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ZERO-SUM BIPARTITE RAMSEY NUMBERS

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1. INTRODUCTION

The starting point of almost all the recent combinatorial research on zero-sum problems is the following theorem.

Theorem A. (Erdös, Ginzburg, Ziv [EGZ]). *Let $m \geqslant k \geqslant 2$ be two integers such that $k \mid m$. Then any sequence of $m + k - 1$ integers contains a subsequence of cardinality m the sum of whose elements is divisible by k.*

There is a rapidly growing literature on zero-sum problems. As can be see in the list of references, most of them dealt with the so-called zero-sum Ramsey numbers, a concept first introduced by Bialostocki and Dierker ([BD1] [BD2]). To describe this concept as well as the bipartite variant we need a few definitions. Let $\mathbb{Z}_k$ denote the cyclic additive group of order k. A $\mathbb{Z}_k$-coloring of the edges of a graph $G = (V, E)$ is a function $f\colon E(G) \to \mathbb{Z}_k$. If $\sum\limits_{e \in E(G)} f(e) = 0$ (in $\mathbb{Z}_k$), we say that G is a zero-sum graph $(\bmod k)$ (with respect to f). If k divides the number, $e(G)$, of edges of G, then the zero-sum Ramsey number $R(G, \mathbb{Z}_k)$ is the smallest integer t such that for every $\mathbb{Z}_k$-coloring of $E(K_t)$ there is a zero-sum $(\bmod k)$ copy of G in K_t.

If G is bipartite and $k \mid e(G)$, then the zero-sum bipartite Ramsey number $B(G, \mathbb{Z}_k)$ is the smallest integer t such that for every $\mathbb{Z}_k$-coloring of $E(K_{t,t})$ (the complete bipartite graph) there is a zero-sum $(\bmod k)$ copy of G in $K_{t,t}$.

The existence of $B(G, \mathbb{Z}_k)$ follows from the trivial inequality $B(G, \mathbb{Z}_k) \leqslant B(G, k)$, where $B(G, k)$ is the classical bipartite Ramsey number using k colors (see e.g. [GRS]).

The first problem we consider here, in section 2, is that of estimating $B(G, \mathbb{Z}_2)$. As shown in [ALCA] $R(G, \mathbb{Z}_2) \leqslant |G| + 2$.

Define $m(G) = \min\{|A|, V(G) = A \cup B, |A| \geqslant |B|\}$ where the minimum is taken over all the representations of G as a bipartite graph with classes A and B, (e.g., $m(K_{1,n}) = n$, $m(K_{2,3} \cup K_{4,7}) = 9$).

We prove that $B(G,\mathbb{Z}_2) \leqslant m(G)+1$ and discuss some exact cases. The second problem we consider here, in Section 3, is that of estimating $B(K_{n,m},\mathbb{Z}_k)$. We prove that if $K \mid n$ ($n \geqslant m$) then $B(K_{n,m},\mathbb{Z}_k) \leqslant n+k-1$ and explore some cases in which this bound is sharp. In contrast we prove that $B(K_{n,n},\mathbb{Z}_{n^2})$ grows exponentially. Essentially the same behaviour is known to hold for $R(K_n,\mathbb{Z}_k)$ vs. $R(K_n,\mathbb{Z}_{\binom{n}{2}})$ as proved in [CAR1] [CAR5] [ALCA].

The third problem considered in Section 3, is to evaluate $B(tK_2,\mathbb{Z}_k)$ when $k \mid t$ and tK_2 is the disjoint union of t edges. Using theorem A and a construction we prove $B(tK_2,\mathbb{Z}_k)=t+k-1$. Some related problems will be considered.

