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CLOSURE SPACES AND CHARACTERIZATIONS OF FILTERS IN TERMS OF THEIR STONE IMAGES

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Abstract. Fréchet, strongly Fréchet, productively Fréchet, weakly bisequential and bisequential filters (i.e., neighborhood filters in spaces of the same name) are characterized in a unified manner in terms of their images in the Stone space of ultrafilters. These characterizations involve closure structures on the set of ultrafilters. The case of productively Fréchet filters answers a question of S. Dolecki and turns out to be the only one involving a non topological closure structure.

Keywords: filters, ultrafilters, Frechet, closure spaces

MSC 2000: 54A05, 54A20, 54D55

1. Introduction

Local topological properties of a topological space X can be interpreted in terms of properties of the neighborhood filters. Such properties of filters usually make sense for general filters and consequently can be intepreted also in terms of the Stone images of such filters, that is, the corresponding sets of finer ultrafilters in the space βX of ultrafilters on X endowed with the usual Stone topology.[1] The general problem is to characterize properties of a filter $\mathscr{F}$ on X in terms of $\beta\mathscr{F} = \{\mathscr{U} \in \beta X\colon \mathscr{U} \geqslant \mathscr{F}\}$ or of $\beta_0\mathscr{F} = \beta\mathscr{F} \cap \beta_0 X$, where $\beta_0 X$ denotes the set of *free* ultrafilters on X. V. Malyhin first followed this line of investigation [11] and more recently revisited the idea with A. Bella in a more systematic manner [2]. Among others, they characterize Fréchet, strongly Fréchet and bisequential filters in terms of their Stone images. Let us recall how these filters are defined:

[1] β denotes the usual Stone topology on βX for which $\{\beta A\colon A \subset X\}$ is a base of open (and of closed) sets. Notice that a non empty subset of βX is closed if and only if it is of the form $\beta\mathscr{F}$ for some filter $\mathscr{F}$ on X (with the convention that $\beta(\{\emptyset\}^{\uparrow}) = \emptyset$).

Two families $\mathscr{A}$ and $\mathscr{B}$ of subsets of a set X are *meshing*—in symbols $\mathscr{A} \# \mathscr{B}$—if $A \cap B \neq \emptyset$ whenever $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Let $\mathbb{F}_\omega$ denote the class of countably based filters. A topological space X is respectively *bisequential, strongly Fréchet, Fréchet* [12], [3] if

$$x \in \operatorname{adh}\mathscr{H} \Longrightarrow \exists \mathscr{L} \in \mathbb{F}_\omega,\ \mathscr{L} \# \mathscr{H},\ \mathscr{L} \to x, \tag{1}$$

for every filter $\mathscr{H}$, every countably based $\mathscr{H}$, every principal filter $\mathscr{H}$ respectively. Obviously, a space X is respectively bisequential, strongly Fréchet, Fréchet if and only if every neighborhood filter in X satisfies the following property of $\mathscr{F}$

$$\mathscr{H} \# \mathscr{F} \Longrightarrow \exists \mathscr{L} \in \mathbb{F}_\omega,\ \mathscr{L} \# \mathscr{H},\ \mathscr{L} \geqslant \mathscr{F}, \tag{2}$$

for every filter $\mathscr{H}$, every countably based $\mathscr{H}$, every principal filter $\mathscr{H}$ respectively. Accordingly a filter $\mathscr{F}$ is called respectively *bisequential, strongly Fréchet, Fréchet* if it satisfies (2) for every filter $\mathscr{H}$, every countably based $\mathscr{H}$, every principal filter $\mathscr{H}$ respectively. *Productively Fréchet spaces* were introduced in [7], [8] as the spaces whose product with every strongly Fréchet space is (strongly) Fréchet and characterized as the spaces satisfying (1) for every strongly Fréchet filter $\mathscr{H}$. *Weakly bisequential spaces* were introduced in [1] as the spaces satisfying (1) for every *countably deep* filter[2] $\mathscr{H}$, and they were studied more systematically in [10]. Naturally, we call *productively Fréchet* the filters satisfying (2) for every strongly Fréchet filter $\mathscr{H}$ and *weakly bisequential* the filters satisfying (2) for every countably deep filter $\mathscr{H}$. S. Dolecki asked [4] if there is a characterization of productively Fréchet filters similar to the known characterizations of Fréchet, strongly Fréchet and bisequential filters. It is the aim of this paper to answer this question. We present a unified theorem characterizing all five classes of filters discussed above in terms of their Stone images. To this end, we need to consider not only topologies, but also closure space structures (defined by a closure operator sharing all the properties of a topological one, save additivity) on βX. While the general context of closure spaces allows a unified treatment of various classes of filters, all the closure considered turn out to be topological, except for the one appearing in the characterization of productively Fréchet filters. This might be one of the reasons why productively Fréchet spaces were introduced so late.

[2] A filter $\mathscr{F}$ is *countably deep* if $\bigcap \mathscr{A} \in \mathscr{F}$ whenever $\mathscr{A}$ is a countable subfamily of $\mathscr{F}$.

