Luis A. Cordero; Manuel de León; Isabel Méndez; Modesto R. Salgado
Existence of p-almost tangent structures

Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 2, 225–234

Persistent URL: http://dml.cz/dmlcz/128337

Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

EXISTENCE OF p-ALMOST TANGENT STRUCTURES

Luis A. Cordero, Santiago de Compostela, Manuel de León, Madrid,
Isabel Méndez, Modesto Salgado, Santiago de Compostela

(Received November 1, 1987)

1 Introduction

In [8], p-almost tangent structures were introduced as a natural generalization of almost tangent structures. A p-almost tangent structure consists of a p-tuple of tensor fields $\{J_1, \dots, J_p\}$ of type (1,1) on a $(p+1)n$-dimensional manifold verifying some compatibility conditions. The tangent bundle of p^1-velocities $T_p^1 N$ of any differentiable manifold N carries a canonical p-almost tangent structure (hence the name of the structure).

In this paper we study the existence of p-almost tangent structures. In §3, we state a theorem on the existence of associated structures in the sense of Bernard [1]. In §4 and §5, obstructions to the existence of p-almost tangent structures on certain manifolds are stablished in terms of their Euler, Stiefel-Whitney, Chern and Pontrjagin classes.

The results in this paper may be closely compared with the corresponding ones for almost tangent structures of higher order (see [6]). However, there are significant differences between almost tangent structures of order p and p-almost tangent structures. For instance, the geometrical model for a p-almost tangent structure is $T_p^1 N$, the tangent bundle of p^1-velocities, manifold which possesses a canonical structure of vector bundle (see [9]), while the geometrical model for an almost tangent structure of order p is $T^p N$, the tangent bundle of order p, manifold which is no longer a vector bundle. (A linear connection on N induces a vector bundle structure on $T^p N$ and allows the construction of a vector bundle isomorphism between $T^p N$ and $T_p^1 N$, but it essentially depends on the linear connection (see [3]).)

Let M be a $(p+1)n$-dimensional manifold and let there be given a p-tuple of tensor fields $\{J_1, \ldots, J_p\}$ of type (1,1) such that, for $1 \leqslant a,\, b \leqslant p$,

(1) $J_a J_b = J_b J_a = 0$;

(2) $\operatorname{rank} J_a = n$, and

(3) $\operatorname{Im} J_a \cap \left(\bigoplus_{b \neq a} \operatorname{Im} J_b\right) = 0$ for all a.

Such a p-tuple of tensor fields $\{J_1, \ldots, J_p\}$ defines a *p-almost tangent structure* on M which is then called a *p-almost tangent manifold.*

Let us remark that a 1-almost tangent structure is an almost tangent structure in the usual sense (see [2], for instance).

Example. Let be N an n-dimensional manifold and $T^1_p N$ the *tangent bundle of p^1-velocities of N* (see [4]). If (x^i) is a coordinate system on N, then $(x^i, x^i_a; 1 \leqslant a \leqslant p)$ will denote the induced coordinates on $T^1_p N$. Define

$$J_a = \sum_{i=1}^{n} \frac{\partial}{\partial x^{i+an}} \otimes \mathrm{d}x^i.$$

Then it is easy to prove that $\{J_1, \ldots, J_p\}$ defines a p-almost tangent structure on $T^1_p N$ (for a more detailed study see [8]).

If we put $V_a = \operatorname{Im} J_a$, then V_a is an n-dimensional distribution on M. Therefore

$$V = \bigoplus_{a=1}^{n} V_a$$

is a pn-dimensional distribution on M.

