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BERNSTEIN'S ANALYTICITY THEOREM FOR QUANTUM DIFFERENCES

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Abstract. We consider real valued functions f defined on a subinterval I of the positive real axis and prove that if all of f's quantum differences are nonnegative then f has a power series representation on I. Further, if the quantum differences have fixed sign on I then f is analytic on I.

Keywords: difference, quantum difference, quantum derivative, power series

MSC 2000: 26A48, 26A24, 26E05

1. Introduction

Many types of differences and derivatives have been used to study smoothness of real valued functions of one or several real variables. Perhaps the best known are the differences $\Delta_h^n f(x)$ defined by $\Delta_h^1 f(x) = f(x+h) - f(x)$ and $\Delta_h^{n+1} f(x) = \Delta_h^n f(x+h) - \Delta_h^n f(x)$, $n = 1, 2, \ldots$. The most general differences of this type were studied in [4], see also [8].

In this paper we consider the quantum differences defined in [3], p. 22 and further studied in [6]. See also [5] for an introduction to quantum calculus. Let f be a real valued function of a real variable and define the quantum derivative $D_1 f(q,x) = (f(qx) - f(x))/(qx - x)$ and inductively

$$D_{n+1} f(q,x) = \frac{D_n f(q,qx) - D_n f(q,x)}{qx - x},$$

$n = 1, 2, \ldots$. Then

$$D_n f(q,x) = \frac{\Delta_n f(q,x)}{q^{n(n-1)/2}\,(q-1)^n\,x^n},$$

where $\Delta_n f(q,x) = \sum_{k=0}^{n} (-1)^k [n/k]_q \cdot q^{k(k-1)/2} f(q^{n-k}x)$ and $[n/k]_q$ denotes the q-binomial coefficients, see [6], p. 114. We call $\Delta_n f(q,x)$ the nth quantum difference of f. We will see (Lemma 1 below) that $\Delta_n f(q,x)$ satisfies the recursion formula

$$\Delta_{n+1} f(q,x) = \Delta_n f(q,qx) - q^n \cdot \Delta_n f(q,x), \quad n = 1,2,\ldots. \tag{1}$$

The quantum differences $\Delta_n f(q,x)$ are not of the general type studied in [4] and [8], since their coefficients depend on the step parameter q.

A classical theorem of S. G. Bernstein [1], p. 190 states that if f has all differences $\Delta_h^n f(x) \geqslant 0$ on [0,1) then f has a power series representation $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on that interval. The analogous result for binary differences was proved in [7] and the most general result for this kind of differences follows from [4]. See also [8], where a slightly less general result is obtained by different methods. We prove the corresponding result for the quantum differences through a series of lemmas that might be of independent interest. Our theorems are in Section 2, while the lemmas and the proofs of the theorems are contained in Section 3.

All functions are defined on some interval $I \subset [0,\infty]$. We say $\Delta_n f(q,x) \geqslant 0$ in I if $\Delta_n f(q,x) \geqslant 0$ for all $x \in I$ and $q > 1$ such that $q^n x \in I$. A function f defined on an open interval I is analytic on I if every $x \in I$ has a neighbourhood where f is represented by a power series.

2. Main result

The following theorems are our main result.

Theorem 1. *Let $f\colon [0,1) \to \mathbb{R}$ and assume that $\Delta_n f(q,x) \geqslant 0$ on $[0,1)$, $n = 1,2,\ldots$. Then f is infinitely differentiable on $[0,1)$ and f has a power series representation*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad 0 \leqslant x < 1,$$

for some sequence $\{c_n\}_0^\infty$ of nonnegative numbers.

Corollary 1. *Let $f\colon [a,b) \to \mathbb{R}$, for some $0 \leqslant a < b \leqslant \infty$, and assume that $\Delta_n f(q,x) \geqslant 0$ on $[a,b)$, $n = 1,2,\ldots$. Then f is infinitely differentiable on $[a,b)$ and*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad a \leqslant x < b,$$

for some sequence $\{c_n\}_0^\infty$ of nonnegative numbers.

Theorem 2. *Let* $f\colon [a,b) \to \mathbb{R}$ *for some* $0 \leqslant a < b \leqslant \infty$ *and assume that* $\Delta_n f(q,x)$ *has fixed sign on* $[a,b)$*, only depending on* n*, for* $n = 1,2,\ldots$. *Then* f *is analytic on* (a,b)*.*

Theorem 3. *Let* $f\colon [a,b) \to \mathbb{R}$*, where* $0 < a < b \leqslant \infty$ *and assume that there are* $C > 0$ *and* $q_0 > 1$ *such that*

$$|\Delta_n f(q,x)| \leqslant (q-1)(q^2-1)\ldots(q^n-1)q^{\binom{n}{2}} \cdot C^n,$$

for all $a < x < q^n x < b$ *and* $1 < q < q_0$*,* $n = 1,2,\ldots$. *Then* f *is analytic on* (a,b)*.*

3. Proofs

We start with the recursion formula (1) for the quantum differences. The proof follows easily from the definition.