2. Classes of filters and closures on βX

A *closure space* $(X,\mathscr{C})$ is a set endowed with a family $\mathscr{C} \subset 2^X$ of *closed sets*, which is closed under arbitrary intersection and contains the empty set (complements of closed sets are called *open*). A closure space $(X,\mathscr{C})$ defines a *closure operator* $\mathrm{cl}_{\mathscr{C}}$ defined by

$$\mathrm{cl}_{\mathscr{C}}\colon\ 2^X \to 2^X,$$
$$\mathrm{cl}_{\mathscr{C}} A = \bigcap_{A\subset C\in\mathscr{C}} C.$$

This closure operator is expansive (i.e., $A \subset \mathrm{cl}\, A$), isotone (i.e., $A \subset B \Longrightarrow \mathrm{cl}\, A \subset \mathrm{cl}\, B$) and idempotent. Conversely, an expansive, isotone and idempotent map $\mathrm{cl}\colon\ 2^X \to 2^X$ such that $\mathrm{cl}\,\emptyset = \emptyset$ is the closure operator of a closure structure $\mathscr{C} = \{C \subset X\colon\ C = \mathrm{cl}\, C\}$ on X. To a closure operator cl, we can associate an *interior operator*

$$\mathrm{int}\, A = (\mathrm{cl}\, A^c)^c\,.$$

This operator is contractive, isotone, idempotent, and $\mathrm{int}\, X = X$. These properties characterize the closure structure as well.

If λ is a closure structure on X, let $\mathscr{N}_\lambda(A) = \{U \subset X\colon\ A \subset \mathrm{int}\, U\}$ and $\mathscr{N}_\lambda(\mathscr{A}) = \bigcup_{A\in\mathscr{A}} \mathscr{N}_\lambda(A)$. See for instance [5] for more details on closure spaces.

Lemma 1.

$$\mathscr{B} \# \mathscr{N}_\lambda(\mathscr{A}) \Longleftrightarrow \mathrm{cl}_\lambda \mathscr{B} \# \mathscr{A}\,.$$

To a class $\mathbb{D}$ of filters on X, we associate a closure structure $\mathbb{D}^*$ on βX by declaring $\{\beta\mathscr{D}\colon\ \mathscr{D} \in \mathbb{D}\}$ a base of open sets for the closure space $(\beta X, \mathbb{D}^*)$. Therefore

$$\mathrm{int}_{\mathbb{D}^*}\, \beta\mathscr{F} = \bigcup_{\substack{\mathscr{D}\in\mathbb{D}\\ \mathscr{F}\leqslant\mathscr{D}}} \beta\mathscr{D} \tag{3}$$

for any filter $\mathscr{F}$.

Let $\mathbb{D}$ and $\mathbb{M}$ be two classes of filters. As a common generalization of Fréchet, strongly Fréchet, productively Fréchet, weakly bisequential and bisequential filters, we call a filter $\mathscr{F}$ $\mathbb{D}$ *to* $\mathbb{M}$ *meshable-refinable*—in symbols, $\mathscr{F} \in (\mathbb{D}/\mathbb{M})_{\#\geqslant}$—if

$$\mathscr{D} \in \mathbb{D},\ \mathscr{D} \# \mathscr{F} \Longrightarrow \exists \mathscr{M} \in \mathbb{M},\ \mathscr{M} \# \mathscr{D},\ \mathscr{M} \geqslant \mathscr{F}.$$

See [6] for details and variants. Let $\mathbb{D} \vee \mathbb{M}$ denote the class of filters of the type $\mathscr{D} \vee \mathscr{M}$ where $\mathscr{D} \in \mathbb{D}$ and $\mathscr{M} \in \mathbb{M}$. Following the terminology of [9], we denote by $\mathbb{F}_1$ the class of principal filters and call $\mathbb{F}_1$-*steady* a class $\mathbb{M}$ satisfying $\mathbb{F}_1 \vee \mathbb{M} \subset \mathbb{M}$.

Lemma 2. *If $\mathbb{D}$ and $\mathbb{M}$ are two $\mathbb{F}_1$-steady classes of filters, then the class $(\mathbb{D}/\mathbb{M})_{\#\geqslant}$ is also $\mathbb{F}_1$-steady.*

Proof. Let $\mathscr{F} \in (\mathbb{D}/\mathbb{M})_{\#\geqslant}$, $A \in \mathbb{F}_1$ be such that $A \# \mathscr{F}$ and $\mathscr{D} \in \mathbb{D}$ be such that $\mathscr{D} \# (\mathscr{F} \vee A)$. Then $\mathscr{F} \# (\mathscr{D} \vee A)$ and $\mathscr{D} \vee A \in \mathbb{D}$ by $\mathbb{F}_1$-steadiness of $\mathbb{D}$. Thus, there exists an $\mathbb{M}$-filter $\mathscr{M} \# (\mathscr{D} \vee A)$ such that $\mathscr{M} \geqslant \mathscr{F}$. Since $\mathscr{M} \vee A \in \mathbb{M}$ by $\mathbb{F}_1$-steadiness of $\mathbb{M}$, $\mathscr{D} \# (\mathscr{M} \vee A)$ and $\mathscr{M} \vee A \geqslant \mathscr{F} \vee A$, we conclude that $\mathscr{F} \vee A \in (\mathbb{D}/\mathbb{M})_{\#\geqslant}$. $\square$

A filter is *free* if its intersection is empty. We denote by $\mathscr{F}^\circ$ the free part $\mathscr{F} \vee (\bigcap \mathscr{F})^c$ of a filter $\mathscr{F}$, and by $\mathscr{F}^\bullet$ its principal part $\bigcap \mathscr{F}$. One or the other may be the degenerate filter $\{\emptyset\}^\uparrow = 2^X$. We always have $\mathscr{F} = \mathscr{F}^\circ \wedge \mathscr{F}^\bullet$, with the convention that $\mathscr{G} \wedge \{\emptyset\}^\uparrow = \mathscr{G}$. By convention, we assume that the degenerate filter $\{\emptyset\}^\uparrow$ is an element of every class of filters we may consider.

