr{C}$.

(In (α), $\mathscr{A}$ will be called a *subalgebra extension* of $\mathscr{C}$ by $\mathscr{B}$, in (β), $\mathscr{A}$ will be called an *ideal extension* of $\mathscr{C}$ by $\mathscr{X}$.)

Let us remark that a subalgebra need not be an ideal and an ideal need not be a subalgebra, thus the problems (α) and (β) are independent (cf. also Section 4). The present paper is devoted to the problem (α); (β) will be dealt with elsewhere.

1. Preliminaries

Monounary and partial monounary algebras play a significant role in the study of algebraic structures (cf. e.g., Jónsson [3], M. Novotný [7]).

A *partial monounary algebra* $\mathscr{A}$ is a pair (A, f_A), where A is a nonempty set and f_A is a partial unary operation on A. If $\operatorname{dom} f_A = A$, then $\mathscr{A}$ is called *complete*; if $\operatorname{dom} f_A \neq A$, then $\mathscr{A}$ is said to be *incomplete*.

Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$, $x, y \in A$. Put $f_A^0(x) = x$ and $f_A^{-1}(x) = \{z \in \operatorname{dom} f_A\colon f_A(z) = x\}$. If $n \in \mathbb{N}$, $f_A^{n-1}(x)$ is defined and $f_A^{n-1}(x) \in \operatorname{dom} f_A$, then we put $f_A^n(x) = f_A(f_A^{n-1}(x))$. Next we put $x \sim y$ if there are $m, n \in \mathbb{N} \cup \{0\}$ such that $f_A^n(x)$, $f_A^m(y)$ are defined and $f_A^n(x) = f_A^m(y)$. Then $\sim$ is an equivalence on the set A and the elements of $A/{\sim}$ are called *connected components* of $\mathscr{A}$. Further, $\mathscr{A}$ is said to be *connected* if it has only one connected component. An element $c \in A$ is called *cyclic* if $f_A^k(c) = c$ for some $k \in \mathbb{N}$. The set of all cyclic elements of some connected component of $\mathscr{A}$ is called a *cycle* of $\mathscr{A}$. An element $c \in A$ is called a *top* of $\mathscr{A}$ if $\mathscr{A}$ is connected and either $c \notin \operatorname{dom} f_A$ or $\{c\}$ is a cycle.

Let $\mathscr{A} = (A, f_A)$, $\mathscr{B} = (B, f_B) \in \mathscr{U}$. Let $B \subseteq A$, $\operatorname{dom} f_B \subseteq \operatorname{dom} f_A$ and if $x \in B \cap \operatorname{dom} f_A$ then $x \in \operatorname{dom} f_B$, $f_B(x) = f_A(x)$. Then $\mathscr{B}$ is called a *subalgebra* of $\mathscr{A}$.

Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$, $\emptyset \neq X \subseteq A$. We will denote by $f_A \restriction X$ the partial operation on X defined as follows: $\operatorname{dom}(f_A \restriction X) = \{x \in X \cap \operatorname{dom} f_A\colon f_A(x) \in X\}$ and if $x \in \operatorname{dom}(f_A \restriction X)$ then $(f_A \restriction X)(x) = f_A(x)$. The partial algebra $(X, f_A \restriction X)$ is called the *relative subalgebra* of $\mathscr{A}$ with carrier X.

Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$. An equivalence θ on A is said to be a *congruence* of $\mathscr{A}$ if $\{x, y\} \subseteq \operatorname{dom} f_A$, $(x, y) \in \theta$ implies $(f_A(x), f_A(y)) \in \theta$. For $x \in \mathscr{A}$, the block (equivalence class) of θ containing x is denoted by $[x]_\theta$ or simply $[x]$. A *quotient* *algebra* $\mathscr{A}/\theta = (A/\theta, f_{A/\theta})$ is such that $\operatorname{dom} f_{A/\theta} = \{[x] \in A/\theta \colon [x] \subseteq \operatorname{dom} f_A\}$ and if $[x] \in \operatorname{dom} f_{A/\theta}$, then $f_{A/\theta}([x]) = [f_A(x)]$.

1.1 Notation. Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$, $\emptyset \neq B \subseteq A$. We denote by θ_B the smallest congruence relation of $\mathscr{A}$ such that if $x, y \in B$ belong to the same connected component of $\mathscr{A}$, then x, y belong to the same equivalence class of the congruence θ_B.