We follow the standard notation of [BOL1]. In particular $e(G)$ denotes the number of edges of G. S_n denotes the group of permutations of n-element set. C_n denotes the cyclic group of permutations of n-element set. For a finite set S let

$$\delta(S)=\begin{cases}1 & \text{if } |S|\equiv 0 \pmod 2\\ 0 & \text{if } |S|\equiv 1 \pmod 2).\end{cases}$$

2. An Upper Bound for $B(G,\mathbb{Z}_2)$

The essence of this section can be summarized as:

Theorem 1. *Let G be a bipartite graph such that $2 \mid e(G)$.*
(i) *if $m(G)\equiv 1 \pmod 2$ then $B(G,\mathbb{Z}_2)=m(G)$.*
(ii) *if $m(G)\equiv 0 \pmod 2$ then $B(G,\mathbb{Z}_2)\leqslant m(G)+1$.*
(iii) *if $m(G)\equiv 0 \pmod 2$ and A realizes $m(G)$, $|A|>|B|$, and for every $x\in A$ $\deg x \equiv 0 \pmod 2$ then $B(G,\mathbb{Z}_2)=m(G)$.*

For the proof we apply a method developed in [ALCA]. We need Lemma and the following definition.

Definition. Suppose $H_1, H_2, \ldots, H_n$ is a family of subgraphs of $K_{t,t}$. Then the sum modulo-2 of $H_1,\ldots,H_n$ denoted by $\oplus\sum_{i=1}^{n} H_i$, is the subgraph of $K_{t,t}$ whose edges are all those edges of $K_{t,t}$ belonging to an odd number of H_i-s.

Observe that this is exactly the sum (in $\mathbb{Z}_2$) of the vectors corresponding to the H_i-s, where to each H_i is associated the characteristic vector, of length t^2, of its edges. (Exactly $e(G)$ places are 1 and the others are 0.)

In the case that $\oplus\sum_{i=1}^{n} H_i$ is the empty graph we write $\oplus\sum_{i=1}^{n} H_i=\underline{0}$.

Lemma. (Parity Lemma.) *Let G be a bipartite graph so that $2 \mid e(G)$. Then $B(G,\mathbb{Z}_2)$ is the least integer t such that $K_{t,t}$ contains a family $H_1,\ldots,H_n$ of subgraphs isomorphic to G, n is odd and $\oplus\sum_{i=1}^{n} H_i=\underline{0}$.*

P r o o f . Let $I_t(G)$ be the family of all subgraphs of $K_{t,t}$ isomorphic to G. To each member $H \in I_t(G)$ we make correspond an equation with $e(G)$ variables, namely $\sum_{e\in E(H)} x_e = 1$ (in $\mathbb{Z}_2$).

This system of equations has no solution if $t \geqslant B(G,\mathbb{Z}_2)$, because in this case a zero-sum (mod 2) copy of G will not satisfy its equation. Hence $B(G,\mathbb{Z}_2)$ is the least such t.

Recall a basic result from linear algebra: The system $Ax = b$ has no solution iff the Gaussian elimination procedure results in a row of the form $(0,0,0\ldots,0,t)$ where $t \neq 0$ (see e.g. [STE] p. 142–143). We find that the above system has no solution iff there is an odd number of equations whose sum (in $\mathbb{Z}_2$) gives $\underline{0} = 1$, and the Lemma follows. □

P r o o f o f T h e o r e m 1 . Suppose $f\colon E(K_{t,t}) \to \mathbb{Z}_2$ where $t = |A| + \delta(A)$, $|A| = m(G)$. Observe that $t = |A| + \delta(A) \equiv 1 \pmod 2$.

Fix a copy of G in $K_{t,t}$, and consider the direct product group $C_t^{(1)} \times C_t^{(2)} := H$ acting on $V(K_{t,t})$, where $C_t^{(1)}$ acts cyclically on one class of $K_{t,t}$ and $C_t^{(2)}$ on the other class.

How many copies of G do we get from the action of H?

Exactly $t^2 \equiv 1 \pmod 2$.

On the other hand as $2\,|\,e(G)$ every edge of $E(K_{t,t})$ appears in exactly $e(G)$ copies of G, under the action of H. Hence $\oplus \sum_{\sigma\in H} \sigma(G) = \underline{0}$, $|H| = t^2 \equiv 1 \pmod 2$ and by the Parity Lemma $B(G,\mathbb{Z}_2) \leqslant t = |A|+\delta(A)$, $|A| = m(G)$ which completes the proof of parts (i) and (ii).