Lemma 3. *Let $\mathbb{D}$ and $\mathbb{M}$ be two $\mathbb{F}_1$-steady classes of filters such that $\mathbb{F}_1 \subset \mathbb{M}$. A filter $\mathscr{F}$ is $\mathbb{D}$ to $\mathbb{M}$ meshable-refinable if and only if its free part $\mathscr{F}^\circ$ is.*

Proof. By Lemma 2, $\mathscr{F}^\circ$ is $\mathbb{D}$ to $\mathbb{M}$ meshable-refinable whenever $\mathscr{F}$ is. Conversely, assume that $\mathscr{F}^\circ \in (\mathbb{D}/\mathbb{M})_{\#\geqslant}$ and let $\mathscr{D} \in \mathbb{D}$ be such that $\mathscr{D} \# \mathscr{F}$. Then $\mathscr{D} \# (\mathscr{F}^\circ \wedge \mathscr{F}^\bullet)$, hence either $\mathscr{D} \# \mathscr{F}^\circ$ or $\mathscr{D} \# \mathscr{F}^\bullet$. In the former case, there exists $\mathscr{M} \in \mathbb{M}$ such that $\mathscr{M} \# \mathscr{D}$ and $\mathscr{M} \geqslant \mathscr{F}^\circ \geqslant \mathscr{F}$. In the later case, $\mathscr{F}^\bullet \in \mathbb{F}_1 \subset \mathbb{M}$ is an $\mathbb{M}$-filter meshing with $\mathscr{D}$ and finer than $\mathscr{F}$. Thus, $\mathscr{F} \in (\mathbb{D}/\mathbb{M})_{\#\geqslant}$. $\square$

Proposition 4. *Let $\mathbb{D}$ be a class of filters on X. The closure space $(\beta X, \mathbb{D}^*)$ is a topological space if and only $\mathbb{D} \vee \mathbb{D} \subset (\mathbb{F}/\mathbb{D})_{\#\geqslant}$.*

If moreover $\mathbb{D} = (\mathbb{L}/\mathbb{M})_{\#\geqslant}$ for some classes $\mathbb{L}$ and $\mathbb{M}$ of filters, then $(\beta X, \mathbb{D}^*)$ is a topological space if and only $\mathbb{D} \vee \mathbb{D} \subset \mathbb{D}$.

Proof. To show that $\mathbb{D}^*$ is a topology, we only need to show that a finite intersection of open sets is open in $\mathbb{D}^*$. But

$$\left(\bigcup_{\alpha\in I} \beta\mathscr{D}_\alpha\right) \cap \left(\bigcup_{\gamma\in J} \beta\mathscr{D}_\gamma\right) = \bigcup_{\substack{\alpha\in I\\ \gamma\in J}} (\beta\mathscr{D}_\alpha \cap \beta\mathscr{D}_\gamma) = \bigcup_{\substack{\alpha\in I\\ \gamma\in J}} \beta(\mathscr{D}_\alpha \vee \mathscr{D}_\gamma),$$

with the convention that $\beta(\mathscr{D}_\alpha \vee \mathscr{D}_\gamma) = \emptyset$ if $\mathscr{D}_\alpha$ and $\mathscr{D}_\gamma$ do not mesh. Now, if $\mathscr{U} \in \left(\bigcup_{\alpha\in I} \beta\mathscr{D}_\alpha\right) \cap \left(\bigcup_{\gamma\in J} \beta\mathscr{D}_\gamma\right)$, there is α and γ such that $\mathscr{U} \geqslant \mathscr{D}_\alpha \vee \mathscr{D}_\gamma$. By assumption, $\mathscr{D}_\alpha \vee \mathscr{D}_\gamma \in (\mathbb{F}/\mathbb{D})_{\#\geqslant}$, so that there exists a $\mathbb{D}$-filter $\mathscr{D}_{\alpha\gamma}$ such that $\mathscr{U} \geqslant \mathscr{D}_{\alpha\gamma} \geqslant \mathscr{D}_\alpha \vee \mathscr{D}_\gamma$. Therefore, $\mathscr{U} \in \beta\mathscr{D}_{\alpha\gamma} \subset \left(\bigcup_{\alpha\in I} \beta\mathscr{D}_\alpha\right) \cap \left(\bigcup_{\gamma\in J} \beta\mathscr{D}_\gamma\right)$.