1.2 Lemma. *Suppose that $\mathscr{A} = (A, f_A) \in \mathscr{U}$, $\mathscr{B} = (B, f_B)$ is a subalgebra of $\mathscr{A}$. Let $x, y \in A$. Then $(x, y) \in \theta_B$ if and only if either x, y belong to the same connected component of $\mathscr{A}$ and $\{x, y\} \subseteq B$ or $x = y$.*

Proof. First let us show that if we put $(x, y) \in \delta$ whenever either x, y belong to the same connected component of $\mathscr{A}$ and $\{x, y\} \subseteq B$, or $x = y$, then δ is a congruence of $\mathscr{A}$. Obviously, δ is an equivalence. Assume that $\{x, y\} \subseteq \operatorname{dom} f_A$, $(x, y) \in \delta$. If $x = y$, then $f_A(x) = f_A(y)$ and $(x, y) \in \delta$. Suppose that $x \neq y$. Then x and y belong to the same connected component of $\mathscr{A}$ and $\{x, y\} \subseteq B$. Since $\mathscr{B}$ is a subalgebra of $\mathscr{A}$, this implies that $\{f_A(x), f_A(y)\} \subseteq B$, $f_A(x)$ and $f_A(y)$ belong to the same connected component of $\mathscr{A}$. Therefore $(f_A(x), f_A(y)) \in \delta$, thus δ is a congruence of $\mathscr{A}$.

From the definition of δ it is obvious that δ is the smallest equivalence relation on A such that if $x, y \in B$ belong to the same connected component of $\mathscr{A}$ then x, y belong to the same equivalence class of δ.

We have proved that $\delta = \theta_B$. □

1.3 Corollary. *Let $\mathscr{A} \in \mathscr{U}$ be connected, and $\mathscr{B} = (B, f_B)$ be a subalgebra of $\mathscr{A}$, $|B| > 1$. Then the unique nontrivial equivalence class of θ_B is equal to B.*

1.4 Notation. Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$ and let $\mathscr{B} = (B, f_B)$ be a subalgebra of $\mathscr{A}$. By a *quotient partial monounary algebra* $\mathscr{A}/\mathscr{B} = (A/B, f_{A/B})$ we understand the partial algebra $\mathscr{A}/\theta_B$.

1.5.1 Corollary. *Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$ be connected and complete, and $\mathscr{B} = (B, f_B)$ be its subalgebra. Then*

(i) $f_{A/B}(\{x\}) = \{f_A(x)\}$ *if* $x \in A$, $f_A(x) \notin B$,
(ii) $f_{A/B}(\{x\}) = B$ *if* $x \in A$, $f_A(x) \in B$,
(iii) $f_{A/B}(B) = B$.

1.5.2 Corollary. *Let* $\mathscr{A} = (A, f_A) \in \mathscr{U}$ *be connected and incomplete, and* $\mathscr{B} = (B, f_B)$ *be its subalgebra. Then*

(i) $f_{A/B}(\{x\}) = \{f_A(x)\}$ *if* $x \in \operatorname{dom} f_A$, $f_A(x) \notin B$,

(ii) $f_{A/B}(\{x\}) = B$ *if* $x \in \operatorname{dom} f_A$, $f_A(x) \in B$,

(iii) $B \notin \operatorname{dom} f_{A/B}$.

Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$. If $x, y \in A$, then we set $x \leqslant y$ if $f_A^k(x) = y$ for some $k \in \mathbb{N} \cup \{0\}$. Notice that the relation $\leqslant$ is a quasi-order on the set A. The notion of an ideal of a lattice is well known. Let us modify the definition for lattices to the following definition for quasi-ordered sets: Let $(Q, \leqslant)$ be a quasi-ordered set, $\emptyset \neq X \subseteq Q$. Then $(X, \leqslant)$ is called an ideal in $(Q, \leqslant)$ if the following conditions are satisfied:

(1) if $a \in X$, $b \leqslant a$, then $b \in X$,

(2) if $a, b \in X$ and $c \in Q$ is a minimal upper bound of $\{a, b\}$, then $c \in X$.

1.6 Definition. Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$, $\emptyset \neq X \subseteq A$. If $(X, \leqslant)$ is an ideal of $(A, \leqslant)$, then the relative subalgebra $\mathscr{X} = (X, f_A \upharpoonright X)$ of $\mathscr{A}$ with carrier X is called an *ideal* of $\mathscr{A}$.

1.7 Notation. Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$ and suppose that $\mathscr{X} = (X, f_X)$ is an ideal of $\mathscr{A}$. We put

$$\mathscr{A}/\mathscr{X} = (A/X, f_{A/X}) = \mathscr{A}/\theta_X.$$

2. The connected case

In this section we will deal with the problem (α) in the case when the partial algebras under consideration are connected.

First let us describe the following construction.

Let $\mathscr{B} = (B, f_B)$, $\mathscr{C} = (C, f_C)$ be connected partial monounary algebras such that $B \cap C = \emptyset$, $|C| > 1$ and that $c \in C$ is a top of $\mathscr{C}$. Next suppose that either

(a) $\mathscr{B}$, $\mathscr{C}$ are complete or

(b) $\mathscr{B}$, $\mathscr{C}$ are incomplete.