For part (iii) observe that $m(G) = |A| \geqslant |B| + \delta(B)$, (by assumption). Let $f\colon E(K_{t,t}) \to \mathbb{Z}_2$, where $t = m(G)$ and fix a copy of $K_{t,q}$ in $K_{t,t}$ where $q = |B| + \delta(B)$. In $K_{t,q}$ fix a copy of G in such a way that A is in the class of order t and B in the class of order q.

Consider the action of the permutation group C_q on the class of order q. As $q \equiv 1 \pmod 2$ we get a family of q copies of G. On the other hand consider an edge $e = (x,y) \in E(K_{t,q})$, where $x \in A$. Clearly e appears in exactly $\deg x$ copies of G under the action of C_q, but $\deg x \equiv 0 \pmod 2$ hence $\oplus \sum_{\sigma\in C_q} \sigma(G) = \underline{0}$, $q \equiv 1 \pmod 2$ and by the parity lemma we are done. □

A simple observation [ALCA] states that if $2 \mid \binom{n}{2}$ then $R(K_n,\mathbb{Z}_2) = n+2$. Here we derive a similar result for the complete bipartite graph $K_{m,n}$ when $2 \mid mn$.

Theorem 2. *Let $n \geqslant m \geqslant 1$ be integers. Then*

$$B(K_{m,n},\mathbb{Z}_2) = \begin{cases} n+1 & \text{if } 2 \mid m,\ m = n \\ n & \text{if } 2 \mid m,\ n > m \\ n+1 & \text{if } 2 \mid n \text{ and } 2 \nmid m. \end{cases}$$

Proof. (i) Suppose first $2 \mid m$, $m = n$. Let $f\colon E(k_{n+1,n+1}) \to \mathbb{Z}$. Take n vertices of one side of $K_{n+1,n+1}$, say $u_1, \ldots, u_n$ and all the $n+1$ vertices of the other side, say $w_1, \ldots, w_{n+1}$.

Define a sequence of $n+1$ integers as follows: for $1 \leqslant i \leqslant n+1$, $a_i = \sum_{j=1}^{n} f(w_i, u_j)$. By Theorem A there are n terms whose sum is 0 (mod 2), namely $\sum_{i\in I} a_i \equiv 0 \pmod 2$, $|I| = n$. Now $u_1, \ldots, u_n$ and $\{w_i, i \in I\}$ form a zero-sum copy of $K_{n,n}$. Hence $B(K_{n,n}, \mathbb{Z}_2) \leqslant n+1$. For the lower bound consider $K_{n,n}$ with classes $A = \{u_1, \ldots, u_n\}$ and $B = \{w_1, \ldots, w_n\}$. Define $f\colon E(K_{n,n}) \to \mathbb{Z}_2$ by

$$f(u_i, w_j) = \begin{cases} 1 & i = j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This $\mathbb{Z}_2$-coloring implies $B(K_{n,n}, \mathbb{Z}_2) > n$, hence $B(K_{n,n}, \mathbb{Z}_2) = n+1$.

(ii) Suppose $2 \mid m$, $n > m$. Repeat the argument above for $f\colon E(K_{n,n}) \to \mathbb{Z}_2$ obtain, in exactly the same way, $B(K_{m,n}, \mathbb{Z}_2) \leqslant n$ and clearly $B(K_{m,n}, \mathbb{Z}_2) \geqslant n$, hence $B(K_{m,n}, \mathbb{Z}_2) = n$.

(iii) Suppose $2 \mid n$, $n > m$ and $2 \nmid m$. For the upper bound repeat the argument of (i) to obtain $B(K_{m,n}, \mathbb{Z}_2) \leqslant n+1$.

For the lower bound consider $K_{n,n}$ with classes $A = \{u_1, \ldots, u_n\}$, $B = \{w_1, \ldots, w_n\}$ and define $f\colon E(K_{n,n}) \to \mathbb{Z}_2$ by

$$f(u_i, w_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Clearly no zero-sum copy of $K_{n,m}$ exists because for every such copy H, $\sum_{e\in E(H)} f(e) = m \equiv 1 \pmod 2$. Hence $B(K_{m,n}, \mathbb{Z}_2) = n+1$ completing the proof. □

3. Estimations of $B(K_{m,n}, \mathbb{Z}_k)$

Let's first extend the argument used in the proof of theorem 2 to investigate $B(K_{m,n}, \mathbb{Z}_k)$ where $k \mid m$ or $k \mid n$.

