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A TOPOLOGICAL SPACE IS STRONGLY PARACOMPACT IF AND ONLY IF FOR ANY MONOTONE INCREASING OPEN COVER OF IT THERE EXISTS A STAR-FINITE OPEN REFINEMENT

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Abstract. We get the following result. A topological space is strongly paracompact if and only if for any monotone increasing open cover of it there exists a star-finite open refinement. We positively answer a question of the strongly paracompact property.

Keywords: paracompact property, strongly paracompact property

MSC 2000: 54C35, 54E32

In this paper we assume all spaces are T_2.

$|u|$ is the cardinal number of a set u.

A set u which consists of some sunsets of a topological space X is locally finite if for any $x \in X$ there exists an open subset U of X, such that $x \in U$ and $|\{V;\ V \cap U \neq \varphi, V \in u\}|$ is a finite cardinal number.

A set u which consists of subsets of a topological space X is star-finite if for any $U \in u$, $|\{V;\ V \cap U \neq \varphi, V \in u\}|$ is a finite cardinal number.

A set u which consists of subsets of a topological space X is star $< k$ (k is a cardinal number) if for any $U \in u$, $|\{V;\ V \cap U \neq \varphi, V \in u\}| < k$.

A topological space is paracompact if for any open cover of it there exists a locally finite open refinement. A topological space is strongly paracompact if for any open cover of it there exists a star-finite open refinement.

On the paracompactness of a topological space there is the following theorem.

Theorem 1. *A topological space is paracompact if and only if for any monotone increasing open cover of it there exists a locally finite open refinement.*

Comparing the strong paracompactness with the paracompactness, there is the following question.

Question 1. If for any monotone increasing open cover of a topological space there exists a star-finite open refinement, is it strongly paracompact?[1]

In the nineties of the last century, the following question on strong paracompactness was posed.

Question 2. If for any infinite open cover u of a topological space X there exists a star $< |u|$ open refinement, is it strongly paracompact?

Theorem 2. *If X is a topological space, the following conditions are equivalent.*

(1) *For any infinite open cover u of X there exists a* star $< |u|$ *open refinement.*

(2) *For any infinite monotone increasing open cover u of X there exists a* star $< |u|$ *open refinement.*

(3) *For any infinite monotone increasing open cover u of X there exists a star-finite open refinement.*

Proof. (1) $\Longrightarrow$ (2). This is obvious.

(2) $\Longrightarrow$ (3). Let u be any infinite monotone increasing open cover of X. We make the proof by transfinite induction with respect to $|u| = k$.

(a) If $k = \omega$, a star $< |u|$ open refinement of u is a star-finite open refinement of u.

(b) If $k > \omega$, we assume that for any infinite monotone increasing open cover of u, if $|u| < k$, there exists a star-finite open refinement.

$|u| = k$. If $\mathrm{cof}(k) < k$, then there is a subcover u_1 of u such that $|u_1| = \mathrm{cof}(k)$. According to the assumption of the transfinite induction, there exists a star-finite open refinement of u_1. It is a star-finite open refinement of u.

If $\mathrm{cof}(k) = k$, there exists a star $< |u|$ open refinement v of u. In elements of v we introduce the equivalence relation as follows.

$V \approx V' \Longleftrightarrow$ there exists a finite subset $\{V_i;\ 1 \leqslant i \leqslant n\}$ of v, such that $V \cap V_1 \neq \varphi$, $V_n \cap V' \neq \varphi$, $1 \leqslant i \leqslant n-1$, $V_i \cap V_{i+1} \neq \varphi$. Let $v = \bigcup\{v_\alpha;\ \alpha < k'\}$, with v_α the equivalence classes of v.

For any $\alpha < k', W_\alpha = \bigcup\{V;\ V \in v_\alpha\}$. W_α is closed and open. Because $W_\alpha = X - \bigcup\{W_\beta;\ \beta \neq \alpha, \beta < k'\}$. $|v_\alpha| < k$. if $|v_\alpha| = k, \mathrm{cof}(k) = \omega < k$. This is a contradiction with $\mathrm{cof}(k) = k$. That is, $k = k'$.

For any $\alpha < k$, $v_\alpha = \{V_{\alpha,\beta};\ \beta < k_\alpha\}$, $k_\alpha < k$. $\beta = 0$, $V'_{\alpha,0} = V_{\alpha,0} \cup (X - W_\alpha)$; $0 < \beta < k_\alpha$, $V'_{\alpha,\beta} = (\bigcup\{V_{\alpha,\gamma};\ \gamma < \beta\}) \cup (X - W_\alpha)$.