Let μ be a mapping of the set $f_C^{-1}(c) - \{c\}$ into B; it will be called *critical.* Define an algebra $\mathscr{P} = (P, f_P) = s(\mathscr{C}, \mathscr{B}, \mu)$ where

$$P = (C - \{c\}) \cup B,$$

$$P - \operatorname{dom} f_P = B - \operatorname{dom} f_B,$$

$$f_P(x) = \begin{cases} f_C(x) & \text{if } x \in C - \{c\},\ f_C(x) \neq c, \\ \mu(x) & \text{if } x \in C - \{c\},\ f_C(x) = c, \\ f_B(x) & \text{if } x \in \operatorname{dom} f_B. \end{cases}$$

It is easy to see that $\mathscr{B}$ is a subalgebra of $\mathscr{P}$ and $\mathscr{P}$ is complete if (a) is valid and incomplete if (b) holds. The construction described above will be expressed as follows: The algebra $\mathscr{P}$ is constructed by *replacing the top in* $\mathscr{C}$ *by* $\mathscr{B}$ *using the critical mapping* μ.

Let us remark that if $|B| = 1$ then $\mathscr{P} \cong \mathscr{C}$.

2.1 Lemma. *Let* $\mathscr{B}$, $\mathscr{C}$, μ *be as above,* $\mathscr{P} = s(\mathscr{C}, \mathscr{B}, \mu)$. *Then* $\mathscr{P}/\mathscr{B} \cong \mathscr{C}$.

Proof. Let us define a mapping $\varphi\colon C \to P/B$ by putting

$$\varphi(x) = \begin{cases} \{x\} & \text{if } x \in C - \{c\}, \\ B & \text{if } x = c. \end{cases}$$

By 1.3, φ is a bijection of C onto P/B.

1) Suppose that (a) holds. We will use 1.5.1.

If $x \in C - \{c\}$, $f_C(x) \neq c$, then $\varphi(f_C(x)) = \{f_C(x)\} = f_{P/B}(\{x\}) = f_{P/B}(\varphi(x))$.

If $x \in C - \{c\}$, $f_C(x) = c$, then $\varphi(f_C(x)) = \varphi(c) = B = f_{P/B}(\{x\}) = f_{P/B}(\varphi(x))$.

If $x = c$, then $\varphi(f_C(x)) = \varphi(c) = B = f_{P/B}(B) = f_{P/B}(\varphi(c))$.

2) Now suppose that (b) is valid; we will apply 1.5.2.

If $x \in C - \{c\}$, then as above, $\varphi(f_C(x)) = f_{P/B}(\varphi(x))$.

If $x = c$, then $x \notin \operatorname{dom} f_C$ and $\varphi(x) = B \notin \operatorname{dom} f_{P/B}$.

Thus, φ is a homomorphism and, therefore, an isomorphism of $\mathscr{C}$ onto $\mathscr{P}/\mathscr{B}$. $\square$

2.2 Lemma. *Let* $\mathscr{A} = (A, f_A)$, $\mathscr{B} = (B, f_B)$, $\mathscr{C} = (C, f_C)$ *be connected partial monounary algebras such that* $\mathscr{C}$ *has a top* c, $|C| > 1$, $B \cap C = \emptyset$. *Next suppose that* $\mathscr{B}$ *is a subalgebra of* $\mathscr{A}$ *and that* $\mathscr{A}/\mathscr{B} \cong \mathscr{C}$. *Then* $(A - B,\ f_A \restriction (A - B))$ *and* $(C - \{c\},\ f_C \restriction (C - \{c\}))$ *are isomorphic.*

Proof. We have $A/B = \{B\} \cup \{\{x\}\colon x \in C - B\}$ by 1.3. Furthermore, there exists an isomorphism i of $\mathscr{C}$ onto $\mathscr{A}/\mathscr{B}$. Clearly, $i(c) = B$, since B is the top of $\mathscr{A}/\mathscr{B}$ in view of 1.5.1 or 1.5.2.

If $x \in C - \{c\}$, then there exists exactly one $y \in A - B$ such that $i(x) = \{y\}$. Put $j(x) = y$. Obviously, j is a bijection of the set $C - \{c\}$ onto $A - B$.

Let $x \in C - \{c\}$, $y = j(x)$. If $x \notin \operatorname{dom} f_C \restriction (C - \{c\})$, then $f_C(x) \notin C - \{c\}$, i.e., $f_C(x) = c$, thus

$$B = i(c) = i(f_C(x)) = f_{A/B}(i(x)) = f_{A/B}(y) = f_A(y),$$

i.e., $y \notin \operatorname{dom} f_A \restriction (A - B)$. Suppose that $x \in \operatorname{dom} f_C \restriction (C - \{c\})$. Then there is $z \in A - B$ with $i(f_C(x)) = \{z\}$, which yields $j(f_C(x)) = z$. Since i is an isomorphism, we obtain

$$\{z\} = i(f_C(x)) = f_{A/B}(i(x)) = f_{A/B}(\{y\}) = \{f_A(y)\}$$

and, therefore, $z = f_A(y)$, which implies

$$j(f_C(x)) = z = f_A(y) = f_A(j(x)).$$

Thus j is an isomorphism. □

2.3 Lemma. *Let $\mathscr{A} = (A, f_A)$, $\mathscr{B} = (B, f_B)$, $\mathscr{C} = (C, f_C)$ be connected partial monounary algebras such that $\mathscr{C}$ has a top c, $|C| > 1$, $B \cap C = \emptyset$. Next suppose that $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ and that $\mathscr{A}/\mathscr{B} \cong \mathscr{C}$. Then either* (a) *or* (b) *is valid and $\mathscr{A}$ is isomorphic to an algebra constructed by replacing the top in $\mathscr{C}$ by $\mathscr{B}$ using a critical mapping.*

