Jorge Martinez

Locally conditioned radical classes of lattice-ordered groups

Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 1, 25–34

Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

LOCALLY CONDITIONED RADICAL CLASSES OF LATTICE-ORDERED GROUPS

JORGE MARTINEZ, Gainesville

1. INTRODUCTION

As is the custom, we shall employ the abbreviation "ℓ-group" for lattice-ordered group. We think that the reader will attach the obvious significance to the terms "ℓ-homomorphism", "ℓ-subgroup", etc. For example, a subgroup C of the ℓ-group G is called a convex ℓ-subgroup if it is order convex as well as an ℓ-subgroup.

Our main references for background material on the theory of ℓ-groups will be [AF] and [BKW]. We shall also draw some additional material from [M], on split subgroups, which we shall presently define.

If G is an ℓ-group and H is an ℓ-subgroup, we say that H is a *split subgroup* provided that for each $g \in G$ we can write $g = g_1 + g_2$, where $g_1 \wedge g_2 = 0$, $g_1 \in H$ and every value of g_2 contains H. Some background material is now required: the convex ℓ-subgroup K is said to be a *value* of $a \in G$ if it is maximal with respect to not containing a. The values of an ℓ-group are precisely the meet-irreducible elements of the algebraic lattice $C(G)$ of convex ℓ-subgroups; this is well known, and may be found in the standard references quoted above, or else in [C1], where it originally appeared.

It is also well known that a value of an ℓ-group is prime; that is, if $a \wedge b = 0$ then either a or b belong to the subgroup in question. A normal convex ℓ-subgroup K is prime if and only if G/K is totaly ordered. (A normal convex ℓ-subgroup is customarily called an ℓ-ideal.)

Here are some of the basic results on split subgroups from [M4].

1.1 *Every cardinal summand of an ℓ-group is a split subgroups.* (Note: $A \in C(G)$ is a cardinal summand of G if $G = A + B$ and $A \cap B = 0$, for some $B \in C(G)$, where the ordering is taken coordinatewise. The notation for cardinal summation is $G = A \boxplus B$.)

1.2 *If $G = \text{lex}(A)$ then A is a split subgroup.* ($G = \text{lex}(A)$, for $A \in C(G)$, means that $G \neq A$, and each positive element outside A exceeds every element in A. Obviously, every subgroup of this kind is prime.)

1.3 *A split subgroup is necessarily convex.*

1.4 *$A \in C(G)$ is a split subgroup if and only if for each $g \in G$ we have $g = a + b$, disjointly, with $a \in A$ and*

$$(*) \qquad G(b) \cap A \subseteq N_b$$

(Note: For $x \in G$, $G(x)$ stands for the convex ℓ-subgroup generated by x. N_x stands for the intersection of all the values of x in $G(x)$. The condition $(*)$ is referred to in [M4] as "A lies below b"; we caution the reader that this termonology is somewhat at variance with that of [BCD]; they agree for normal-valued ℓ-groups. Recall that G is said to be normal-valued if every value of G if normal in its cover in $C(G)$.)

1.5 If G is archimedean then $A \in C(G)$ is a split subgroup precisely when it is a cardinal summand.

1.6 *If A is a split subgroup then the expression of each $g \in G$ as prescibed by 1.4 is unique.*

1.7 *Every split subgroup is order-closed.* (But the converse is false.)

Let us close this introduction with one of the main results from [M4], which points out our interest in local residues. First, a reminder: each $G(x)/N_x$ is called a local residue of G.