Lemma 1. *For any positive integer* n *and any function* f *it holds*

$$\Delta_{n+1} f(q,x) = \Delta_n f(q,qx) - q^n \cdot \Delta_n f(q,x).$$

The next lemma is a useful integral formula for the quantum differences of sufficiently smooth functions.

Lemma 2. *Let* $f\colon I \to \mathbb{R}$ *have* m *continuous derivatives, for some positive integer* m*, then*

$$\Delta_m f(q,x) = q^{\binom{m}{2}} \int_x^{qx} \mathrm{d}t_1 \int_{t_1}^{qt_1} \mathrm{d}t_2 \ldots \int_{t_{m-1}}^{qt_{m-1}} \mathrm{d}t_m \, f^{(m)}(t_m),$$

for all $x, q^m x$ *in* I*.*

P r o o f. The cases $m = 1$ and $m = 2$ are easy. Assume the lemma is true for some $m = k \geqslant 2$ and note that if we define $f_q(x) = f(qx)$ then $\Delta_m f_q(q,x) = \Delta_m f(q,qx)$. Therefore we get

$$\begin{aligned}
\Delta_{k+1} f(q,x) &= \Delta_k f_q(q,x) - q^k \cdot \Delta_k f(q,x)\\
&= q^{\binom{k}{2}} q^k \left(\int_x^{qx} \mathrm{d}t_1 \int_{t_1}^{qt_1} \mathrm{d}t_2 \ldots \int_{t_{k-1}}^{qt_{k-1}} \mathrm{d}t_k \left(f^{(k)}(qt_k) - f^{(k)}(t_k)\right)\right)\\
&= q^{\binom{k+1}{2}} \int_x^{qx} \mathrm{d}t_1 \int_{t_1}^{qt_1} \mathrm{d}t_2 \ldots \int_{t_{k-1}}^{qt_{k-1}} \mathrm{d}t_k \int_{t_k}^{qt_k} \mathrm{d}t_{k+1} \, f^{(k+1)}(t_{k+1})
\end{aligned}$$

and the general case follows by induction over m. □

Remark. Let $f(x) = x^k$, for some positive integer k, then clearly $\Delta_m f(q,x) = 0$, if $k < m$, by Lemma 2. Simple calculations using Lemma 2 show that $\Delta_m f(q,x) = q^{\binom{m}{2}}(q-1)(q^2-1)\dots(q^m-1)x^m$, for $k = m$, and $\Delta_m f(q,x) = q^{\binom{m}{2}} \cdot (q^2-1)(q^3-1)\dots(q^{m+1}-1)x^{m+1}$, for $k = m+1$.

Lemma 3. *Assume that f has m continuous derivatives on $(0,\infty)$, for some positive integer m, and that $f_+^{(m+1)}(x)$ exists. Then*

$$\frac{\Delta_{m+1} f(q,x)}{(q-1)^{m+1}x^{m+1}} \to f_+^{(m+1)}(x),$$

as $q \searrow 1$.

P r o o f. Lemma 2 gives that

$$\begin{aligned}\Delta_m f(q,qx) &= q^{\binom{m}{2}} \int_{qx}^{q^2x} \mathrm{d}t_1 \int_{t_1}^{qt_1} \mathrm{d}t_2 \dots \int_{t_{m-1}}^{qt_{m-1}} \mathrm{d}t_m\, f^{(m)}(t_m) \\ &= q^m q^{\binom{m}{2}} \int_{x}^{qx} \mathrm{d}u_1 \int_{u_1}^{qu_1} \mathrm{d}u_2 \dots \int_{u_{m-1}}^{qu_{m-1}} \mathrm{d}u_m\, f^{(m)}(qu_m)\end{aligned}$$

by the change of variables $t_i = qu_i$, $1 \leqslant i \leqslant m$. Hence

$$\Delta_{m+1} f(q,x) = q^m q^{\binom{m}{2}} \int_{x}^{qx} \mathrm{d}u_1 \int_{u_1}^{qu_1} \mathrm{d}u_2 \dots \int_{u_{m-1}}^{qu_{m-1}} \mathrm{d}u_m\, (f^{(m)}(qu_m) - f^{(m)}(u_m))$$

by Lemma 1. From the existence of $f_+^{(m+1)}(x)$ we get $f^{(m)}(qu_m) = f^{(m)}(x) + f_+^{(m+1)}(x)(qu_m - x) + o(qu_m - x)$ and analogously for $f^{(m)}(u_m)$. This gives

$$f^{(m)}(qu_m) - f^{(m)}(u_m) = (q-1)u_m(f_+^{(m+1)}(x) + o(1)),$$

as $q \searrow 1$, since $x \leqslant u_m \leqslant q^m x$. By the remark above we get

$$\begin{aligned}\Delta_{m+1} f(q,x) = q^{\binom{m+1}{2}}&(q-1)(q^2-1)\dots(q^{m+1}-1) \\ &\times \frac{1}{(m+1)!}\, x^{m+1}(f_+^{(m+1)}(x) + o(1))\end{aligned}$$

and Lemma 3 follows. $\square$

The next lemma contains the main step in the proofs of Theorems 1 and 2.