Proof. Since $\mathscr{A}/\mathscr{B} \cong C$, 1.5.1 and 1.5.2 imply that either (a) or (b) is valid. By 2.2 there is an isomorphism ι of $(A-B,\ f_A \restriction (A-B))$ onto $(C-\{c\},\ f_C \restriction (C-\{c\}))$. Consider $x \in C-\{c\}$ such that $f_C(x) = c$. Then $\iota(x) \in A-B$ and $f_A(\iota(x)) \notin A-B$, i.e., $f_A(\iota(x)) \in B$. Put $\mu(x) = f_A(\iota(x))$.

Let $\mathscr{P} = s(\mathscr{C}, \mathscr{B}, \mu)$. Then $P = (C - \{c\}) \cup B$. We define a mapping $\varphi\colon (C - \{c\}) \cup B \to A$ as follows:

$$\varphi(x) = \begin{cases} x & \text{if } x \in B, \\ \iota(x) & \text{if } x \in C - \{c\}. \end{cases}$$

Clearly, φ is a bijection of $(C - \{c\}) \cup B$ onto A.

Let $x \in C - \{c\}$, $f_C(x) \in C - \{c\}$. The definition of φ yields $f_P(x) = f_C(x)$ and $\varphi(f_P(x)) = \iota(f_P(x)) = \iota(f_C(x)) = f_A(\iota(x)) = f_A(\varphi(x))$, because ι is an isomorphism.

Let $x \in C - \{c\}$, $f_C(x) = c$. Then $f_P(x) = \mu(x) = f_A(\iota(x)) \in B$ which implies $\varphi(f_P(x)) = f_A(\iota(x)) = f_A(\varphi(x))$.

Let $x \in \operatorname{dom} f_B$. Then $f_P(x) = f_B(x) \in B$ and $\varphi(f_P(x)) = f_P(x) = f_B(x) = f_A(x) = f_A(\varphi(x))$.

Finally, let $x \in B - \operatorname{dom} f_B$. By the definition of $\mathscr{P}$ we see that $x \in P - \operatorname{dom} f_P$.

Therefore φ is an isomorphism of $\mathscr{P}$ onto $\mathscr{A}$. □

2.4 Theorem. *Let $\mathscr{B} = (B, f_B)$, $\mathscr{C} = (C, f_C)$ be connected partial monounary algebras, $|C| > 1$, $B \cap C = \emptyset$. Suppose that $\mathscr{C}$ has a top c and that either* (a) *or* (b) *is valid. The following conditions are equivalent:*

(i) *$\mathscr{A}$ is isomorphic to an algebra constructed by replacing the top of $\mathscr{C}$ by $\mathscr{B}$ using a critical mapping;*
(ii) *$\mathscr{A}$ is a subalgebra extension of $\mathscr{C}$ by $\mathscr{B}$.*

Proof. This is a corollary of 2.1 and 2.3. □

2.5 Theorem. *Let $\mathscr{B} = (B, f_B)$, $\mathscr{C} = (C, f_C)$ be connected partial monounary algebras, $|C| > 1$, $B \cap C = \emptyset$. A subalgebra extension of $\mathscr{C}$ by $\mathscr{B}$ exists if and only if there is $c \in C$ such that c is a top of $\mathscr{C}$ and either* (a) *$\mathscr{B}$, $\mathscr{C}$ are complete or* (b) *$\mathscr{B}$, $\mathscr{C}$ are incomplete.*

Proof. Let $\mathscr{A}$ be a subalgebra extension of $\mathscr{C}$ by $\mathscr{B}$, i.e., $\mathscr{B}$ be a subalgebra of $\mathscr{A}$ and $\mathscr{A}/\mathscr{B} \cong \mathscr{C}$. By 1.5.1, 1.5.2, B is the top of $\mathscr{A}/\mathscr{B}$, thus there exists a top in $\mathscr{C}$. Further, $\mathscr{C}$ is complete iff $\mathscr{B}$ is complete.

The converse implication follows from 2.4. □

2.6 Theorem. *Let $\mathscr{B} = (B, f_B)$, $\mathscr{C} = (C, f_C)$ be connected partial monounary algebras, $|C| = 1$, $B \cap C = \emptyset$. A subalgebra extension $\mathscr{A}$ of $\mathscr{C}$ by $\mathscr{B}$ exists if and only if either* (a) *or* (b) *is valid; in this case $\mathscr{A} = \mathscr{B}$.*

Proof. If $\mathscr{A}$ is a subalgebra extension of $\mathscr{C}$ by $\mathscr{B}$ and $|C| = 1$, then $\mathscr{A} = \mathscr{B}$ by 1.5.1, 1.5.2. Obviously, then either (a) or (b) is valid.

