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ON A VARIETY OF INFINITE ALGEBRAS

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In this paper we shall deal with the following varieties of algebras. For each natural number $n > 0$ a variety V_n with one n-ary operation h and n unary operations f_i for $i = 1, \ldots, n$, will be considered. The defining system of identities is

$$\text{(i)} \qquad h(f_1(x), \ldots, f_n(x)) = x,$$

$$\text{(ii)} \qquad f_i(x_1, \ldots, x_n) = x_i \quad \text{for} \quad i = 1, \ldots, n.$$

The aim of this paper is to show that the varieties V_n for $n > 1$ all algebras are one-element or infinite, to describe V_n-free algebras and to solve the word problem.

When there is no danger of confusion, we will not distinguish between the notation of an algebra and its basic set.

Propositon 1. *For each algebra $A \in V_n$ the operation $h\colon A^n \to A$ is a bijection, and for any set A and any bijection $\varphi\colon A^n \to A$ there exists an algebra $A \in V_n$ such that $h = \varphi$ in the unique way.*

Proof. Let $A \in V_n$ be an arbitrary algebra and let $a \in A$ be an arbitrary element. Then $a = h(f_1(a), \ldots, f_n(a))$, thus h is surjective. Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in A$ and $h(a_1, \ldots, a_n) = h(b_1, \ldots, b_n)$. Then $a_i = f_i h(a_1, \ldots, a_n) = f_i h(b_1, \ldots, b_n) = b_i$ for $i = 1, \ldots, n$. Therefore h is injective.

Let A be an arbitrary set and let $\varphi\colon A^n \to A$ be a bijection. For an arbitrary element $a \in A$ define

$$f_i(a) = e_i(\varphi^{-1}(a)) \quad \text{for} \quad i = 1, \ldots, n$$

where $e_i(x_1, \ldots, x_n) = x_i$. Now we show that for arbitrary $a, a_1, \ldots, a_n \in A$ we have

$$\varphi(f_1(a), \ldots, f_n(a)) = a$$

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and

$$f_i(\varphi(a_1,\ldots,a_n)) = a_i \quad \text{for} \quad i = 1,\ldots,n.$$

Let a, a_1, …, $a_n \in A$, then $\varphi(f_1(a),\ldots,f_n(a)) = \varphi(e_1\varphi^{-1}(a),\ldots,e_n\varphi^{-1}(a)) = a$ and $f_i(\varphi(a_1,\ldots,a_n)) = e_i\varphi^{-1}(\varphi(a_1,\ldots,a_n)) = e_i(a_1,\ldots,a_n) = a_i$ for $i = 1$, …, n.

Let us suppose that there exist operations g_i on A for $i = 1$, …, n such that for arbitrary a, a_1, …, $a_n \in A$

$$\begin{aligned} \varphi(g_1(a),\ldots,g_n(a)) &= a \\ g_i(\varphi(a_1,\ldots,a_n)) &= a_i \quad \text{for} \quad i = 1,\ldots,n \end{aligned}$$

holds. Then $g_i(a) = g_i(\varphi(f_1(a),\ldots,f_n(a))) = f_i(a)$ for $i = 1$, …, n. □

Corollary. *If $n \neq 1$ then each algebra $A \in V_n$ is one-element or infinite.*

Proof. It follows immediately for the fact that $h\colon A^n \to A$ is a bijection. □

Proposition 2. *Let $A \in V_n$ be an arbitrary algebra. For arbitrary elements a_1, …, $a_n \in A$ there exists a unique element $a \in A$ such that $f_i(a) = a_i$ for $i = 1$, …, n.*

Let A be any set and let $\varphi_i\colon A \to A$ for $i = 1$, …, n be arbitrary mapping with the property that for any elements a_1, …, $a_n \in A$ there exists a unique element $a \in A$ such that $\varphi_i(a) = a_i$ for $i = 1$, …, n. Then there exists a unique algebra $A \in V_n$ on the set A such that $f_i = \varphi_i$ for $i = 1$, …, n.

Proof. Let $A \in V_n$ be an arbitrary algebra and let a_1, …, $a_n \in A$ be arbitrary elements. Then $f_i(a_1,\ldots,a_n) = a_i$ for $i = 1$, …, n. Suppose that there exists an element $a \in A$ with the property $f_i(a) = a_i$ for $i = 1$, …, n. Since $f_i(h(a_1, \ldots, a_n)) = a_i$ for $i = 1$, …, n and such an element is unique, it follows that $a = h(a_1, \ldots, a_n)$.

Let A be an arbitrary set and let $\varphi_i\colon A \to A$ for $i = 1$, …, n be arbitrary mappings with the properties as in the proposition. For arbitrary elements a_1, …, $a_n \in A$ denote by $h(a_1, \ldots, a_n)$ the element for which $\varphi_i(h(a_1, \ldots, a_n)) = a_i$ for $i = 1$, …, n. Let $a \in A$, then $\varphi_i h(\varphi_1(a),\ldots,\varphi_n(a)) = \varphi_i(a)$ for all i. Since the element with this property is unique, it follows that $h(\varphi_1(a),\ldots,\varphi_n(a)) = a$. It follows also that the algebra $A \in V_n$ on the set A with the property $f_i = \varphi_i$ for $i = 1$, …, n exists and is unique. □

Corollary. *All operations f_i on algebras $A \in V_n$ are surjective.*

Now we describe the free algebras in the varieties V_n. The case $n = 1$ is very simple, because all algebras in V_1 are unary. The V_1-free algebra on a one-element set is an infinite chain, where f is the function of the successor and h is the function of the predecessor. The V_1-free algebra on an arbitrary set X is the disjoint system of V_1-free algebras on the one-element set. The cardinality of this system is the same as the cardinality of X.