Conversely, if $\mathbb{D}^*$ is a topology and $\mathscr{D}_1$ and $\mathscr{D}_2$ are two $\mathbb{D}$-filters such that $\mathscr{D}_1 \# \mathscr{D}_2$, then $\beta\mathscr{D}_1 \cap \beta\mathscr{D}_2 = \beta(\mathscr{D}_1 \vee \mathscr{D}_2)$ is open. Therefore,

$$\beta(\mathscr{D}_1 \vee \mathscr{D}_2) = \bigcup_{\substack{\mathscr{D}\in\mathbb{D} \\ \mathscr{D}_1\vee\mathscr{D}_2\leqslant\mathscr{D}}} \beta\mathscr{D}.$$

For every $\mathscr{F} \# (\mathscr{D}_1 \vee \mathscr{D}_2)$, there exists $\mathscr{U} \in \beta(\mathscr{D}_1 \vee \mathscr{D}_2) \cap \beta\mathscr{F}$. In view of the above description of $\beta(\mathscr{D}_1 \vee \mathscr{D}_2)$, there exists $\mathscr{D} \in \mathbb{D}$ such that $\mathscr{U} \geqslant \mathscr{D} \geqslant \mathscr{D}_1 \vee \mathscr{D}_2$. Since $\mathscr{D} \# \mathscr{F}$, we conclude that $\mathscr{D}_1 \vee \mathscr{D}_2 \in (\mathbb{F}/\mathbb{D})_{\#\geqslant}$.

Obviously, if $\mathbb{D} \vee \mathbb{D} \subset \mathbb{D}$ then $\mathbb{D} \vee \mathbb{D} \subset (\mathbb{F}/\mathbb{D})_{\#\geqslant}$. We show that the converse is true if $\mathbb{D} = (\mathbb{L}/\mathbb{M})_{\#\geqslant}$. Indeed, if $\mathscr{D}_1$ and $\mathscr{D}_2$ are $\mathbb{D}$-filters and $\mathscr{L}$ is a $\mathbb{L}$-filter such that $\mathscr{L} \# (\mathscr{D}_1 \vee \mathscr{D}_2)$, then there exists a $\mathbb{D}$-filter $\mathscr{D} \# \mathscr{L}$ such that $\mathscr{D} \geqslant \mathscr{D}_1 \vee \mathscr{D}_2$ because $\mathbb{D} \vee \mathbb{D} \subset (\mathbb{F}/\mathbb{D})_{\#\geqslant}$. Since $\mathscr{D} \in (\mathbb{L}/\mathbb{M})_{\#\geqslant}$, there exists a $\mathbb{M}$-filter $\mathscr{M}$ such that $\mathscr{M} \# \mathscr{L}$ and $\mathscr{M} \geqslant \mathscr{D} \geqslant \mathscr{D}_1 \vee \mathscr{D}_2$. Therefore $\mathscr{D}_1 \vee \mathscr{D}_2 \in (\mathbb{L}/\mathbb{M})_{\#\geqslant} = \mathbb{D}$. ☐

Let $\mathbb{F}$, $\mathbb{F}_\omega$, $\mathbb{F}_{\wedge\omega}$, $\mathbb{F}_1$ denote respectively the class of all, of countably based, of countably deep and of principal filters respectively. By the above observation, the closure structures $\mathbb{F}^*$, $\mathbb{F}^*_\omega$, $\mathbb{F}^*_{\wedge\omega}$ and $\mathbb{F}^*_1$ are topological because $\mathbb{F}$, $\mathbb{F}_\omega$, $\mathbb{F}_{\wedge\omega}$ and $\mathbb{F}_1$ are stable under finite suprema. More specifically, $\mathbb{F}^*$ is the discrete topology on βX, $\mathbb{F}^*_\omega$ is the G_δ-topology $G_\delta\beta$ associated to β,[3] and $\mathbb{F}^*_1$ is the usual Stone topology β of βX. On the other hand,

$$\mathscr{F} \in \mathbb{F}_{\wedge\omega} \iff \mathscr{N}_\beta(\beta\mathscr{F}) = \mathscr{N}_{G_\delta\beta}(\beta\mathscr{F}).$$

Hence the topology $\mathbb{F}^*_{\wedge\omega}$ is generated by β-closed sets having the same β and $G_\delta\beta$ neighborhood filters. More generally, if τ denotes a topology on X and $G_\delta\tau$ denotes the associated G_δ-topology, a closure structure $\tau^\perp$ can be defined on X by declaring $\tau^\perp$-open unions of τ-closed sets F satisfying $\mathscr{N}_\tau(F) = \mathscr{N}_{G_\delta\tau}(F)$. We call a topological space (X, τ) *δ-normal* if for every pair of disjoint closed sets F_1, F_2 there exists disjoint G_δ-sets G_1 and G_2 such that $F_1 \subset G_1$ and $F_2 \subset G_2$.[4]

Lemma 5. *Let (X, τ) be a δ-normal topological space. Then the closure structure $\tau^\perp$ is a topology on X.*

Proof. Assume (X, τ) is δ-normal. We only need to show that if F_1, F_2 are τ-closed sets such that $\mathscr{N}_\tau(F_i) = \mathscr{N}_{G_\delta\tau}(F_i)$ for $i = 1, 2$ then $\mathscr{N}_\tau(F_1 \cap F_2) = \mathscr{N}_{G_\delta\tau}(F_1 \cap F_2)$.

[3] Because it is easy to see that the G_δ-topology of βX has a base consisiting of *β-closed* G_δ-sets.

[4] The authors want to thank Francis Jordan (Georgia Southern University) for helpful discussions that led to Lemma 5.