Conversely, if (a) or (b) is valid, then $\mathscr{A}$ is the unique subalgebra extension of $\mathscr{C}$ by $\mathscr{B}$. □

3. Subalgebra extension—the nonconnected case

The aim of the present section is to investigate the problem (α) if the partial algebras under consideration are not assumed to be connected.

3.1 Notation. Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$ and let $\{A_j\}_{j \in J}$ be the system of connected components of $\mathscr{A}$. Then $\mathscr{A}_j = (A_j, f_A \restriction A_j)$ for $j \in J$ is a subalgebra of $\mathscr{A}$. We will write

$$A = \sum_{j \in J} A_j, \quad \mathscr{A} = \sum_{j \in J} \mathscr{A}_j.$$

3.2 Lemma. *Let $\mathscr{A} = \sum_{j \in J} \mathscr{A}_j$, $\mathscr{B}$ be a subalgebra of $\mathscr{A}$ and let $\mathscr{C} = \mathscr{A}/\mathscr{B}$. Then $\mathscr{C} = \sum_{j \in J} \mathscr{C}_j$, $\mathscr{B} = \sum_{l \in L} \mathscr{B}_l$, $L \subseteq J$. Further,*

(1) *if $j \in J - L$, then $\mathscr{C}_j \cong \mathscr{A}_j$,*

(2) *if $j \in L$, then $\mathscr{A}_j$ is a subalgebra extension of $\mathscr{C}_j$ by $\mathscr{B}_j$.*

Proof. For $j \in J$ we denote $B_j = B \cap A_j$. Let $L = \{j \in J \colon B_j \neq \emptyset\}$. Then $\mathscr{B}_l = (B_l, f_A \restriction B_l)$ for $l \in L$ is a subalgebra of $\mathscr{A}_l$ and $\mathscr{B} = \sum_{l \in L} \mathscr{B}_l$. From the definition of θ_B it follows that if $(x, y) \in \theta_B$, $x \neq y$, then x, y belong to the same connected component of $\mathscr{A}$. Therefore $\mathscr{C} = \sum_{j \in J} \mathscr{C}_j$. The assertions (1) and (2) then hold in view of the definition. □

3.3 Theorem. *Let* $\mathscr{B} = \sum_{l \in L} \mathscr{B}_l$, $\mathscr{C} = \sum_{j \in J} \mathscr{C}_j$, $\mathscr{A} \in \mathscr{U}$. *The following conditions are equivalent:*

(i) $\mathscr{A}$ *is a subalgebra extension of* $\mathscr{C}$ *by* $\mathscr{B}$;

(ii) $\mathscr{A} = \sum_{j \in J} \mathscr{A}_j$ *and there is an injection* $\tau \colon L \to J$ *such that for* $j \in J$,

(1) *if* $j \neq \tau(l)$ *for each* $l \in L$, *then* $\mathscr{A}_j \cong \mathscr{C}_j$,

(2) *if* $j = \tau(l)$, $l \in L$, *then* $\mathscr{A}_j$ *is a subalgebra extension of* $\mathscr{C}_j$ *by* $\mathscr{B}_l$.

P r o o f. Suppose that (i) is valid, i.e., $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ and $\mathscr{C} \cong \mathscr{A}/\mathscr{B}$. By 3.2 we have $\mathscr{A} = \sum_{j \in J} \mathscr{A}_j$. Further, since $\mathscr{B}$ is a subalgebra of $\mathscr{A}$, for each $l \in L$ there is a uniquely determined $j \in J$ such that $\mathscr{B}_l$ is a subalgebra of $\mathscr{A}_j$; put $\tau(l) = j$. Then $\tau \colon L \to J$ is an injection.

Let $j \in J$. If $j \neq \tau(l)$ for each $l \in L$, then $\mathscr{C}_j \cong \mathscr{A}_j$ by 3.2. If $j = \tau(l)$, then 3.2 implies that $\mathscr{A}_j$ is a subalgebra extension of $\mathscr{C}_j$ by $\mathscr{B}_l$.

Conversely, assume that (ii) holds. Then $\mathscr{B}$ is a subalgebra of $\mathscr{A}$. Denote $\mathscr{D} = \mathscr{A}/\mathscr{B}$. In view of 3.2, $\mathscr{D} = \sum_{j \in J} \mathscr{D}_j$. Further, $\mathscr{B}$ can be by 3.2 written in the form $\mathscr{B} = \sum_{k \in K} \mathscr{E}_k$, $K \subseteq J$ and

(3) if $j \in J - K$, then $\mathscr{D}_j \cong \mathscr{A}_j$,

(4) if $j \in K$, then $\mathscr{A}_j$ is a subalgebra extension of $\mathscr{D}_j$ by $\mathscr{E}_j$. According to the assumption, $\mathscr{B} = \sum_{l \in L} \mathscr{B}_l$, thus there is a bijection $\tau \colon L \to K$ such that $\mathscr{B}_l = \mathscr{E}_{\tau(l)}$ for each $l \in L$. Then τ is an injection of L into J.