Theorem 3. *Let $n \geqslant m \geqslant 1$ be integers. Then*

(i) $$B(K_{m,n}, \mathbb{Z}_k) \leqslant \begin{cases} m+k-1 & \text{if } k \mid m,\ m \leqslant n \leqslant m+k-2 \\ n & \text{if } k \mid m,\ n \geqslant m+k-1 \\ n+k-1 & \text{if } k \mid n \text{ and } k \nmid m. \end{cases}$$

(ii) *put* $f(k) = \begin{cases} k-1 & \text{if } k \text{ is a prime} \\ \lfloor\sqrt{k-1}\rfloor & \text{otherwise} \end{cases}$ *then*

$$B(K_{m,n}, \mathbb{Z}_k) \geqslant \max\{m + f(k), n\}.$$

P r o o f. Suppose $k|m$ and $m \leqslant n \leqslant m+k-2$. Consider $f\colon E(K_{m+k-1,m+k-1}) \to \mathbb{Z}_k$. Take n vertices at one class of $K_{m+k-1,m+k-1}$ say $\{u_1, \ldots, u_n\}$ and all the $m+k-1$ vertices from the other class $\{w_1, \ldots, w_{m+k-1}\}$ Define $a_i = \sum_{j=1}^{n} f(w_i, u_j)$, $1 \leqslant i \leqslant m+k-1$. By theorem A there exists $I \subset \{1, 2, \ldots, m+k-1\}$, $|I| = m$ such that $\sum_{i\in I} a_i \equiv 0 \pmod{k}$. Clearly $\{u_i\}_{i=1}^{n}$ and $\{w_i, i \in I\}$ form a zero-sum copy $(\bmod\, k)$ of $K_{m,n}$. The two other cases follow easily along the same line, proving (i). For (ii) consider the following $\mathbb{Z}_k$-coloring.

Take a copy of $K_{m+f(k)-1,m+f(k)-1}$ with classes $\{u_1, \ldots, u_{m+f(k)-1}\}$ and $\{w_1, \ldots, w_{m+f(k)-1}\}$. Define a $\mathbb{Z}_k$-coloring as follows.

$$f(u_i, w_j) = \begin{cases} 1 & \text{iff } i \geqslant m \text{ and } j \geqslant m \\ 0 & \text{otherwise.} \end{cases}$$

Any copy of $K_{m,n}$ must contain some of the u_i, $i \geqslant m$, say a of them and some of the w_j, $j \geqslant m$, say b of them.

For such a copy we have $\sum_{e\in E(K_{m,n})} f(e) = ab \not\equiv 0 \pmod{k}$ because of the definition of $f(k)$, and the fact that $a, b \leqslant f(k)$. Hence we must have $B(K_{m,n}, \mathbb{Z}_k) \geqslant \max\{m + f(k), n\}$. □

An immediate corollary of Theorem 3 is:

Theorem 4. *Let $n \geqslant m \geqslant 1$ be integers and k be a prime. Then*

$$B(K_{m,n}, \mathbb{Z}_k) = \begin{cases} m+k-1 & \text{if } k \mid m \quad m \leqslant n \leqslant m+k-2. \\ n & \text{if } k \mid m \quad n \geqslant m+k-1 \\ & \text{(holds even if } k \text{ is not a prime).} \end{cases}$$

R e m a r k. The main consequence of Theorem 3 is that if $k \mid mn$ and $k \leqslant \max\{m, n\}$ then $B(K_{m,n}, \mathbb{Z}_k)$ is small. So it is inevitable to ask what if $k \mid mn$ but $k > \max\{m, n\}$. Moreover even after Theorems 3 and 4 we have not yet determined $B(K_{1,n}, \mathbb{Z}_k)$ although we know that it is at most $n + k - 1$. We shall take a closer look at these problems.