Theorem 1.8. *(2.8 of [M4]) Suppose that G is normal-valued. Then the convex ℓ-subgroup A of G is split if and only if, for each $x \geqslant 0$, $\left[\left(G(x) \cap A\right) + N_x\right]/N_x$ is a cardinal summand of $G(x)/N_x$.*

2. Locally conditioned radical classes

Let us begin by giving the definition of a radical class; it is not the standard one used by ring theorists, rather one more suited to the theory of ℓ-groups. All classes are assumed to be closed under the formation of ℓ-isomorphic images. U is a radical class if it is closed with respect to taking convex ℓ-subgroups, and whenever $\{C_i : i \in I\}$ is a family of convex ℓ-subgroups of G belonging to U then $C = \vee C_i$, the supremum in $C(G)$, also belongs to U. If, in addition, U is closed under forming ℓ-homomorphic images, we say that U is a torsion class. A quasi-torsion class is a radical class which is closed with respect to forming ℓ-homomorphic images under maps which preserve all suprema and infima. (Note: such maps are called complete ℓ-homomorphisms, and by 6.1.5 in [BKW], they are the ℓ-homomorphisms for which the kernels are order-closed.)

For a thorough treatment of torsion classes we refer the reader to [M2], and to [M1] for more particular information about the class of hyper-archimedean ℓ-groups. Conrad gives a substantial account of radical classes in [C4]. We proceed to give a list of the radical classes which are most natural in the theory, or else some of the classes which we shall need in the sequel.

Ar, the quasi-torsion class of all archimedean ℓ-groups. (See [Ke]) E, the torsion class of all hyper-archimedean ℓ-groups. (Recall: an ℓ-group G is hyper-archimedean if every ℓ-homomorphic image of G is archimedean. This class is extensively studied in [C3] and [M1].) N, the torsion class and variety of all normal-valued ℓ-groups.

(Note: by a theorem of Holland [H], every variety of ℓ-groups is a torsion class.)

Pr, the class of all projectable ℓ-groups. (Recall: G is projectable if every principal polar of G is a cardinal summand. We remind the reader that $P^{\perp} = \{g \in G\colon |g| \wedge |x| = 0, \text{ for all } x \in P\}$ is called the *polar* of $P \subseteq G$. Each polar is an (order-closed) convex ℓ-subgroup of G. The set $P(G)$ of all polars of G, under inclusion, is a boolean algebra; this was first observed by Šik [Š]. Each $x^{\perp\perp} = [G(x)]^{\perp\perp}$ is the principal polar generated by x.) Pr is a radical class, but not a quasi-torsion class; (see [A].)

Sp, the torsion class of all Specker groups. (Recall that an element $0 < s \in G$ is singular if $x \wedge (s - x) = 0$, for each $0 \leqslant x \leqslant s$. An ℓ-group is called a *Specker group* if it is generated—as a group—by its singular elements. In [C3] Conrad gives a thorough account of Specker groups.)

F, the torsion class of all finite-valued ℓ-groups. (As the name suggests, an ℓ-group G is finite-valued if each nonzero element has at most finitely many values. See section 6.4 of [BKW] for a treatment of finite-valued ℓ-groups; most of the results in that section are due to Conrad, and they appear in [C1].)

Sv, the quasi-torsion class of all spaecial-valued ℓ-groups. (A special element is one which has exactly one value. An ℓ-group G is special-valued if every positive element can be expressed as a disjoint supremum of special elements. It is easy to see that a finite-valued ℓ-group is special-valued. This class was first introduced by Conrad in [C2]. That Sv is indeed a quasi-torsion class is shown in [C4]; [BD] gives a more direct proof of this fact.)

Suppose now that U is any radical class. For any ℓ-group G, there is, by definition, a largest convex ℓ-subgroup $\mathsf{U}(G)$ of G belonging to U. $\mathsf{U}(G)$ is called the U-radical of G. As proved in [M2] for torsion classes, the U-radical satisfies the following properties: (a) $\mathsf{U}(C) = C \cap \mathsf{U}(G)$, for each $C \in C(G)$, and (b) $\mathsf{U}(K) = V\mathsf{U}(K_\alpha)$, whenever $K = VK_\alpha$ in $C(G)$. Moreover, a radical class can be defined by its "radical function", satisfying the two preceding contitions. We then recover U as the class of all ℓ-groups G for which $\mathsf{U}(G) = G$.