Let G be a $G_\delta\tau$-set containing $F_1 \cap F_2$. We need to find an open set O such that $F_1 \cap F_2 \subset O \subset G$. The sets $G \cup F_1^c$ and $G \cup F_2^c$ are G_δ-sets containing F_2 and F_1 respectively. By assumption, there exists τ-open sets U_1 and U_2 such that $F_1 \subset U_1 \subset G \cup F_2^c$ and $F_2 \subset U_2 \subset G \cup F_1^c$. Then $F_2 \cap U_1^c$ and $F_1 \cap U_2^c$ are two disjoint closed sets. By δ-normality, there exists two disjoint G_δ-sets $G_2 \supset F_2 \cap U_1^c$ and $G_1 \supset F_1 \cap U_2^c$. Notice that $G_i \supset F_i \cap G^c$ so that F_i is included in the G_δ-set $G_i \cup G$ for $i = 1, 2$. Therefore, there exists open sets O_i such that $F_i \subset O_i \subset G_i \cup G$. Hence $F_1 \cap F_2 \subset O_1 \cap O_2 \subset (G_1 \cup G) \cap (G_2 \cup G) \subset G$. □

As observed above, the closure $\mathbb{F}^*_{\wedge\omega}$ is $\beta^\perp$. Since β is compact and Hausdorff, hence normal, Lemma 5 gives another proof of the topological nature of $\mathbb{F}^*_{\wedge\omega}$.

Theorem 6. *Let $\mathbb{D}$ and $\mathbb{M}$ be two classes of filters and let $\mathscr{F}$ be a filter.*

$$\mathscr{F} \in (\mathbb{D}/\mathbb{M})_{\#\geqslant} \Longleftrightarrow \beta\mathscr{F} \subset \mathrm{cl}_{\mathbb{D}^*}\left(\mathrm{int}_{\mathbb{M}^*}\left(\beta\mathscr{F}\right)\right).$$

P r o o f. Assume that $\mathscr{F} \in (\mathbb{D}/\mathbb{M})_{\#\geqslant}$ and that $\mathscr{U} \in \beta\mathscr{F}$. Let $\beta\mathscr{D}$ be a $\mathbb{D}^*$-neighborhood of $\{\mathscr{U}\}$,where $\mathscr{D} \in \mathbb{D}$ and $\mathscr{U} \in \beta\mathscr{D}$. Then $\mathscr{D} \# \mathscr{F}$. Therefore, there exists an $\mathbb{M}$-filter $\mathscr{M}$ that meshes with $\mathscr{D}$ and is finer than $\mathscr{F}$. Thus $\beta\mathscr{D} \cap \beta\mathscr{M} \neq \emptyset$ and $\beta\mathscr{M} \subset \mathrm{int}_{\mathbb{M}^*}\left(\beta\mathscr{F}\right)$. Hence, $\mathscr{N}_{\mathbb{D}^*}\left(\mathscr{U}\right) \# \mathrm{int}_{\mathbb{M}^*}\left(\beta\mathscr{F}\right)$. In view of Lemma 1, $\mathscr{U} \in \mathrm{cl}_{\mathbb{D}^*}\left(\mathrm{int}_{\mathbb{M}^*}\left(\beta\mathscr{F}\right)\right)$.

Conversely, if $\beta\mathscr{F} \subset \mathrm{cl}_{\mathbb{D}^*}\left(\mathrm{int}_{\mathbb{M}^*}\left(\beta\mathscr{F}\right)\right)$ and if $\mathscr{D}$ is a $\mathbb{D}$-filter meshing with $\mathscr{F}$, then there exists $\mathscr{U} \in \beta\mathscr{F} \cap \beta\mathscr{D} \subset \mathrm{cl}_{\mathbb{D}^*}\left(\mathrm{int}_{\mathbb{M}^*}\left(\beta\mathscr{F}\right)\right)$. Therefore, $\mathscr{N}_{\mathbb{D}^*}\left(\mathscr{U}\right) \# \mathrm{int}_{\mathbb{M}^*}\left(\beta\mathscr{F}\right)$. In particular, $\beta\mathscr{D} \in \mathscr{N}_{\mathbb{D}^*}\left(\mathscr{U}\right)$ so that $\beta\mathscr{D} \cap \mathrm{int}_{\mathbb{M}^*}\left(\beta\mathscr{F}\right) \neq \emptyset$. In view of (3), there exists an $\mathbb{M}$-filter $\mathscr{M} \geqslant \mathscr{F}$ such that $\beta\mathscr{D} \cap \beta\mathscr{M} \neq \emptyset$, that is, such that $\mathscr{D} \# \mathscr{M}$. Thus $\mathscr{F} \in (\mathbb{D}/\mathbb{M})_{\#\geqslant}$. □

In view of Lemma 3, we can assume the filter $\mathscr{F}$ to be free in the statement above (hence $\beta\mathscr{F} = \beta_0\mathscr{F}$), provided that $\mathbb{D}$ and $\mathbb{M}$ are both $\mathbb{F}_1$-steady and that $\mathbb{F}_1 \subset \mathbb{M}$, which is statisfied by any classes $\mathbb{D}$ and $\mathbb{M}$ we consider in this paper. Therefore, we assume in the sequel that $\mathscr{F}$ is a *free* filter. Theorem 6 gives in particular:

$\mathscr{F}$ is	iff
bisequential	$\beta\mathscr{F} \subset \mathrm{int}_{G_\delta\beta}\left(\beta\mathscr{F}\right)$,i.e., $\beta\mathscr{F}$ is G_δ-open
weakly bisequential	$\beta\mathscr{F} \subset \mathrm{cl}_{\mathbb{F}^*_{\wedge\omega}}\left(\mathrm{int}_{G_\delta\beta}\left(\beta\mathscr{F}\right)\right)$
productively Fréchet	$\beta\mathscr{F} \subset \mathrm{cl}_{(\mathbb{F}_\omega/\mathbb{F}_\omega)^*_{\#\geqslant}}\left(\mathrm{int}_{G_\delta\beta}\left(\beta\mathscr{F}\right)\right)$
strongly Fréchet	$\beta\mathscr{F} \subset \mathrm{cl}_{G_\delta\beta}\left(\mathrm{int}_{G_\delta\beta}\left(\beta\mathscr{F}\right)\right)$
Fréchet	$\beta\mathscr{F} \subset \mathrm{cl}_{\beta}\left(\mathrm{int}_{G_\delta\beta}\left(\beta\mathscr{F}\right)\right)$

Notice that the class $(\mathbb{F}_\omega/\mathbb{F}_\omega)_{\#\geqslant}$ of strongly Fréchet filters is *not* stable under suprema, as shown for instance by an example of Isbell, first presented in [13] and detailed in [2, Example 3.2]. In view of Proposition 4, $(\beta X, (\mathbb{F}_\omega/\mathbb{F}_\omega)^*_{\#\geqslant})$ is *not* topological, and therefore the representation of productively Fréchet filters in βX is not of topological nature.

In [2], Bella and Malyhin obtained similar characterizations of Fréchet and strongly Fréchet filters (though in a somewhat different language). More specifically, their results can be rephrased as follows:

$\mathscr{F}$ is	iff
strongly Fréchet	$\beta\mathscr{F} \subset \mathrm{cl}_{G_\delta\beta}(\mathrm{int}_\beta(\beta\mathscr{F}))$
Fréchet	$\beta\mathscr{F} \subset \mathrm{cl}_\beta(\mathrm{int}_\beta(\beta\mathscr{F}))$

To clarify the relationships between our results and those of [2], we describe more explicitely the role of sequences.

Since $\beta_0\left((x_n)_{n\in\mathbb{N}}\right) = \beta_0(\{x_n\colon n\in\mathbb{N}\})$ and since any sequence finer than a free filter is free, we have

$$\bigcup_{\mathscr{F}\leqslant(x_n)_{n\in\mathbb{N}}} \beta_0\left((x_n)_{n\in\mathbb{N}}\right) \subset \mathrm{int}_\beta\beta\mathscr{F} \subset \mathrm{int}_{G_\delta\beta}\,\beta\mathscr{F}, \tag{4}$$

for any filter $\mathscr{F}$ on X. The simple observation that the first inclusion in (4) can be reversed if the underlying set X is countable is essentially due to Malykhin [11].

Moreover, denoting by $\mathscr{E}(\mathscr{F})$ the set of (automatically free) sequences finer than $\mathscr{F}$, we have:

Proposition 7. *The following are equivalent:*

1. $\mathscr{E}(\mathscr{F}) \neq \emptyset$;
2. $\bigcup_{\mathscr{F}\leqslant(x_n)_{n\in\mathbb{N}}} \beta\left((x_n)_{n\in\mathbb{N}}\right) \neq \emptyset$;
3. $\mathrm{int}_\beta\,\beta\mathscr{F} \neq \emptyset$;
4. $\mathrm{int}_{G_\delta\beta}\,\beta\mathscr{F} \neq \emptyset$.

P r o o f. $(1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4)$ are obvious.

$(4 \Longrightarrow 1)$. If $\mathrm{int}_{G_\delta\beta}\,\beta\mathscr{F} \neq \emptyset$ then there exists $\mathscr{H} \in \mathbb{F}_\omega$ such that $\mathscr{F} \leqslant \mathscr{H}$. Therefore $\mathscr{E}(\mathscr{F}) \neq \emptyset$ because $\mathscr{E}(\mathscr{H}) \neq \emptyset$. □

However, none of the inclusions of (4) can be reversed in general.

Example 8 ($\mathscr{U} \in \mathrm{int}_{G_\delta\beta}\,\beta\mathscr{F} \setminus \mathrm{int}_\beta\,\beta\mathscr{F}$)**.** Let $\mathscr{F}$ be a non almost principal[5] but countably based filter. If the cofinite filter $\mathscr{C}_A$ of some infinite subset A of X is

[5] A filter $\mathscr{F}$ is called *almost principal* [9] if there exists $F_0 \in \mathscr{F}$ such that $|F_0 \setminus F| < \omega$ for every $F \in \mathscr{F}$. Principal and cofinite filters are almost principal.

finer than $\mathscr{F}$, then $A^c \# \mathscr{F}$ because $\mathscr{F}$ is not almost principal. Moreover, there exists an ultrafilter $\mathscr{U}_0$ of $\mathscr{F}$ such that $A^c \in \mathscr{U}$ whenever $\mathscr{F} \leqslant \mathscr{C}_A$. Otherwise, for every $\mathscr{U} \in \beta\mathscr{F}$ there exists $A_\mathscr{U} \in \mathscr{U}$ such that $\mathscr{F} \leqslant \mathscr{C}_{A_\mathscr{U}}$. Then we would have finitely many $A_\mathscr{U}$'s whose union belongs to $\mathscr{F}$. But $\mathscr{F}$ would be cofinite on that set and therefore would be almost principal. By construction $\mathscr{U}_0 \notin \operatorname{int}_\beta \beta\mathscr{F}$, but $\mathscr{U}_0 \in \beta\mathscr{F} = \operatorname{int}_{G_\delta\beta} \beta\mathscr{F}$.