Now, let be $z \in M$; then V_z is a pn-dimensional subspace of the tangent space $T_z M$. Choose a complement H_z of V_z in $T_z M$, and let $\{e_i\}$ be a basis of H_p. Then

$$\{e^i, e^{i+an} = J_a e^i; 1 \leqslant a \leqslant p\}$$

is a basis for $T_z M$, that is a frame at z which will be said *adapted* to the p-almost tangent structure. If

$$\{\bar{e}^i, \bar{e}^{i+an}; 1 \leqslant a \leqslant p\}$$

is another such frame at z, $\{e^i\}$ being a basis for a different complement of V_z, then there are $n \times n$ square matrices $A, A_1, \ldots, A_p$, with $A \in \mathrm{Gl}(n, \mathbf{R})$, such that

$$\bar{e}^i = A^i_j e^i + (A_1)^i_j e^{j+n} + \cdots + (A_p)^i_j e^{j+pn},$$

and hence

$$\bar{e}^{i+an} = A^i_j e^{j+an}, \quad 1 \leqslant a \leqslant p.$$

Therefore, two adapted frames are related by a $(p+1)n \times (p+1)n$-square matrix of the form

$$\begin{pmatrix} A & 0 & \dots & 0 \\ A_1 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_p & 0 & \dots & A \end{pmatrix}. \tag{1}$$

The set of such matrices is a closed Lie subgroup $G \subset \mathrm{Gl}((p+1)n, \mathbf{R})$, and hence the set of all the adapted frames at all points of M defines a G-structure on M. The following results have been obtained in [8]:

Theorem 2.1. (1): *Given a $(p+1)n$-dimensional manifold M, there is a natural one-to-one correspondence between the p-almost tangent structures and the G-structures on M.*

(2): *A p-almost tangent structure $\{J_1, \dots, J_p\}$ on M is integrable if and only if $\{J_a, J_b\} = 0$, where $\{J_a, J_b\}$ is the tensor field of type $(1,2)$ on M given by*

$$\{J_a, J_b\}(X,Y) = [J_a X, J_b Y] - J_a[X, J_b Y] - J_b[J_a X, Y].$$

3 Existence of p-almost tangent structures

Let M be an n-dimensional manifold.

Definition 3.1. A G'-structure $P'(M, G^n)$ on M is said to be equivalent to a G-structure $P(M,G)$ if there exists $E \in \mathrm{Gl}(n, \mathbf{R})$ such that $P' = PE$. In such case P' is a G-structure if and only if $E \in N(G)$ (the normalizer of G in $\mathrm{Gl}(n,\mathbf{R})$). We then say that P' is a G-structure *associated* to P.

Proposition 3.1. [1] *Let $P'(M, G')$ be a principal bundle over M with structural group G'. Then there is an one-to-one correspondence between closed subbundles of P' and the sections of P'/G, G being an arbitrary closed subgrouop of G'. Furthermore, there is and one-to-one correspondence between the closed subbundles of P' with structural group G and the sections of P'/G.*

Consequently, the problem of determining all the possible G-structures on M is solved when $N(G) \neq G$, because there is an one-to-one correspondence between these G-structures and the sections of FM/G.

For the case of the p-almos tangent structures, that we are considering here, we have

Proposition 3.2. $N(G) \neq G$.

Proof. Let $\tilde{G}$ be the subgroup of $\mathrm{Gl}((p+1)n, \mathbb{R})$ of matrices of the form

$$\begin{pmatrix} A & 0 & \dots & 0 \\ A_1 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_p & 0 & \dots & \lambda A \end{pmatrix},$$

where $A \in \mathrm{Gl}(n, \mathbb{R})$ and $\lambda \in \mathbb{R} - \{0\}$. A direct computation shows that $G \prec \tilde{G} \subset N(G)$. □

Therefore

Corollary 3.1. *If there exists a p-almost tangent structure on a manifold M, then all its associated structures correspond biunivocally to the sections of FM/G.*

4 Orientability, Euler nad Stifel-Whitney classes of p-almost tangent manifolds

Let $\{J_1, \dots, J_p\}$ be a p-almost tangent structure on M. We shall define a Riemannian metric $\mathbf{g}$ on M adapted to the structure as follows.