Let $j \in J - K$, i.e., $j \neq \tau(l)$ for each $l \in L$. By (1) and (3) we obtain

(5) $\mathscr{C}_j \cong \mathscr{A}_j \cong \mathscr{D}_j$.

Let $j \in K$, i.e., $j = \tau(l)$ for some $l \in L$. From (2) and (4) we obtain

(6) $\mathscr{A}_j$ is a subalgebra extension of $\mathscr{C}_j$ by $\mathscr{B}_l$,

(7) $\mathscr{A}_j$ is a subalgebra extension of $\mathscr{D}_j$ by $\mathscr{E}_{\tau(l)} = \mathscr{B}_l$.

Therefore

(8) $\mathscr{B}_l$ is a subalgebra of $\mathscr{A}_j$ and $\mathscr{A}_j/\mathscr{B}_l \cong \mathscr{C}_j$,

(9) $\mathscr{B}_l$ is a subalgebra of $\mathscr{A}_j$ and $\mathscr{A}_j/\mathscr{B}_l \cong \mathscr{D}_j$,

hence

(10) $\mathscr{C}_j \cong \mathscr{D}_j$.

Then (5) and (10) imply that $\mathscr{C} \cong \mathscr{D}$ and that $\mathscr{A}$ is a subalgebra extension of $\mathscr{C}$ by $\mathscr{B}$. □

3.4 Theorem. *Let* $\mathscr{B} = \sum_{l \in L} \mathscr{B}_l$, $\mathscr{C} = \sum_{j \in J} \mathscr{C}_j$, $B \cap C = \emptyset$. *A subalgebra extension* $\mathscr{A}$ *of* $\mathscr{C}$ *by* $\mathscr{B}$ *exists if and only if there is an injection* $\tau \colon L \to J$ *such that if* $j = \tau(l)$ *for some* $l \in L$, *then there exists a top* c_j *in* $\mathscr{C}_j$ *and either both partial algebras* $\mathscr{B}_l$, $\mathscr{C}_j$ *are complete or both partial algebras* $\mathscr{B}_l$, $\mathscr{C}_j$ *are incomplete.*

4. Remark to the problem (β)

Let us notice that a subalgebra $\mathscr{B}$ of $\mathscr{A} \in \mathscr{U}$ need not be an ideal of $\mathscr{A}$ and that an ideal $\mathscr{X}$ of $\mathscr{A}$ need not be a subalgebra of $\mathscr{A}$:

Example 1. Let $\mathscr{A} = (A, f_A)$ $A = \{0,1,2,3\} = \operatorname{dom} f_A$, $f_A(2) = f_A(3) = 1$, $f_A(0) = f_A(1) = 0$, $B = \{0,1\}$, $X = \{1,2,3\}$, $f_B = f_A \restriction B$, $f_X = f_A \restriction X$. Then $\mathscr{B} = (B, f_B)$ is a subalgebra of $\mathscr{A}$ which is not an ideal of $\mathscr{A}$, $\mathscr{X} = (X, f_X)$ is an ideal of $\mathscr{A}$ which is not a subalgebra of $\mathscr{A}$.

4.1 Lemma. *Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$ be connected, $x \in A$, $y \in A$, $x \neq y$. Then there exists a minimal upper bound of the set $\{x, y\}$.*

P r o o f. There exist nonnegative integers m, n such that $f^m(x) = f^n(y)$. The assertion holds if either $m = 0$ or $n = 0$. In the remaining cases we have $m \geqslant 1$ and $n \geqslant 1$. Thus, there exists an integer $m \geqslant 1$ such that $f^m(x) = f^n(y)$ for some integer $n \geqslant 1$. Denote by m_0 the least integer $m \geqslant 1$ such that there exists an integer $n \geqslant 1$ with $f^m(x) = f^n(y)$ and put $z = f^{m_0}(x)$. Then z is an upper bound of the set $\{x, y\}$. Let t be an upper bound of the set $\{x, y\}$. Then there exist nonnegative integers m_1, n_1 with $f^{m_1}(x) = t = f^{n_1}(y)$. By our hypothesis, we have $m_1 \geqslant 1$, $n_1 \geqslant 1$. The minimality of m_0 implies $m_0 \leqslant m_1$ and the existence of a nonnegative integer p such that $m_1 = m_0 + p$. It follows that $f^p(z) = f^p(f^{m_0}(x)) = f^{m_1}(x) = t$, hence $z \leqslant t$ and z is a minimal upper bound of the set $\{x, y\}$. $\square$

4.2 Lemma. *Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$ be connected and let $\mathscr{X} = (X, f_X)$ be an ideal of $\mathscr{A}$, $|X| > 1$. Then θ_X contains only one nontrivial equivalence class; this class is equal to the set $X \cup \{f_A^n(x)\colon\ x \in X, n \in \mathbb{N}, f_A^{n-1}(x) \in \operatorname{dom} f_A\}$.*