We need one additional definition from the literature—see [M2]; a radical class U is said to be *complete* if it is closed with respect to forming extensions; that is to say, whenever G is an ℓ-group and A is an ℓ-ideal so that both A and G/A belong to U, then $G \in \mathsf{U}$.

We shall call an ℓ-homomorphism a split homomorphism if its kernel is a split subgroup. The following observation is an immediate consequence of result in [M4], among them 1.7, quoted here.

Lemma 2.1. *Suppose that $\varphi: G \to H$ is an ℓ-homomorphism. It is a split homomorphism if and only if for each $g \in G$ there is an $x \in G$ so that $\varphi g = \varphi x$ and every value of x contains* Ker(φ). *Every split homomorphism is complete, but not conversely.*

We also record two more remarks, the proofs of which are straighforward.

2.2. If $\varphi: G \to H$ is a split homomorphism and $a, b > 0$ are such that each one of their values contain Ker(φ), then $\varphi a \wedge \varphi b = 0$ implies that $a \wedge b = 0$.

2.3. Suppose that $\varphi: G \to H$ is a split homomorphism onto H. Then the map $K \to \varphi^{-1}(K)$ describes an order-isomorphism between the lattices of split subgroups of H and that of the split subgrups of G which contain Ker(φ).

(Note: This result fails for complete ℓ-homomomrphisms; it is known that if φ is complete then $\varphi^{-1}(K)$ is order-closed whenever K is—it is not hard to prove directly—but the converse is false.)

Let us call a radical class U a split class if it closed with respect to forming split-homomorphic images. From what has already been said, every quasi-torsion class is a split class. Here is an example of a split class which is not quasi-torsion.

Proposition 2.4. *The class* Pr *of projectable ℓ-groups is a split class.*

(Note: We have already remarked that Pr is not a quasi-torsion class.)

Proof. We already know that Pr is a radical class. So all that is left to do is to prove that the class is closed under forming split-homomorphic images. Suppose that G is a projectable ℓ-group and that K is a split ℓ-ideal in G. We take a coset $K + x$, and assume—in view of Lemma 2.1—without loss of genarality, that every value of x contains K, with $x > 0$. If $0 \leqslant g \in G$ we may write $g = g_1 + g_2$, so that $0 \leqslant g_1 \in x^{\perp\perp}$ and $g_2 \wedge x = 0$. Now, $K + g = K + g_1 + K + g_2$, and once again we may assume that the values of all these elements contain K. Then it is clear that $K + g_2 \in (K + x)^{\perp}$. On the other hand, if each value of y contains K and $K + y$ is disjoint from $K + x$, then by 2.2, y and x are disjoint, which implies that $y \wedge g_1 = 0$, proving that $K + g_1 \in (K + x)^{\perp\perp}$, and that G/K is projectable. □

We come now to the principal construction of this article. Let us begin with an arbitrary class X of archimedean ℓ-groups. Let Loc(X) stand for the class of all ℓ-groups G for which every local residue $G(x)/N_x$ belongs to X. We shall say that a class U is *locally conditioned* if U = Loc(X) for a suitable X.

The next result gives an account of the basic closure properties of a locally conditoned class.

Proposition 2.5. *Suppose* U = Loc(X). *Then* U *is closed under forming:* (a) *convex ℓ-subgroups,* (b) *unions of chains of convex ℓ-subgroups belonging to* U *and*

(c) *images under split homomorphisms.*

If, in addition, X *satisfies:*

$$(\Sigma) \qquad G = A + B, \text{ with } A, B \in C(G) \text{ and } A, B \in \mathsf{X} \rightarrow G \in \mathsf{X},$$

Then U *is a split class.*

Proof. (a) and (b) are clear, since if $0 < x \in C \in C(G)$ then $G(x) \subseteq C$, by definition. Now suppose that K is a split ℓ-ideal of G. In examining local residues of G/K and cosets $K + x$, we assume without loss of generality that every value of x contains K. So pick such an $x > 0$ and let N/K be the intersection of all the maximal convex ℓ-subgroups of $[G(x) + K]/K$; these are the values of $K + x$ in $[G(x) + K]/K$. Since every value of x contains K, $G(x) \cap N_x$, while

$$[G(x) + K]/N = [G(x) + N]/N \cong G(x)/N_x,$$

and the latter belongs to X. This proves that (c) holds.