Let H be an n-dimensional distribution on M complementary to $V = V_1 \oplus \dots \oplus V_p$ (for instance, we can take H being an orthogonal complement of V with respect to some Riemannian metric on M). Let $\mathbf{g}_0$ be a metric on the vector bundle H. Since for each a, $1 \leqslant a \leqslant p$,

$$J_{a|H} \colon H \to V_a$$

is a vector bundle isomorphism, we may define a metric $\mathbf{g}_a$ on the vector bundle V_a such that

$$\mathbf{g}_a(J_a X, J_a Y) = \mathbf{g}_0(X, Y), \quad X, Y \in H.$$

Therefore

$$\mathbf{g}(X, Y) = \mathbf{g}_0(X_0, Y_0) + \sum_{a=1}^{p} \mathbf{g}_a(X_a, Y_a),$$

defines a metric on M, where $X = X_0 + \sum_{a=1}^{p} X_a$, $Y = Y_0 + \sum_{a=1}^{p} Y_a$, $Y_0 \in H$, $X_a, Y_a \in V_a$.

Let $\{e^i\}$ be an orthogonal basis of H_z, $z \in M$, with respect to $\mathbf{g}_0$. Then $\{e^{i+an} = J_a e^i\}$ is a basis of $(V_a)_z$ orthonormal with respect to $\mathbf{g}_a$ and $\{e^\alpha; 1 \leqslant \alpha \leqslant (p+1)n\}$ is a frame at z orthonormal with respect to $\mathbf{g}$. Let $\{\bar{e}^\alpha\}$ be another such orhonormal frame, $\{\bar{e}^i\}$ being a different orhonormal basis of H_z; then there exists an $n \times n$ square matrix $A \in \mathrm{Gl}(n, \mathbf{R})$ such that

$$\bar{e}^i = A^i_j e^j$$

and hence

$$\bar{e}^{i+an} = A^i_j e^{i+an}, \quad 1 \leqslant a \leqslant p.$$

Hence the orthonormal frames are related by a $(p+1)n \times (p+1)n$ square matrix

$$\begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix}$$

with $A \in O(n)$.

The set of such matrices is a closed Lie subgroup $G' \subset \mathrm{Gl}\big((p+1)n, \mathbf{R}\big)$, and the set of all such frames at all points of M defines a G'-structure on M. Conversely, it is easy to prove that given a G'-structure P' on M, then P' defines a p-almost tangent structure on M.

Summing up, we have

Proposition 4.1. *Given a $(p+1)n$-dimensional manifold M, there is a natural one-to-one correspondence between the p-almost tangent structures and the G'-structures on M.*

Let us remark that $G' = O(n) \times \cdots \times O(n) \subset O\big((p+1)n\big)$. Furthermore, if p is odd, then $G' \subset SO\big((p+1)n\big)$. Thus, we have

Proposition 4.2. *A p-almost tangent manifold is orientable if p is odd.*

Corollary 4.1. *The real projective space $\mathbf{RP}^{(p+1)n}$ does not admit p-almost tangent structures for odd p.*

Remark 1. As a consequence of Proposition 4.1 it follows that a p-almost tangent structure and an almost tangent structure of order p have a common subordinate G'-structure (see [6]).

Let M be a p-almost tangent manifold. The tangent bundle TM of M can be decomposed as the Whitney sum

$$TM = H \oplus V_1 \oplus \cdots \oplus V_p; \tag{2}$$

therefore, the Euler class of TM is given by

$$\mathbf{e}(TM) = \mathbf{e}(H) \cup \mathbf{e}(V_1) \cup \ldots \cup \mathbf{e}(V_p), \tag{3}$$

where $\cup$ denotes the cup product. Hence

Proposition 4.3. *Let M be a $(p+1)n$-dimensional manifold such that $\mathbf{e}(TM) \neq 0$, and $\mathbf{H}^n(M, \mathbb{Z}) = 0$. Then M does not admit p-almost tangent structures.*

Proof. Let M be a p-almost tangent manifold. From (3), one deduces $\mathbf{e}(TM) = 0$, since $\mathbf{e}(H), \mathbf{e}(V_a) \in \mathbf{H}^n(M, \mathbb{Z})$. □

Corollary 4.2. *Let M be a connected, compact and oriented $(p+1)n$-dimensional manifold with non-zero Euler charasteristic $\chi(M)$ and such that $\mathbf{H}^n(M, \mathbb{Z}) = 0$. Then M does not admit p-almost tangent structures.*