Let's first derive a lower bound for $B(K_{m,n}, \mathbb{Z}_k)$ for large k.

Theorem 5. *Suppose $k \mid n^2$ and further $n^2/k = t$ where t is a fixed integer. Then $B(K_{n,n}, \mathbb{Z}_k) \geqslant \frac{n}{2e} e^{n/4t^2}$.*

P r o o f. We apply the "second moment" probabilistic argument.

Let $f\colon E(K_{m,m}) \to \mathbb{Z}_k$ be a random mapping, (m to be determined later), given by the rule

$$f(e) = \begin{cases} 1 & \text{with probability } \frac{k}{2n^2} = \frac{1}{2t} \\ 0 & \text{with probability } 1 - \frac{1}{2t}. \end{cases}$$

For every copy of $K_{n,n}$ in $K_{m,m}$ let $Y = \sum_{e \in E(K_{n,n})} f(e)$ be the edge-sum random variable. Then $Y \sim B(n^2, \frac{k}{2n^2})$, $E(Y) = n^2 \cdot \frac{k}{2n^2} = \frac{k}{2}$ and $\sigma(Y) = \sqrt{n^2 \frac{k}{2n^2}(1 - \frac{k}{2n^2})} < \sqrt{\frac{k}{2}}$, ($Y$ is a binomial random variable). By the standard approximation of the binomial distribution (see e.g. [BOL2] p. 11–12) the probability that $Y \equiv 0 \pmod k$ (i.e., will deviate by at least $\sqrt{\frac{k}{2}}$ standard deviations from its expectation) is

$$\leqslant \operatorname{Prob}\left(|Y - E(Y)| \geqslant \frac{k}{2}\right) \leqslant 2e^{-2k^2/4n^2} = 2e^{-k^2/2n^2}.$$

Hence if we choose m, such that $\binom{m}{n}^2 < \frac{1}{2} e^{k^2/2n^2}$ then we infer that $B(K_{n,n}, \mathbb{Z}_k) > m$.

A simple calculation gives $m \leqslant \frac{n}{2e} e^{k^2/4n^3} = \frac{n}{2e} e^{n/4t^2}$. Hence $B(K_{n,n}, \mathbb{Z}_k) \geqslant \frac{n}{2e} e^{n/4t^2}$. □

R e m a r k. The same argument gives an exponential lower bound for $B(K_{n,n}, \mathbb{Z}_k)$ if $k \mid n^2$ and $k > n^{1.5+\varepsilon}$, $\varepsilon > 0$ fixed.

Let's now derive an upper bound for $B(K_{m,n}, \mathbb{Z}_{mn})$.

Theorem 6.

$$B(K_{m,n}, \mathbb{Z}_{mn}) \leqslant \min\left\{(2n-2)\binom{2m-1}{m} + 1, (2m-2)\binom{2n-1}{n} + 1\right\}$$

P r o o f. Set $1 + (2n-2)\binom{2m-1}{m} = q$ and let $f\colon E(K_{q,q}) \to \mathbb{Z}_{mn}$. Choose $2m-1$ vertices $A = \{v_1, \ldots, v_{2m-1}\}$ from one class of $K_{q,q}$, and let B denote the set of vertices of the other class. By theorem A, for each $u \in B$ there is a subset $A_u \subset A$ such that $|A_u| = m$ and $\sum_{v \in A_u} f(u,v) \equiv 0 \pmod m$.

But there are $\binom{2m-1}{m}$ subsets of cardinality m of A, and $|B| = q = (2n-2)\binom{2m-1}{m} + 1$, hence there are $2n-1$ vertices of B, say $u_1, u_2, \ldots, u_{2n-1}$ such that $A_{u_1} = A_{u_2} = \ldots = A_{u_{2n-1}} := D$, $(D \subset A)$. For each $1 \leqslant i \leqslant 2n-1$ put $a_i = \frac{1}{m} \sum_{v \in D} f(u_i, v)$ and observe that a_i must we an integer for $1 \leqslant i \leqslant 2n-1$.