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$v_{\alpha'} = \{V'_{\alpha,\beta};\ \beta < k_\alpha\}$ is a monotone increasing open cover of X and $|v_\alpha| < k$. According to the assumption of the transfinite induction there exists a star-finite open refinement w'_α of v_α. $w_\alpha = \{W' \cap W_\alpha;\ W' \in w'_\alpha\}$ is a star-finite opencover of W_α and a refinement of v_α. $w = \bigcup\{w_\alpha;\ \alpha < k\}$ is a star-finite open refinement of u.

(3) $\Longrightarrow$ (1) $u = \{U_\alpha;\ \alpha < k\}$ is an infinite open cover of X such that $|u| = k$. $V_0 = U_0$; $0 < \alpha < k$, $V_\alpha = \bigcup\{U_\beta;\ \beta < \alpha\}$. $v = \{V_\alpha;\ \alpha < k\}$ is a monotone increasing infinite open cover of X. Then there exists a star-finite open refinement $w = \{W_\beta;\ \beta < k'\}$ of v such that there exists a function $f\colon\ k' \to k$ such that for any $\beta < k', W_\beta \subset V_{f(\beta)}$.$\{U_\alpha \cap W_\beta;\ \alpha < f(\beta), \beta < k'\}$ is a star $< |u| = k$ open refinement of u. For any $U_\alpha \cap W_\beta$, because w is star-finite, $\{W;\ W \cap W_\beta \neq \varphi, W \in w\} = \{W_{\beta_1}, \ldots, W_{\beta_n}\}$. Then $|\bigcup\{\{U_\alpha \cap W_{\beta_i};\ \alpha < f(\beta_i)\}, i \leqslant n\}| \leqslant |f(\beta_1)| + \ldots + |f(\beta_n)| = \max\{|f(\beta_i)\}|\ ;\ i \leqslant n\} < k$. That is, it is star $< |u| = k$.

According to Theorem 1 and Theorem 2 the following result is obtained.

Lemma 1. *If for any infinite open cover u of a topological space there exists a* star $< |u|$ *open refinement, it is paracompact.*

On the star-finite property, we have the following results.

Lemma 2. *Let u, v consist of some subsets of a topological space X. If u, v are star-finite, then*

(1) $u \wedge v = \{U \cap V;\ U \in u, V \in v\}$ *is star-finite;*

(2) $\{\bigcap \Phi;\ \Phi \subset u, |\Phi| < \omega\}$ *is star-finite;*

(3) $\{\mathrm{star}\{x, u\} = \bigcup\{U;\ x \in U, U \in u\};\ x \in X\}$ *is star-finite;*

(4) $\{\mathrm{star}\{U, u\} = \bigcup\{V;\ V \cap U \neq \varphi, V \in u\};\ U \in u\}$ *is star-finite.*

Theorem 3. *If for any infinite monotone increasing open cover of a topological space there exists a star-finite open refinement, it is strongly paracompact.*

P r o o f. Let X be a topological space which satisfies the condition of Theorem 3. Let $u = \{U_\alpha;\ \alpha < k\}(|u| = k)$ be any infinite open cover of X. We make the proof by transfinite induction with respect to $k = |u|$.

If $k = \omega$, $\alpha = 0$, $V_0 = U_0$; $0 < \alpha < \omega$, $V_\alpha = \bigcup\{U_\beta;\ \beta < \alpha\}$. $v = \{V_\alpha;\ \alpha < \omega\}$ is a monotone increasing infinite open cover of X. There exists a star-finite open refinement $w = \{W_\beta;\ \beta < k'\}$ of v such that there exists a function $f\colon\ k' \to \omega$ such that for any $\beta < k'$, $W_\beta \subset V_{f(\beta)}$. $\{U_\alpha \cap W_\beta;\ \alpha < f(\beta), \beta < k'\}$ is a star-finite open refinement of u. For any $U_\alpha \cap W_\beta$, because w is star-finite, $\{W;\ W \cap W_\beta \neq \varphi, W \in w\} = \{W_{\beta_1}, \ldots, W_{\beta_n}\}$. Thus $|\bigcup\{\{U_\alpha \cap W_{\beta_i};\ \alpha < f(\beta_i)\}, i \leqslant n\}| \leqslant |f(\beta_1)| + \ldots + |f(\beta_n)| = \max\{|f(\beta_i)\}|\ ;\ i \leqslant n\} < \omega$. That is, it is star-finite.

If $k > \omega$, we suppose that for any infinite open cover of u, which satisfies the hypothesis of Theorem 3, if $|u| < k$, there exists a star-finite open refinement.

For any infinite open cover u of X, which satisfies the condition of Theorem 3, $|u| = k$, $\alpha = 0$, $V_0 = U_0$; if $0 < \alpha < k$, $V_\alpha = \bigcup\{U_\beta\,;\ \beta < \alpha\}$. $v = \{V_\alpha\,;\ \alpha < k\}$ is a monotone increasing infinite open cover of X. There exists a star-finite open refinement $w' = \{W'_\beta\,;\ \beta < k'\}$ of v such that there exists a function $f\colon\ k' \to k$ such that for any $\beta < k', W'_\beta \subset V_{f(\beta)}$. According to Lemma 1, X is paracompact. Then there is an open cover $w = \{W_\beta\,;\ \beta < k'\}$ of w' such that for any $\beta < k'$, $\overline{W_\beta} \subset W'_\beta$.