To prove that U is a split class, it suffices to show that (Σ) holds for U. So suppose that $G = A + B$, with $A, B \in C(G)$, and that both belong to X. (Note: since each local residue is archimedean G is normal-valued, and consequently $+$ is the supremum in $C(G)$!) Suppose that $0 \leqslant g \in G$, and write $g = a + b$ with $0 \leqslant a \in A$ and $0 \leqslant b \in B$. It should be clear that, without loss of generality, we may assume that $G = G(g)$, $A = G(a)$ and $B = G(b)$; denote $N = N_g$. Then observe that $N_a = N \cap A$ and that $N_b = N \cap B$. Now

$$G/N = [A + N]/N + [B + N]/N,$$

and $[A + N]/N \cong A/N_a$, while $[B + N]/N \cong B/N_b$; since both A/N_a and B/N_b belong to X and X satisfies (Σ), it follows that G/N is in X, hence that $G \in$ U. □

2.6. Table 2.6 now gives a number of examples of locally conditioned radical classes. Some have already been mentioned; others are not very widely known. The last entry in the table is new. The third column in the table provides a reference for comments on the class involved, either from the literature or else in the discussion which follows.

X	Loc(X)	Remarks
Ar	N	G is normal-valued if and only if each local residue of G is archimedean
E	Pw	See 2.6.1
O_R	F	See 2.6.2 and 6.4.1, [BKW]
Sp	Sd	See 2.6.3 and [CM2]
B	πSv	See Proposition 2.8

2.6.1. Pw stands for the class of all pairwise splitting ℓ-groups; these first appeared in [M3]. G is a pairwise splitting ℓ-group if for each $0 \leqslant x,\ y \in G$ we can write $y = y_1 + y_2$, disjointly, so that $y_1 \leqslant mx$, for some natural number m, and so that $x \wedge y_2 \ll y_2$. It should be easy for the reader to see that G is pairwise splitting if and only if each $G(x)$ is a split subgroup. From the discussion in [M3] on the various ways one can decompose an element by another in a pairwise splitting ℓ-group, it follows that in such an ℓ-group each N_x is split as well; (the converse is false, as in any archimedean ℓ-group each $N_x = 0$ and hence a split subgroup.)

Lemma 3 in [M3] shows that G is pairwise splitting if and only if each local residue is hyper-archimedean.

2.6.2. O_R stands for the class of finite cardinal sums of archimedean o-groups. It is an immediate consequence of Conrad's decomposition theorem for finite-valued ℓ-groups—6.4.1. in [BKW]—that G is finite-valued precisely when each local residue of G belongs to O_R.

2.6.3. We have already brought up Sp, the torsion class of Specker groups. Sd = Loc(Sp) was introduced in [CM2]; Theorem 15 of that paper gives several equivalent conditions for and ℓ-group to lie in Sd. Recall that $0 < d \in G$ is discrete if for each value V of d, the coset $V+d$ covers V in the chain G/V. One of the conditions in the theorem allued to here is this: each $0 < g \in G$ can be written in the following way: $g = n_1 d_1 + w_1 + \ldots + n_k d_k + w_k$, where the d_i are discrete and pairwise disjoint, each n_i is a natural number and $w_i \ll d_i$, for each $i = 1, \ldots, k$. We shall see presently that the class Sv of special-valued ℓ-groups is not locally conditioned. Now we introduce a slightly bigger class which is. We call a normal-valued ℓ-group G pseudo-special-valued if each $g > 0$ has a special value. Clearly then, every special-valued ℓ-group is pseudo-special-valued.