Proof. Let μ be the fundamental class of M, $\mu \in \mathbf{H}^{(p+1)n}(M, \mathbb{Z})$. Then the Euler class of M is given by

$$\mathbf{e}(TM) = \mu(M) = 0.$$

The result follows now from Proposition 4.3. □

Corollary 4.3. (1): *If p is odd or n is even, then $S^{(p+1)n}$ does not admit p-almost tangent structures.*

(2): *If both p and n are odd, then $\mathbf{CP}^{\frac{(p+1)n}{2}}$ does not admit p-almost tangent structures.*

Proof. (1) in fact, $\mathbf{H}^n(S^{(p+1)n}, \mathbb{Z}) = 0$ and $\chi(S^{(p+1)n}) = 2$. The result then follows from Corollary 4.2. (2) is proved in a similar way. □

Next, we shall study the Stifel-Whitney classes of a p-almost tangent manifold.

From (2) we deduce that the total Stifel-Whitney class of TM is given by

$$\mathbf{w}(TM) = \left(\mathbf{w}(H)\right)^{p+1} = \left(\mathbf{w}(V_a)\right)^{p+1}. \tag{4}$$

Thus,

Proposition 4.4. *The total Stifel-Whitney class of a p-almost tangent manifold M can be expressed as a power of order $p+1$.*

Corollary 4.4. *Let M be a p-almost tangent manifold. If p is odd, then all the odd Stifel-Whitney classes of M vanish.*

Proof. The result follows by straighforward computation from (4). □

R e m a r k 2. Since M is orientable if and only if $\mathbf{w}_1 = 0$, Proposition 4.2 is, in fact, a consequence of Corollary 4.4.

5 Pontrjagin and Chern classes of p-almost tangent structures

Let M be a $(p+1)n$-dimensional manifold. For each point $z \in M$, $T_z^{\mathbf{C}}M$ will denote the complexification of T_zM.

Assume given a p-almost tangent structure $\{J_1, \dots, J_p\}$ on M, and for each J_a let us still denote as J_a its canonical extension to $T_z^{\mathbf{C}}M$; then, the extended tensor fields J_a, $1 \leqslant a \leqslant p$ still verify (1)–(3) in §2. Note that each extended J_a has complet rank n.

It is clear that the existence of a p-almost tangent structure on M induces a reduction of the structural group of the complexified tangent bundle $T^{\mathbf{C}}M$ of M to the subgroup $G^{\mathbf{C}}$ of $\mathrm{Gl}((p+1)n, \mathbf{C})$ of complex matrices of the same form of those in (1).

Let $P(M, \mathrm{Gl}((p+1)n, \mathbf{C}))$ be the principal bundle of frames of $T^{\mathbf{C}}M$, and let $P'(M, G^{\mathbf{C}})$ be the reduction of P to $G^{\mathbf{C}}$. A connection on P is said a *p-almost tangent connection* if it is reducible to a connection on P'.

From [5, Prop. 5.2] it follows easily:

Proposition 5.1. *Let Γ be a connection on P, with connection form ω and covariant derivative ∇. Then Γ is a p-almost tangent connection if and only if $\nabla J_a = 0$, $1 \leqslant a \leqslant p$.*

Suppose that Γ is a p-almost tangent connection on M with connection form ω. Since ω, restricted to P', takes values in the Lie algebra of $G^{\mathbf{C}}$ it follows that the matrix form of ω is as follows:

$$\begin{pmatrix} \omega^i_j & 0 & \dots & 0 \\ \omega^i_{j+n} & \omega^i_j & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \omega^i_{j+pn} & 0 & \dots & \omega^i_j \end{pmatrix}.$$

Consequently, the curvature form Ω of Γ, restricted to P' has the form

$$\begin{pmatrix} \Omega^i_j & 0 & \dots & 0 \\ \Omega^i_{j+n} & \Omega^i_j & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega^i_{j+pn} & 0 & \dots & \Omega^i_j \end{pmatrix}.$$