Lemma 4. *Assume that* $f\colon (0,\infty) \to \mathbb{R}$ *satisfies* $\Delta_k f(q,x) \geqslant 0$*, for* $x>0$*,* $q>1$ *and* $k=1,2,\dots n$*, for some* $n \geqslant 2$*. Then the following statements hold:*

A_n: *$f(x)$ has $n-2$ continuous derivatives for $x>0$,*

B_n: *$f_+^{(n-1)}(x)$ and $f_-^{(n-1)}(x)$ exist for all $x>0$,*

C_n: *$f_-^{(n-1)}(x) \leqslant f_+^{(n-1)}(x) \leqslant f_-^{(n-1)}(y) \leqslant f_+^{(n-1)}(y)$, for all $0<x<y$,*

D_n: *$\lim\limits_{q\to 1} \Delta_{n-2} f(q,x)/(q-1)^{n-2}x^{n-2} = f^{(n-2)}(x)$, for all $x>0$.*

Proof. The proof is by induction over n. First let $n=2$ and note that f is nondecreasing and that $\Delta_2 f(q,x) \geqslant 0$ is equivalent to $\Delta_1 f(q,qx) \geqslant q\cdot\Delta_1 f(q,x)$ by Lemma 1. Fix $0<x<y$ and find $q>1$ and a positive integer m such that $q^m x < y < q^{m+1}x$. Then

$$f(y)-f(x) \geqslant \sum_{i=1}^{m}(f(q^i)-f(q^{i-1}x)) \geqslant (1+q+\ldots+q^{m-1})(f(qx)-f(x))$$

which gives $0 \leqslant f(qx)-f(x) \leqslant (f(y)-f(x))/m$. Letting $q\to 1$ and $m\to\infty$ proves that f is right contiunous at x. A similar argument proves left continuity and A_2 follows. The condition $\Delta_2 f(q,x) \geqslant 0$ can also be written

$$f(qx) \leqslant \frac{1}{q+1}\, f(q^2x) + \frac{q}{q+1}\, f(x).$$

It follows that f is convex and therefore B_2 and C_2 hold, while D_2 is empty. This completes the proof in the case $n=2$.

Now assume that the lemma holds for some $n\geqslant 2$ and that $\Delta_k f(q,x) \geqslant 0$, $1\leqslant k\leqslant n+1$. Then $A_n - D_n$ hold. Fix any $x>0$ then by Lemma 3

$$\Delta_n f(q,x) \leqslant q^{-n}\cdot\Delta_n f(q,qx) \leqslant \ldots \leqslant q^{-\nu n}\cdot\Delta_n f(q,q^\nu x) \leqslant \ldots. \tag{2}$$

The νth term in (2) equals

$$q^{-\nu n}(\Delta_{n-1} f(q,q^{\nu+1}x) - q^{n-1}\cdot\Delta_{n-1} f(q,q^\nu x)) \tag{3}$$

by Lemma 1. Let k and l be positive integers (to be defined below) and add the $k+1$ succesive terms in (2) starting from $\nu=0$ and $\nu=l$ respectively. Then we get

$$\sum_{\nu=0}^{k} q^{(n-1)\nu}q^{-\nu n}\cdot\Delta_n f(q,q^\nu x) \leqslant \sum_{\nu=l}^{k+l} q^{(n-1)(\nu-l)}q^{-\nu n}\cdot\Delta_n f(q,q^\nu x).$$

We now replace the nth order differences in the last equation by differences of order $n-1$ using (3) and we get

$$\begin{aligned} &q^{(n-1)k}q^{-kn}\cdot\Delta_{n-1} f(q,q^{k+1}x) - q^{n-1}\cdot\Delta_{n-1} f(q,x) \\ &\leqslant q^{(n-1)k}q^{-(k+l)n}\cdot\Delta_{n-1} f(q,q^{k+l+1}x) - q^{-ln}q^{n-1}\cdot\Delta_{n-1} f(q,q^l x) \end{aligned} \tag{4}$$

by the telescoping effect. Fix any $r > 1$ and find an integer k and $q > 1$ such that $q^{k+1} = r$. Let $l = k + 1$. Multiply (4) by $q^k q^{-(n-1)k}$ and use (2) to modify the second and fourth terms of the result to arrive at

$$\tag{5} \begin{aligned} q^k q^{-kn} \cdot \Delta_{n-1} f(q, rx) - q^{n-1} \cdot \Delta_{n-1} f(q, x) \\ \leqslant q^k q^{-(2k+1)n} \cdot \Delta_{n-1} f(q, r^2 x) - q^{-ln} q^{n-1} \cdot \Delta_{n-1} f(q, rx). \end{aligned}$$