Proposition 2.7. *An ℓ-group G is pseudo-special-valued if and only if every positive element of G has a special component.*

(Note: recall that a is a component of $b > 0$ if $b = a + c$, with a and c disjoint.)

Proof. The necessity should be clear. As for the sufficiency, suppose that $g > 0$ has a special value V; then take $s > 0$, having its only value at V. Since V is normal in its cover V^*, there is a natural number n such that $V + ns \geqslant V + g$. Then it is easy to verify that $ns \wedge g$ is special, and is a component of g. □

For the moment let us emphasize, πSv is a split class, and that Sv $\neq \pi$Sv; in the next section we shall give an example of a pseudo-special-valued ℓ-group which is not special-valued.

It should be obvious that if G is pseudo-special-valued then so is each local residue. Thus, each $G(x)/N_x$ is an archimedean ℓ-group in which every positive element has a basic component; ie. each local residue has a basis. (For a reference see 7.3.4 of

[BKW], and recall that $0 < b \in G$ is said to be basic if the interval $[0, b]$ is a chain. A basis is a maximal pairwise disjoint set of elements which happen to be basic. Note that every basic element is special, and that the converse is true in archimedean ℓ-groups.) In Table 2.6 B denotes the class of archimedean ℓ-groups with a basis. What we have shown in this paragraph is that $\pi\mathsf{Sv} \subseteq \mathrm{Loc}(\mathsf{B})$.

Conversely, suppose that each local residue of G is archimedean and has a basis. If $0 < g \in G$, then $N_g + g$ has a basic component $N_g + b$ in $G(g)/N_g$. It is well known that an archimedean ℓ-group with basis is special-valued. Now, without loss of generality, we can arrange things so that b is a component of g; then it should be clear that b is also special. All of which serves to prove:

Proposition 2.8. *$G \in \pi\mathsf{Sv}$ if and only if every local residue of G has a basis.* (Or: $\pi\mathsf{Sv} = \mathrm{Loc}(\mathsf{B})$.)

The reader mat have noticed that in Table 2.6, all the examples save the last one are actually torsion classes. $\pi\mathsf{Sv}$ is not, as we shall see in the next section. The following result spells out when we can expect a locally conditioned class to be a torsion class. The proof is straightforward, and is left to the reader.

Propositon 2.9. *Suppose that X satisfies (Σ) from Proposition 2.5 plus the following: X is closed under taking ℓ-homomorphic images which are archimedean. Then $\mathrm{Loc}(\mathsf{X})$ is a torsion class.*

Incidentally, Proposition 2.9 shows that Sd, from 2.6.3, is a torsion class; this had been claimed in [CM2] without proof. If S stands for the (archimedean) ℓ-groups which are representable as groups of real-valued functions with finite range, then Proposition 2.9 also applies; for nilpotent ℓ-groups Loc(S) was discussed in [CM1], although not using the terminology of this article.

Except for N, none of the classes in Table 2.6 is complete. Our next theorem, which we consider the central result of this paper, says that the locally conditioned classes obtained from archimedean classes satisfying (Σ) of Proposition 2.5 do have some measure of completeness. Let us say that U is split-complete if for each ℓ-group G having a split ℓ-ideal K so that both K and G/K belong to U it follows that $G \in \mathsf{U}$.

Theorem 2.10. *If X is a class of archimedean ℓ-groups which is closed under forming finite cardinal sums, then $\mathsf{U} = \mathrm{Loc}(\mathsf{X})$ is split complete.*

Proof. Suppose that K is a split ℓ-ideal of G such that K and G/K belong to U. Write $0 < g \in G$ as $g = x + y$, disjointly, with x in K, and so that every value of y contains K. On the one hand we have:

$$G(g)/N_g \cong [G(x) + N_g]/N_g \boxplus [G(y) + N_g]/N_g,$$

and $N_x = N_g \cap G(x)$, with $N_y = N_g \cap G(y)$. Now, $[G(x) + N_g]/N_g \cong G(x)/N_x$ and the latter belongs to X because $x \in K \in \mathsf{U}$.