For any $\beta < k'$, $\{W'_\beta \cap U_\alpha\,;\ \alpha < f(\beta)\} \cup \{X - \overline{W_\beta}\}$ is an open cover of X and $|\{W'_\beta \cap U_\alpha\,;\ \alpha < f(\beta)\} \cup \{X - \overline{W_\beta}\}| = |f(\beta)| < k$. According to the hypothesis of the transfinite induction, there exists a star-finite open refinement w'_β of it. Set $w_\beta = \{W\,;\ W \cap \overline{W_\beta} \neq \varphi, W \in w'_\beta\}$.

For any $x \in \overline{W_\beta}, O_{\beta,x} = \bigcap\{W\,;\ x \in W, W \in w_\beta\}$. w_β is star-finite. $|\{W\,;\ x \in W, W \in w_\beta\}| < \omega$. $O_{\beta,x}$ is an open subset of X. Acording to Lemma 2, $\{O_{\beta,x}\,;\ x \in \overline{W_\beta}\}$ is star-finite.

$O_x = \bigcap\{O_{\beta,x}\,;\ x \in \overline{W_\beta}\}$. w is star-finite. $|\{O_{\beta,x}\,;\ x \in \overline{W_\beta}, \beta < k'\}| < \omega$. O_x is an open subset of X.

$G_x = X - \bigcup\{\overline{W_\beta}\,;\ x \notin \overline{W_\beta}, \beta < k'\}$. w' is star-finite. $\bigcup\{\overline{W_\beta}\,;\ x \notin \overline{W_\beta}, \beta < k'\}$ is closed. That is, G_x is open.

According to Lemma 2 $\{\mathrm{star}\{x, \{\overline{W_\beta}\,;\ \beta < k'\}\}\,;\ x \in X\}$ is star-finite. For any $x \in X$, $G_x \subset \mathrm{star}\{x, \{\overline{W_\beta}\,;\ \beta < k'\}\}$. So$\{G_x\,;\ x \in X\}$ is star-finite.

$\{G_x \cap O_x\,;\ x \in X\}$ is star-finite.

For any $x \in X$, if $y \notin \bigcup\{\overline{W_\beta}\,;\ x \in \overline{W_\beta}, \beta < k'\}, G_x \cap G_y = \varphi$. That is, $(G_x \cap O_x) \cap (G_y \cap O_y) = \varphi$.

Because $\{\overline{W_\beta}\,;\ \beta < k'\}$ is star-finite, $\left|\{\overline{W_\beta}\,;\ x \in \overline{W_\beta}, \beta < k'\}\right| < \omega$. That is, $\{\overline{W_\beta}\,;\ x \in \overline{W_\beta}, \beta < k'\} = \{\overline{W_{\beta_1}}, \ldots, \overline{W_{\beta_n}}\}$. For any $1 \leqslant i \leqslant n, \{O_{\beta_i,y}\,;\ y \in \overline{W_{\beta_i}}\}$ is finite. So for any $1 \leqslant i \leqslant n$, $\{O_{\beta_i,y}\,;\ O_{\beta_i,x} \cap O_{\beta_i,y} \neq \varphi\,;\ y \in \overline{W_{\beta_i}}\}$ is finite. $\{O_y\,;\ O_x \cap O_y \neq \varphi, y \in \overline{W_{\beta_1}} \cup \ldots \cup \overline{W_{\beta_n}}\}$ is finite. $\{G_y\,;\ G_x \cap G_y \neq \varphi, y \in \overline{W_{\beta_1}} \cup \ldots \cup \overline{W_{\beta_n}}\}$ is finite. $\{O_y \cap G_y\,;\ O_x \cap G_x \cap O_y \cap G_y \neq \varphi, y \in \overline{W_{\beta_1}} \cup \ldots \cup \overline{W_{\beta_n}}\}$ is finite.

That is, $\{G_x \cap O_x\,;\ x \in X\}$ is star-finite. It is a star-finite open refinement of u.

According to the principle of the transfinite induction, for any infinite open cover of u, there exists a star-finite open refinement.

That is, X is strongly paracompact.

According to Theorem 2, Theorem 3, we can get following.

Theorem 4. *If for any infinite open cover u of a topological space there exists a* star $< |u|$ *open refinement, it is strongly paracompact.*

Theorem 5. *If for any monotone increasing open cover of a topological space there exists a star-finite open refinement, it is strongly paracompact.*

We thus positively answer a question on the strong paracompactness of a topological space.

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