Example 9 **($\mathscr{U} \in \operatorname{int}_\beta \beta\mathscr{F} \setminus \bigcup_{\mathscr{F} \leqslant (x_n)_{n\in\mathbb{N}}} \beta((x_n)_{n\in\mathbb{N}})$).** Let X be an uncountable set and let $\mathscr{F}$ be the cofinite filter on X. A free uniform ultrafilter $\mathscr{U}$ on X (i.e., every element has the cardinality of X) is not finer than any free sequence. Therefore $\mathscr{U} \in \beta_0 X = \beta\mathscr{F}$ but $\mathscr{U} \notin \bigcup_{\mathscr{F} \leqslant (x_n)_{n\in\mathbb{N}}} \beta\left((x_n)_{n\in\mathbb{N}}\right)$.

Proposition 10. *If $\mathbb{D} \subset (\mathbb{F}_\omega/\mathbb{F}_\omega)_{\#\geqslant}$ and $\mathscr{F}$ is a free filter, then*

$$\operatorname{cl}_{\mathbb{D}^*}\left(\bigcup_{\mathscr{F} \leqslant (x_n)_{n\in\mathbb{N}}} \beta\left((x_n)_{n\in\mathbb{N}}\right)\right) = \operatorname{cl}_{\mathbb{D}^*}\left(\operatorname{int}_\beta \beta\mathscr{F}\right) = \operatorname{cl}_{\mathbb{D}^*}\left(\operatorname{int}_{G_\delta\beta} \beta\mathscr{F}\right).$$

Proof. $\operatorname{cl}_{\mathbb{D}^*}\left(\bigcup_{\mathscr{F} \leqslant (x_n)_{n\in\mathbb{N}}} \beta((x_n)_{n\in\mathbb{N}})\right) \subset \operatorname{cl}_{\mathbb{D}^*}\left(\operatorname{int}_{G_\delta\beta} \beta\mathscr{F}\right)$ is always true, regardless of $\mathbb{D}$. Now, if $\mathscr{U} \in \operatorname{cl}_{\mathbb{D}^*}\left(\operatorname{int}_{G_\delta\beta} \beta\mathscr{F}\right)$ then $\mathscr{N}_{\mathbb{D}^*}(\mathscr{U}) \# \operatorname{int}_{G_\delta\beta} \beta\mathscr{F}$. In other words, $\beta\mathscr{D} \cap \operatorname{int}_{G_\delta\beta} \beta\mathscr{F} \neq \emptyset$ for every $\mathscr{D} \in \mathbb{D}$ such that $\mathscr{D} \leqslant \mathscr{U}$. Hence, there exists a countably based filter $\mathscr{L} \geqslant \mathscr{F}$ such that $\beta\mathscr{L} \cap \beta\mathscr{D} \neq \emptyset$, that is, $\mathscr{L} \# \mathscr{D}$. Since $\mathscr{D} \in (\mathbb{F}_\omega/\mathbb{F}_\omega)_{\#\geqslant}$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \geqslant \mathscr{L} \vee \mathscr{D}$. Hence $(x_n)_{n\in\mathbb{N}} \geqslant \mathscr{F}$, $\beta(x_n)_{n\in\mathbb{N}} \cap \beta\mathscr{D} \neq \emptyset$ and $\mathscr{D} \leqslant \mathscr{U}$. Therefore, $\mathscr{U} \in \operatorname{cl}_{\mathbb{D}^*}\left(\bigcup_{\mathscr{F} \leqslant (x_n)_{n\in\mathbb{N}}} \beta((x_n)_{n\in\mathbb{N}})\right)$. □

Therefore, we can refine some of our results, as well as those of [2] described above:

$\mathscr{F}$ is	iff
productively Fréchet	$\beta\mathscr{F} \subset \operatorname{cl}_{(\mathbb{F}_\omega/\mathbb{F}_\omega)^*_{\#\geqslant}}\left(\bigcup_{\mathscr{F} \leqslant (x_n)_{n\in\mathbb{N}}} \beta((x_n)_{n\in\mathbb{N}})\right)$
strongly Fréchet	$\beta\mathscr{F} \subset \operatorname{cl}_{G_\delta\beta}\left(\bigcup_{\mathscr{F} \leqslant (x_n)_{n\in\mathbb{N}}} \beta\left((x_n)_{n\in\mathbb{N}}\right)\right)$
Fréchet	$\beta\mathscr{F} \subset \operatorname{cl}_\beta\left(\bigcup_{\mathscr{F} \leqslant (x_n)_{n\in\mathbb{N}}} \beta\left((x_n)_{n\in\mathbb{N}}\right)\right)$