As for the other component in the above cardinal sum, since every value of y contains K, $K \cap G(y) \subseteq N_y$, which implies that $G(y)/N_y \cong [G(y) + K]/[N_y + K]$, and the latter lies in X since G/K is in U. □

The final result in this section addresses the question of how to describe the largest class of archimedean ℓ-groups which conditions a given locally conditioned class. The proof is straightforward and will be omitted.

Proposition 2.11. *Suppose that* $\mathsf{U} = \mathrm{Loc}(\mathsf{X})$. *Then*
(a) If G is archimedean then $G \in \mathsf{U}$ precisely when each $G(x) \in \mathsf{X}$.
(b) Suppose that V is any quasi-torsion class for which

$$G \in \mathsf{V} \cap \mathsf{Ar} \text{ if and only if each } G(x) \in \mathsf{X},$$

then $\mathsf{V} \subseteq \mathsf{U}$.
(c) If U° denotes the class of archimedean ℓ-groups in U with strong order unit, then $\mathrm{Loc}(\mathsf{U}^\circ) \subseteq \mathsf{U}$. If U is a quasi-torsion class, then equality holds, and U° is the largest class of archimedean ℓ-groups with strong order unit which conditions U.

3. Closing comments and examples

3.1. Sv versus πSv. Consider the following example, which appears in [M4] and is named EC-Z. It is the group of all pairs (s_n, k) in which (s_n) is an eventually constant sequence of integers and k is an integer. Set $(s_n, k) > 0$ if $s_n \geqslant 0$, for each n, and the sequence (s_n) is eventually strictly positive, or (s_n) is finitely nonzero with $k > 0$. It is shown in [M4] that $EC - Z$ is the extension of one special-valued ℓ-group by another with a split kernel. This means that Sv is not a split-complete class. (And then Theorem 2.10 says that Sv is not locally conditioned by any class of archimedean ℓ-groups which is not closed under forming finite cardinal sums, but we can do better.)

It is easy to see that $EC - Z$ is pseudo-special-valued but not special-valued. Also, as we have already observed, Sv is a quasi-torsion class. So, according to Proposition 2.11 (c), if Sv is locally contitioned at all, then $\mathsf{Sv} = \mathrm{Loc}(\mathsf{Sv}^\circ) = \mathrm{Loc}(\mathsf{B}) = \pi\mathsf{Sv}$, which is false.

Incidentally, it is probable that πSv is not a quasi-torsion class, but we have not found an example to prove this.

3.2. Other examples of classes which are not locally conditioned. Let Cd stand for the class of all completelly distributive, normal-valued ℓ-groups. (For a discussion of complete distributivity, the reader is urged to refer to Chapter 6 of [BKW].) Now observe that $\mathsf{Cd} \cap \mathsf{Ar} = \mathsf{Sv} \cap \mathsf{Ar} = \mathsf{B}$. Then, arguing once again through Proposition 2.11 (c), it can be shown that if Cd is locally conditioned then $\mathsf{Cd} = \pi\mathsf{Sv}$. This is not so; one way to see this is to recall from [BCW] that every ℓ-group may be embedded in one with a basis, and in such a way that one can produce normal-valued ℓ-groups with a basis—which are therefore in Cd—which have no special elements modulo their basis subgroups. Such ℓ-groups, evidently, cannot be pseudo-special-valued.

Employing Proposition 2.11 in the same way, one can show that Pr, the class of projectable ℓ-groups, is not locally conditioned; just observe this: $\mathsf{O}_R \subseteq \mathsf{Pr}^\circ$, and so, if Pr is locally conditioned, then $\mathsf{F} = \mathrm{Loc}(\mathsf{O}_R) \subseteq \mathsf{Pr}$, which is absurd.