Therefore (see [7, Ch. XII]), the total Pontrjagin class of TM is given by

$$p(TM) = \det\left(I_{(p+1)n} - (1/2\pi)\Omega^{\alpha}_{\beta}\right) = \left\{\det\left(I_n - (1/2\pi)\Omega^{i}_{j}\right)\right\}^{p+1}$$

Thus,

Theorem 5.1. *The total Pontrjagin class of a p-almost tangent manifold M can be expressed as a power of order $p+1$.*

Now, since the total Chern class of TM is given by

$$c(TM) = \det\left(I_{(p+1)n} - \frac{1}{2\pi\sqrt{-1}}\Omega^{\alpha}_{\beta}\right) = \left\{\det\left(I_n - \frac{1}{2\pi\sqrt{-1}}\Omega^{i}_{j}\right)\right\}^{p+1}.$$

(see [7, Ch. XII]).

Theorem 5.2. *The total Chern class of a p-almost tangent manifold M can be expressed as a power of order $p+1$.*

Corollary 5.1. *The complex projective spaces $\mathbf{CP}^{2s}$ do not admit $(2p-1)$-almost tangent structures.*

P r o o f. In fact, for $p > s$, there exist no such structures by dimensional reasons. Then, suppose that $p \leqslant s$. As it is well known, the total Chern class of $\mathbf{CP}^{2s}$ is

$$c(\mathbf{CP}^{2s}) = (1+\alpha)^{2s+1},$$

where α is a generator of $\mathbf{H}^2(\mathbf{CP}^{2s}, \mathbb{Z})$. Therefore,

$$c(\mathbf{CP}^{2s}) = \sum_{i=0}^{2s+1} \binom{2s+1}{i} \alpha^{2s+1-i} = 1 + (2s+1)\alpha + \cdots.$$

Now, if $\mathbf{CP}^{2s}$ would admit a $(2p-1)$-almost tangent structure, according to Theorem 5.2, we should have

$$c(\mathbf{CP}^{2s}) = \left(\sum_{j=0}^{r} \lambda_j \alpha^j\right)^{2p} = (\lambda_0)^{2p} + 2p(\lambda_0)^{2p-1}\lambda_1\alpha + \cdots$$

for some integer r. By identifying coefficients in both expressions, we deduce

$$1 = (\lambda_0)^{2p}, \quad \lambda_0 = \pm 1, \quad 2s+1 = 2p(\lambda_0)^{2p-1}\lambda_1 = \pm 2p\lambda_1.$$

The last equality gives a contradiction. □

Proposition 5.2. *Let M be a p-almost tangent manifold. If p is odd, then the odd Pontrjagin classes of M can be expressed as twice a polynomial in its Chern classes.*

P r o o f. Let $T^{\mathbf{C}}M$ be the complexification of TM. Then the total Chern class of $T^{\mathbf{C}}M$ can be expressed as a power of order $p+1$. Therefore, when p is odd, the odd Chern classes of $T^{\mathbf{C}}M$ can be expressed as twice a polynomial in these Chern classes. Now, if E is a complex vector bundle and $E_{\mathbf{R}}$ denotes the underlying real vector bundle, we have (see [10, Ch. 15])

$$p_k(E_{\mathbf{R}}) = c_k(E)^2 - 2c_{k-1}(E)c_{k+1}(E) + \cdots \pm 2c_1(E)c_{2k-1}(E) \pm 2c_{2k}(E).$$

Thus, when k is odd, the result follows. □

R e m a r k 3. Proposition 5.2 cna be used to obtain a new proof of Corollary 5.1.