Apply theorem A again on $\{a_1, \ldots, a_{2n-1}\}$. Then there is a subset $I \in \{1, 2, \ldots, 2n-1\}$, $|I| = n$, such that $\sum_{i \in I} a_i \equiv 0 \pmod{n}$.

Now the complete bipartite graph $K_{m,n}$ with classes $V_1 = D$ and $V_2 = \{u_i : i \in I\}$ is a zero-sum copy $(\bmod\, mn)$ of $K_{m,n}$. □

Remark. A rough estimate gives $\frac{n}{2e} e^{n/4} \leqslant B(K_{n,n}, \mathbb{Z}_{n^2}) \leqslant n4^n$, but by the trivial observation that $B(K_{n,n}, \mathbb{Z}_{n^2}) \geqslant B(K_{n,n}, 2)$, and by the standard probabilistic argument we can improve the lower bound to $B(K_{n,n}, \mathbb{Z}_{n^2}) \geqslant \frac{n}{2e} 2^{n/2} \geqslant \frac{n}{2e} e^{n/4}$. Also by standard probabilistic argument one can show $B(K_{n,n}, n^2) \geqslant \frac{1}{3n} n^n$.

Hence $B(K_{n,n}, \mathbb{Z}_{n^2}) \lll B(K_{n,n}, n^2)$.

Our last result is the exact determination of $B(K_{1,n}, \mathbb{Z}_k)$ and $B(nK_2, \mathbb{Z}_k)$.

Theorem 7. *Let $n \geqslant k \geqslant 2$ be integers such that $k \mid n$. Then*

$$B(nK_2, \mathbb{Z}_k) = B(K_{1,n}, \mathbb{Z}_k) = n + k - 1.$$

Proof. Let $f \colon E(K_{n+k-1,n+k-1}) \to \mathbb{Z}_k$. Then trivially by Theorem A (as it contains both a copy of $K_{1,n+k-1}$ and a copy of $(n+k-1)K_2$) there is a zero-sum $(\bmod\, k)$ copy of both $K_{1,n}$ and nK_2. For the lower bound of $B(K_{1,n}, \mathbb{Z}_k)$ take a copy of $K_{n+k-2,n+k-2}$ with classes $\{u_1, u_2, \ldots, u_{n+k-2}\}$ and $\{w_1, \ldots, w_{n+k-2}\}$.

$$\text{Define } f(u_i, w_j) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant n-1 \text{ and } n \leqslant j \leqslant n+k-2 \\ & \text{or } 1 \leqslant j \leqslant n-1 \text{ and } n \leqslant i \leqslant n+k-2 \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that there is no zero-sum copy of $K_{1,n}$. For the lower bound of $B(nK_2, \mathbb{Z}_k)$ take again a copy of $K_{n+k-2,n+k-2}$ with classes as before.

$$\text{Define } f(u_i, w_j) = \begin{cases} 1 & \text{if } n \leqslant i \leqslant n+k-2 \\ 0 & \text{otherwise.} \end{cases}$$

Once again it is easy to see that for every copy of nK_2, $1 \leqslant \sum_{e \in E(nK_2)} f(e) \leqslant k-1$, hence no zero-sum copy of nK_2 exists. □

In closing we suggest some further problems and conjectures, whose solution may contribute to our understanding of the behavior of the zero-sum bipartite Ramsey numbers.

Problem 1. Determine $B(G, \mathbb{Z}_2)$ for every graph G such that $2 \mid e(G)$, or at least if G is connected.

Problem 2. Determine $B(K_{m,n}, \mathbb{Z}_k)$ for $k \mid mn$ and $k \leqslant \max\{m, n\}$. Recall that by Theorem 3 this is a moderate number.

Problem 3. Is it true that $\lim_{n\to\infty} B(K_{n,n}, \mathbb{Z}_{n^2})/B(K_{n,n}, 2) = 1$?

Conjecture. (A. Biallostocki) For $n \geqslant 2$ $B(K_{2,n}, \mathbb{Z}_{2n}) \leqslant 4n-3$.
Observe that by theorem G we only know that $B(K_{2,n}, \mathbb{Z}_{2n}) \leqslant 6n-5$.

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