However, if $\mathbb{D} \nsubseteq (\mathbb{F}_\omega/\mathbb{F}_\omega)_{\#\geqslant}$, we may have $\beta\mathscr{F} \subset \operatorname{cl}_{\mathbb{D}^*}\left(\operatorname{int}_{G_\delta\beta}(\beta\mathscr{F})\right)$ but $\beta\mathscr{F} \nsubseteq \operatorname{cl}_{\mathbb{D}^*}\left(\operatorname{int}_\beta(\beta\mathscr{F})\right)$. For instance, Example 8 gives such a situation for the class $\mathbb{D} = \mathbb{F}$ of all filters, in which case $\mathbb{D}^*$ is the discrete topology. We can more generally characterize classes of filters $\mathbb{D}$ for which interiors of $\beta\mathscr{F}$ for two different closure structures $\mathbb{M}^*$

and $\mathbb{N}^*$ have the same $\mathbb{D}^*$-closures. If $\mathbb{M}$ and $\mathbb{N}$ are two classes of filters, we call a filter $\mathscr{F}$ $\mathbb{M}$ to $\mathbb{N}$ *refinable-meshable*—in symbols $\mathscr{F} \in (\mathbb{M}/\mathbb{N})_{\geqslant\#}$ if

$$\mathscr{M} \in \mathbb{M},\ \mathscr{M} \# \mathscr{F} \Longrightarrow \exists \mathscr{N} \in \mathbb{N}\colon\ \mathscr{N} \geqslant \mathscr{M} \text{ and } \mathscr{N} \# \mathscr{F}.$$

Theorem 11. *Let $\mathbb{M}$ and $\mathbb{N}$ and $\mathbb{D}$ be three classes of filters. The following are equivalent:*

1. $\mathbb{D} \subset (\mathbb{M}/\mathbb{N})_{\geqslant\#}$;
2. $\mathrm{cl}_{\mathbb{D}^*}(\mathrm{int}_{\mathbb{M}^*}\beta\mathscr{F}) \subset \mathrm{cl}_{\mathbb{D}^*}(\mathrm{int}_{\mathbb{N}^*}\beta\mathscr{F})$ *for every filter* $\mathscr{F}$;
3. $\beta\mathscr{M} \subset \mathrm{cl}_{\mathbb{D}^*}(\mathrm{int}_{\mathbb{N}^*}(\beta\mathscr{M}))$ *for every* $\mathbb{M}$*-filter* $\mathscr{M}$;
4. $\mathbb{M} \subset (\mathbb{D}/\mathbb{N})_{\#\geqslant}$.

Proof. $(1 \Longrightarrow 2)$. Assume that $\mathbb{D} \subset (\mathbb{M}/\mathbb{N})_{\geqslant\#}$ and let $\mathscr{U} \in \mathrm{cl}_{\mathbb{D}^*}(\mathrm{int}_{\mathbb{M}}\beta\mathscr{F})$. In other words, $\mathscr{N}_{\mathbb{D}^*}(\mathscr{U}) \# \mathrm{int}_{\mathbb{M}^*}\beta\mathscr{F}$. Let $\mathscr{D} \in \mathbb{D}$ such that $\mathscr{U} \in \beta\mathscr{D}$. Since $\beta\mathscr{D} \# \mathrm{int}_{\mathbb{M}^*}\beta\mathscr{F}$, there exists an $\mathbb{M}$-filter $\mathscr{M} \geqslant \mathscr{F}$ such that $\beta\mathscr{D} \cap \beta\mathscr{M} \neq \emptyset$, that is, $\mathscr{D} \# \mathscr{M}$. Therefore, there exists an $\mathbb{N}$-filter $\mathscr{N}$ such that $\mathscr{N} \geqslant \mathscr{M}$ and $\mathscr{N} \# \mathscr{D}$ because $\mathscr{D} \in (\mathbb{M}/\mathbb{N})_{\geqslant\#}$. Hence, there exists an $\mathbb{N}$-filter $\mathscr{N}$ such that $\mathscr{N} \geqslant \mathscr{F}$ and $\beta\mathscr{D} \cap \beta\mathscr{N} \neq \emptyset$. In other words, $\mathscr{N}_{\mathbb{D}^*}(\mathscr{U}) \# \mathrm{int}_{\mathbb{N}^*}\beta\mathscr{F}$ so that $\mathscr{U} \in \mathrm{cl}_{\mathbb{D}^*}(\mathrm{int}_{\mathbb{N}^*}\beta\mathscr{F})$.

$(2 \Longrightarrow 3)$ is obvious and $(3 \Longrightarrow 4)$ follows from Theorem 6. Finally, $(4 \Longrightarrow 1)$ follows immediately from the definitions of the classes of filters considered. □

In particular, if there exists on X a countably based filter $\mathscr{M}$ not in $(\mathbb{D}/\mathbb{A})_{\#\geqslant}$, where $\mathbb{A}$ denotes the class of almost principal filters, then $\beta\mathscr{M} = \mathrm{int}_{G_\delta\beta}(\beta\mathscr{M})$ but $\beta\mathscr{M} \nsubseteq \mathrm{cl}_{\mathbb{D}^*}(\mathrm{int}_\beta(\beta\mathscr{M}))$. Notice that this is the case for any free countably based filter $\mathscr{M}$ meshing with a $\mathbb{D}$-filter $\mathscr{D}$ such that $\mathscr{E}(\mathscr{D}) = \emptyset$, for instance a uniform filter (if X is uncountable), or an ultrafilter. In particular, the cocountable filter on an uncountable set X is a countably deep and uniform filter, meshing with any other free uniform filter. Therefore, on an uncountable set X, there exist countably based filters $\mathscr{M}$ such that $\beta\mathscr{M} \nsubseteq \mathrm{cl}_{\mathbb{F}^*_{\wedge\omega}}(\mathrm{int}_\beta(\beta\mathscr{M}))$.

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