Corollary 5.2. *Suppose that $\left(\mathbf{w}_{2j}(M)\right)^2 \neq 0$, for some odd integer j. Then M does not admit $(2p-1)$-almost tangent structures.*

P r o o f. As it is well known (see [10, Ch. 15]), if E is a real vector bundle over M, then the mod 2 reduction of the Pontrjagin class $p_i(E)$ is equal to the square of the Stiefel-Whitney class $\mathbf{w}_{2i}(E)$. Therefore, from Proposition 5.2.,

$$\left(\mathbf{w}_{2j}(M)\right)^2 = p_j(M) = 0 \bmod 2.$$

□

Corollary 5.3. *The real projective spaces $\mathbf{RP}^{2s}$ do not admit $(2p-1)$-almost tangent structures.*

P r o o f. In fact, the total Stiefel-Whitney class of $\mathbf{RP}^{2s}$ is given by

$$\mathbf{w}(\mathbf{RP}^{2s}) = (1+\alpha)^{2s+1},$$

where α is a generator of $\mathbf{H}^1(\mathbf{RP}^{2s}, \mathbb{Z})$. □

Finally, we study the existence or not of p-almost tangent structures on several families of symmetric Hermitian irreducible compact spaces.

(1) *The complex Grassmannians* $W(m,n) = U(m+n)/U(m) \times U(n)$. The first Chern class of $W(m,n)$ is

$$c_1\left(W(m,n)\right) = -(m+n)\sigma_1,$$

where σ_1 is a generator of the infinite cyclic group $\mathbf{H}^2(W(m,n),\mathbb{Z})$. Therefore, bearing in mind the proof of Proposition 5.5, we deduce that $W(m,n)$ *does not admit* $(2p-1)$-*almost tangent structures when* $m+n$ *is odd.* Since $W(1,n) = \mathbf{CP}^n$, we reobtain Corollary 5.1.

(2) *The spaces* $G_n = Sp(n)/U(n)$.

For the spaces, we have

$$c_1(G_n) = (n+1)\sigma_1,$$

where σ_1 is the generator of $\mathbf{H}^2(G_n,\mathbb{Z})$. Thus, G_{2n} *does not admit* $(2p-1)$-*almost tangent structures.*

(3) *The complex quadrics* $Q_n = SO(n+2)/SO(2)\times SO(n)$.

In this case, we have

$$c_1(Q_n) = n\alpha.$$

Therefore, Q_n *does not admit* $(2p-1)$-*almost tangent structures.*

References

[1] *D. Bernard*: Sur la géometrie différentielle des G-structures, Ann. Inst. Fourier *10* (1960), 153–273.

[2] *R. S. Clark, M. Bruckheimer*: Sur les structures presque tangentes, C. R. Acad. Sc. Paris *251* (1960), 557–563.

[3] *M. Djaa, J. Gancarziewicz*: The geometry of tangent bundles of order r, Bol. Acad. Galega de Ciencias *IV* (1985), 147–165.

[4] *Ch. Ehresmann*: Les prolongéments d'une variété différentiable, I. Calcul des jets, prolongement principal, C. R. Acad. Sc. Paris *233* (1951), 598–600.

[5] *A. Fujimoto*: Theory of G-structures, Public. of the Study Group of Geometry, Tokyo, 1972.

[6] *P. M. Gadea, J. L. Rosendo*: On almost tangent structures of order k, An. Stiint. Univ. Al. I. Cuza, Iasi *22* (1976), 213–220.

[7] *S. Kobayashi, K. Nomizu*: Foundations of Differential Geometry, I, II, Interscience Publ., New York, 1963–69.

[8] *M. de León, I. Méndez, M. Salgado*: p-almost tangent structures, Rendiconti Circ. Mat. Palermo, Ser. 33 *XXXVII* (1988), 282–294.

[9] *M. de León, I. Méndez, M. Salgado*: Integrable p-almost tangent manifolds and tangent bundles of p^1-velocities, Acta Mathematica Hungarica *58* (1991), 39–47.

[10] *J. W. Milnor, J. D. Stasheff*: Characteristic classes, Ann. Math. Studies, Princeton Univ. Press, Princeton, 1974.

[11] *K. Yano, S. Ishihara*: Tangent and Cotangent Bundles, Marcel Dekker, New York, 1973.

Authors' addresses: M. de León, Unidad de Matemáticas, Consejo Superior de Investigaciones Científicas, Serrano, 123, 28006 Madrid; L. A. Cordero, I. Méndez, M. Salgado, Dept. Geometría y Topología, Fac. de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela, Spain.