Recall that G has property (F) if it has no bounded, infinite pairwise disjoint sets. We let F_0 stand for the class of all ℓ-groups possessing property (F); it is well known that this is a torsion class. However, by arguments very similar to the previous ones, it can be demonstrated that F_0 is not locally conditioned.

Notice that neither Pr nor F_0 are split-complete; the same is true about Cd and Sv. This persuaded us to conjecture that if U is a quasi-torsion class of normal-valued ℓ-groups which is split-complete, then U should be locally contitioned. However, this is false; let Dc be the torsion class of all ℓ-groups which satisfy the descending chain condition on prime convex ℓ-subgroups. Dc is complete! On the other hand. if $\mathsf{U} = \mathsf{Dc} \cap \mathsf{N}$ were locally conditioned then, by Proposition 2.11, $\mathrm{Loc}(\mathsf{U}^\circ) \subseteq \mathsf{U}$, and this would imply that U contains every finite-valued ℓ-group, which is not the case.

References

[AF] *M. Anderson, T. Feil*: Lattice-Ordered Groups, Kluwer Publ., Dordrecht, Boston, 1988.

[A] *M. Anderson*: Subprojectable and localy flat lattice-ordered groups, University of Kansas Dissertation, 1977.

[BCD] *R. Ball, P. Conrad, M. Darnel*: Above and below subgroups of a lattice-ordered group, Trans. Amer. Math. Soc. *297* (1986), 1–40.

[BCW] *A. Bigard, P. Conrad, S. Wolfenstein*: Compactly generated lattice-ordered groups, Math. Zeitschr. *107* (1968), 201–211.

[BKW] *A. Bigard, K. Keimel, S. Wolfenstein*: Groupes et Anneaux Réticulés, Lecture Notes in Math. 608, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[BD] *J. P. Bixler, M. Darnel*: Special-valued ℓ-groups, Alg. Univ. *22* (1986), 172–191.

[C1] *P. Conrad*: The lattice of all convex ℓ-subgroups of a lattice-ordered group, Czech. Math. Jour. *15* (1965), 101–123.

[C2] *P. Conrad*: A characterization of lattice-ordered groups by their convex ℓ-subgroups, Jour. Austral. Math. Soc. *7* (1967), 145–159, MR 35:5371.

[C3] *P. Conrad*: Epi-archimedean groups, Czech. Math. Jour. *24* (1974), 1–27.

[C4] *P. Conrad*: K-radical classes of lattice-ordered groups, Lecture Notes in Math. 848, Springer-Verlag, 1980, pp. 186–206.

[CM1] *P. Conrad, J. Martinez*: Locally finite conditions on lattice-ordered groups, Czech. Math. Jour. *39* (1989), 432–444.

[CM2] *P. Conrad, J. Martinez*: Signatures and S-discrete lattice-ordered groups, Alg. Univ, to appear.

[H] *W. C. Holland*: Varieties of lattice-ordered groups are torsion classes, Czech. Math. Jour. *29* (1979), 11–12.

[Ke] *G. O. Kenny*: Lattice-Ordered Groups, University of Kansas Dissertation, 1975.
[M1] *J. Martinez*: The hyper-archimedean kernel sequence of a lattice-ordered group, Bull. Austral. Math. Soc. *10* (1974), 337–350.
[M2] *J. Martinez*: Torsion theory for lattice-ordered groups, Czech. Math. Jour. *25* (1975), 284–299.
[M3] *J. Martinez*: Pairwise-splitting lattice-ordered groups, Czech. Math. Jour. *27* (1977), 545–551.
[M4] *J. Martinez*: The closed subgroup problem for lattice-ordered groups, Archiv. der Math. *54* (1990), 212–224.
[Š] *F. Šik*: Zur theorie der halbgeordnete Gruppen, Czech. Math. Jour. *10* (1960), 400–424.

Author's address: Department of Mathematics, University of Florida, Gainesville, FL 32611, U.S.A.