We now divide (5) by $q^{n-1}(q-1)^{n-1}x^{n-1}$ to get

$$\begin{aligned} \frac{\Delta_{n-1} f(q, rx)}{(q-1)^{n-1}(rx)^{n-1}} - \frac{\Delta_{n-1} f(q, x)}{(q-1)^{n-1}x^{n-1}} \\ \leqslant \frac{1}{r}\Big(\frac{\Delta_{n-1} f(q, r^2 x)}{(q-1)^{n-1}(r^2 x)^{n-1}} - \frac{\Delta_{n-1} f(q, rx)}{(q-1)^{n-1}(rx)^{n-1}} \Big). \end{aligned}$$

Letting $q \to 1$ gives

$$\tag{6} f_+^{(n-1)}(rx) - f_+^{(n-1)}(x) \leqslant \frac{1}{r}(f_+^{(n-1)}(r^2 x) - f_+^{(n-1)}(rx)),$$

or equivalently $\Delta_2 f_+^{(n-1)}(r, x) \geqslant 0$, by Lemma 3.

Then $f_+^{(n-1)}$ is continuous by the case $n = 2$ and letting $y \to x$ in C_n shows that $f_+^{(n-1)}(x) = f_-^{(n-1)}(x) = f^{(n-1)}(x)$. We can now apply the case $n = 2$ to $f^{(n-1)}$ and conclude that A_{n+1}, B_{n+1} and C_{n+1} hold. It only remains to prove D_{n+1}. We apply Lemma 2, with $m = n - 1$, and note that $f^{(n-1)}(t_{n-1}) = f^{(n-1)}(x) + o(q - 1)$, as $q \to 1$. The remark following Lemma 2, with $m = n - 1$ and $k = n - 1$, proves D_{n+1}. □

P r o o f of Theorem 1. We give the proof from [1], p. 193. Since $f^{(n)}$ is non-negative and continuous on $(0, 1)$ and $f_+^{(n)}$ is continuous on $[0, 1)$, $n = 0, 1, 2, \ldots$, we have by Taylor's formula

$$f(x) - P_n(x) = \frac{x^{n+1}}{n!} \int_0^1 f^{(n+1)}(ux)(1-u)^n \,\mathrm{d}u,$$

where $P_n(x)$ is the Taylor polynomial of degree n. Then for any $0 \leqslant x \leqslant R < 1$ we have

$$0 \leqslant f(x) - P_n(x) \leqslant \Big(\frac{x}{R}\Big)^{n+1} (f(R) - P_n(R))$$

and the theorem follows with $a_n = f^n(0)/n!$. □

P r o o f of Corollary 1. Let f be as in the statement of the corollary and assume that $b < \infty$. Now define $g(x) = f((x - x)/b)$ and apply Theorem 1 to the function g. The case $b = \infty$ follows easily by letting $b \to \infty$. □

Proof of Theorem 2. It is easy to see that $f(x)$ has derivatives of all orders on $[a,b)$ by the proof of Lemma 1. Further, $f^n(x)$ has constant sign, only depending on n, by Lemma 3. Then as in [1], p. 196 we conclude that f is analytic on (a,b). $\square$

Proof of Theorem 3. Let $[u,v] \subset (a,b)$ be such that $v-u < \max(1, u/C)$ and $v/u < q_0$. Choose $R > v$ such that $R-u < \max(1, u/C)$ and define $g(x) = (R-x)^{-1}$ on $[u,v]$. Then $g^{(n)}(x) = n!\,(R-x)^{-n-1}$ and by Lemma 2 we get

$$\begin{aligned}\Delta_n g(q,x) &\geqslant q^{\binom{n}{2}} n!\,(R-u)^{-n-1}\frac{q-1}{1}\frac{q^2-1}{2}\dots\frac{q^n-1}{n}x^n \\ &\geqslant q^{\binom{n}{2}}(q-1)(q^2-1)\dots(q^n-1)C^n,\end{aligned}$$

on $[u,v]$. Hence $\Delta_n(g-f)(q,x) \geqslant 0$ on $[u,v]$, $n = 1,2,\dots$, and therefore $f(x) = g(x) - (g(x) - f(x))$ is analytic on (u,v) by Corollary 1. Varying $[u,v]$ in (a,b) completes the proof. $\square$

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