P r o o f. Since $|X| > 1$, the definition of an ideal and 4.1 imply that there is $x \in X \cap \operatorname{dom} f_A$ such that $f_A(x) \in X$. Consider the congruence relation θ_X; we obtain $(x, f_A(x)) \in \theta_X$. If $f_A(x) \in \operatorname{dom} f_A$, then $(f_A(x), f_A^2(x)) \in \theta_X$. Similarly, if $f_A^{n-1}(x) \in \operatorname{dom} f_A$ for $n \in \mathbb{N}$, then $(f_A^{n-1}(x), f_A^n(x)) \in \theta_X$. Thus the elements $x, f_A(x), f_A^2(x), \ldots$ are in the same congruence class of θ_X. By the minimality of θ_X we get that θ_X contains only one nontrivial equivalence class, and this class is equal to $X \cup \{f_A^n(x)\colon\ n \in \mathbb{N},\ f_A^{n-1}(x) \in \operatorname{dom} f_A\}$. $\square$

4.3 Lemma. *Let $\mathscr{A} = (A, f_A) \in \mathscr{U}$ be connected and let $\mathscr{X} = (X, f_X)$ be an ideal of $\mathscr{A}$, $|X| > 1$. Then there is a unique subalgebra $\mathscr{B}$ of $\mathscr{A}$ such that $\mathscr{A}/\mathscr{B} = \mathscr{A}/\mathscr{X}$.*

P r o o f. Denote $B = X \cup \{f_A^n(x)\colon\ n \in \mathbb{N},\ f_A^{n-1}(x) \in \operatorname{dom} f_A\}$. It is clear that $\mathscr{B} = (B, f_A \restriction B)$ is a subalgebra of $\mathscr{A}$. Further, $\mathscr{B}$ is the unique subalgebra of $\mathscr{A}$ such that $\mathscr{A}/\mathscr{B} = \mathscr{A}/\mathscr{X}$ in view of 1.3 and 4.2. $\square$

4.3.1 Notation. If the assumption of 4.3 is valid, then the algebra $\mathscr{B}$ of 4.3 will be denoted $\mathscr{X}^*$.

4.4 Theorem. *Let $\mathscr{A} = \sum_{j \in J} \mathscr{A}_j$ and let $\mathscr{X} = (X, f_X)$ be an ideal of $\mathscr{A}$. For $j \in J$ let $X_j = X \cap A_j$. Suppose that $K = \{j \in J\colon\ |X_j| > 1\} \neq \emptyset$. If $\mathscr{B} = \sum_{k \in K} (\mathscr{X}_k)^*$, then $\mathscr{B}$ is the unique subalgebra of $\mathscr{A}$ such that $\mathscr{A}/\mathscr{B} = \mathscr{A}/\mathscr{X}$.*

Proof. The assertion follows from 4.3 and from the definitions of θ_B and θ_X. □

4.4.1 Notation. If the assumption of 4.4 is satisfied, then we denote $\mathscr{B} = \mathscr{X}^*$; $\mathscr{X}^*$ will be called the subalgebra of $\mathscr{A}$ generated by the ideal $\mathscr{X}$.

For given $\mathscr{B}, \mathscr{C} \in \mathscr{U}$ let $\mathscr{S}(\mathscr{C}, \mathscr{B})$ be the system of all subalgebra extensions of $\mathscr{C}$ by $\mathscr{B}$. Further, let $\mathscr{I}(\mathscr{C}, \mathscr{B})$ be the system of all ideal extensions of $\mathscr{C}$ by $\mathscr{B}$.

Example 2. Let $\mathscr{C} = (C, f_C)$, $\mathscr{B} = (B, f_B)$, $C = \{c, d\}$, $\operatorname{dom} f_C = \{d\}$, $f_C(d) = c$, $B = \{0, 1, 2\}$, $\operatorname{dom} f_B = \{1, 2\}$, $f_B(1) = f_B(2) = 0$. By 2.5 and 2.4, $\mathscr{S}(\mathscr{C}, \mathscr{B}) \neq \emptyset$ and there are (up to isomorphism) exactly three algebras belonging to $\mathscr{S}(\mathscr{C}, \mathscr{B})$: they have the carrier $P = \{0, 1, 2, d\}$ and their operations f_1, f_2, f_3 have the domain $\{1, 2, d\}$, $f_i(j) = 0$ for $i = 1, 2, 3$, $j = 1, 2$ and $f_1(d) = 0$, $f_2(d) = 1$, $f_3(d) = 2$, since we obtain them using three possible critical mappings. For $i = 1, 2, 3$, (B, f_B) is not an ideal of (P, f_i), thus $(P, f_i) \notin \mathscr{I}(\mathscr{C}, \mathscr{B})$, i.e.,

(1) $\mathscr{S}(\mathscr{C}, \mathscr{B}) \cap \mathscr{I}(\mathscr{C}, \mathscr{B}) = \emptyset$.

Let (Q, f_Q) be such that $Q = \{0, 1, 2, 3, 4, d\}$, $\{4\} = Q - \operatorname{dom} f_Q$, $f_Q(1) = f_Q(2) = 0$, $f_Q(0) = 3$, $f_Q(3) = f_Q(d) = 4$. Then $(Q, f_Q) \in \mathscr{I}(\mathscr{C}, \mathscr{B})$.

This example shows that neither $\mathscr{S}(\mathscr{C}, \mathscr{B})$ nor $\mathscr{I}(\mathscr{C}, \mathscr{B})$ is empty and (1) is valid.

Example 3. Let $\mathscr{C} = (C, f_C)$, $C = \{c, d\}$, $f_C(c) = f_C(d) = c$, $\mathscr{X} = (X, f_X)$, $X = \{0, 1, 2\}$, $\operatorname{dom} f_X = \{1, 2\}$, $f_X(1) = f_X(2) = 0$. By 2.5, $\mathscr{S}(\mathscr{C}, \mathscr{X}) = \emptyset$. Let us consider the system $\mathscr{I}(\mathscr{C}, \mathscr{X})$. If $\mathscr{A} \in \mathscr{I}(\mathscr{C}, \mathscr{X})$, i.e., $\mathscr{X}$ is an ideal of $\mathscr{A}$ and $\mathscr{A}/\mathscr{X} \cong C$, then by 4.3 and 4.3.1 there is a subalgebra $\mathscr{X}^*$ of $\mathscr{A}$ such that $\mathscr{A}/\mathscr{X} = \mathscr{A}/\mathscr{X}^*$. We can try to describe $\mathscr{I}(\mathscr{C}, \mathscr{X})$ using the fact that we already know how to construct $\mathscr{S}(\mathscr{C}, \mathscr{B})$ for given $\mathscr{C}, \mathscr{B}$. Therefore we will try to assign some algebra $\mathscr{B}$ to $\mathscr{X}$, then to construct $\mathscr{S}(\mathscr{C}, \mathscr{B})$ and we will hope it will be useful for describing $\mathscr{I}(\mathscr{C}, \mathscr{X})$.

Since $\mathscr{C}$ is complete, $\mathscr{S}(\mathscr{C}, \mathscr{B}) \neq \emptyset$ only if also $\mathscr{B}$ is complete. In a natural way, to $\mathscr{X}$ there corresponds the following partial monounary algebra $\mathscr{B} = (B, f_B)$:

$$B = X \cup \{f_B(0),\ f_B^2(0),\ f_B^3(0), \ldots\}$$

(with $f_B^k(0) \neq f_B^j(0)$ for $k \neq j$), $f_B(1) = f_B(2) = 0$. (This seems to be the most natural way of assigning $\mathscr{B}$ to $\mathscr{X}$.)

Then $\mathscr{S}(\mathscr{C}, \mathscr{B}) \neq \emptyset$; the algebras belonging to $\mathscr{S}(\mathscr{C}, \mathscr{B})$ are of the form $s(\mathscr{C}, \mathscr{B}, \mu)$, where μ is a critical mapping.

For each $\mathscr{P} \in \mathscr{S}(\mathscr{C}, \mathscr{B})$ we get

(i) $\mathscr{C} \cong \mathscr{P}/\mathscr{B} = \mathscr{P}/\mathscr{X}$,

(ii) $\mathscr{B}$ is a subalgebra of $\mathscr{P}$.

Thus $\mathscr{S}(\mathscr{C}, \mathscr{B})$ consists of algebras with the carrier $B \cup \{d\}$. Let $(P, f_P) \in \mathscr{S}(\mathscr{C}, \mathscr{B})$. If $f_P(d) \notin \{0, 1, 2\}$, then (P, f_P) belongs also to $\mathscr{I}(\mathscr{C}, \mathscr{X})$. If $f_P(d) \in \{0, 1, 2\}$, then $\mathscr{X}$ is not an ideal of (P, f_P), therefore $(P, f_P) \notin \mathscr{I}(\mathscr{C}, \mathscr{X})$. Hence we obtain

(1) $\mathscr{S}(\mathscr{C}, \mathscr{B}) \nsubseteq \mathscr{I}(\mathscr{C}, \mathscr{X})$,

(2) $\mathscr{S}(\mathscr{C}, \mathscr{B}) \cap \mathscr{I}(\mathscr{C}, \mathscr{X}) \neq \emptyset$.

Further, let (Q, f_Q) be as in Example 2. Then $(Q, f_Q) \in \mathscr{I}(\mathscr{C}, \mathscr{X}) - \mathscr{S}(\mathscr{C}, \mathscr{B})$, thus

(3) $\mathscr{I}(\mathscr{C}, \mathscr{X}) \nsubseteq \mathscr{S}(\mathscr{C}, \mathscr{B})$.

The construction of replacing the top of an algebra $\mathscr{C}$ by some algebra $\mathscr{B}$ using critical mappings did not solve the problem (β).

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