The following theorem concerns the V_n-free algebras in the case $n > 1$.

Theorem 1. *Let A be the V_n-free algebra on a set X and let B be the V_n-free algebra on a set Y, where X and Y are arbitrary finite sets. If* $\operatorname{card} X = \operatorname{card} Y \pmod{n-1}$, *then A and B are isomorphic.*

P r o o f. Let A be the V_n-free algebra on a set $X = \{x_1, \dots, x_k\}$. We show, that A is the V_n-free algebra also on the set $Y = \{x_1, \dots, x_{k-1}, f_1(x_k), \dots, f_n(x_k)\}$. Let $C \in V_n$ be an arbitrary algebra and let $\varphi\colon Y \to C$ be an arbitrary mapping. Suppose that φ can be extended to a homomorphism $\psi\colon A \to C$. Denote $x_j\psi = c_j$ for $j = 1, \dots, k-1$ and $f_i(x_k)\psi = c_{k-1+i}$ for $i = 1, \dots, n$. Since $x_k\psi = [h(f_1(x_k), \dots, f_n(x_k))]\psi = h([f_1(x_k)]\psi, \dots, [f_n(x_k)]\psi) = h(c_k, \dots, c_{k+n-1})$ and A is V_n-free on the set X, φ can be uniquely extended to the homomorphism $\psi\colon A \to C$. $\square$

Corollary. *All V_2-free algebras on finite sets are isomorphic.*

In what follows a polynomial symbol will always mean a V_n-polynomial symbol.

Definition. The *lenght* of a polynomial symbol p is the number of occurences of variables and will be denoted by $l(p)$.

Definition. A polynomial symbol p is *minimal* if there exists no polynomial symbol q with the property $q = p$ and $l(q) < l(p)$.

The following lemmas and the theorem concern the word problem in the variety V_n.

Lemma 1. *Let $w_1, \dots, w_m$ be a sequence of polynomial symbols with the property $w_i = w_{i+1}$ for $i = 1, \dots, m-1$ and w_m is minimal. Then there exists a sequence $u_1, \dots, u_k$ of polynomial symbols such that $u_1 = w_1$, $u_k = w_m$, $u_j = u_{j+1}$ and $l(u_j) > l(u_{j+1})$ for $j = 1, \dots, k-1$.*

P r o o f. Note that by extending a polynomial symbol using one of the identities (i) and (ii), just one occurence of the polynomial symbol h arises. Let us denote this occurence by h^0. Let us assume that a polynomial symbol α is extended by using the identity (i). It means that instead of α we will have $h^0(f_1(\alpha), \dots, f_n(\alpha))$. Observe that if h^0 disappears by shortening this polynomial symbol using one of the identities (i) and (ii), the result is just the polynomial symbol α. Now let us assume that a polynomial symbol β_i is extended by using the identity (ii). It means that instead of β_i we will have $f_i h^0(\beta_1, \ldots, \beta_n)$. Observe that if h^0 disappears by shortening this polynomial symbol using the identity (ii), the result is just the polynomial symbol β_i, and if h^0 disappears by shortening the polynomial symbol using the identity (i), then $\beta_j = \beta_i$ for $j = 1, \ldots, n$ and the result is again just the polynomial symbol β_i. We can see that the last extending of a polynomial symbol is useless, and this can be shown for all the extensions. □

Denote by $s_1(\alpha)$ a polynomial symbol which is obtained from the polynomial symbol α by one shortening using the identity (i), and denote by $s_2(\alpha)$ a polynomial symbol which is obtained from the polynomial symbol α by one shortening using the identity (ii).

Lemma 2. *Let α be an arbitrary polynomial symbol. If each of φ and ψ is denoting for s_1 or s_2, then there exists a polynomial symbol which can be obtained by shortening both $\varphi(\alpha)$ and $\psi(\alpha)$.*

Proof. Let β be the shortest subword of α which is changed by applying φ to $\varphi(\beta)$ and let γ be the shortest subword of α which is changed by applying ψ to $\psi(\gamma)$. If β and γ are disjoint subwords, then the assertion of the lemma is trivial. Assume that γ is a subword of β. If $\beta = h(f_1(\xi), \ldots, f_n(\xi))$, then $\varphi(\beta) = \xi$ and γ is a subword of ξ. In this case $\psi\varphi(\alpha) = \varphi\lambda\psi(\alpha)$ where λ denotes applying ψ to each other argument of h in the word $\psi(\beta)$. If $\beta = f_i h(\xi_1, \ldots, \xi_n)$ and γ is a subword of ξ_i, then $\psi\varphi(\alpha) = \varphi\psi(\alpha)$ and if $\beta = f_i h(\xi_1, \ldots, \xi_n)$ and γ is not a subword of ξ_i, then $\varphi(\alpha) = \varphi\psi(\alpha)$. □

Theorem 2. *For each polynomial symbol α there exists a unique minimal polynomial symbol μ which can be obtained by shortening α with the property $\mu = \alpha$.*

Proof. The proof can be carried out by induction on the length k of the polynomial symbol. For $k = 1$ the assertion is trivial. Let the assertion be true for all $k \leqslant m - 1$, we show that the assertion is true for $k = m$. Let β be an arbitrary polynomial symbol with $l(\beta) = m$. Let β_1 and β_2 be such polynomial symbols which can be obtained by shortening β using one of the identities (i) and (ii). According to the induction hypothesis for each of the polynomial symbols β_1 and β_2 there exists a unique minimal polynomial symbol μ_1 and μ_2, respectively, for which $\mu_1 = \beta_1$ and $\mu_2 = \beta_2$. According to Lemma 2 there exists a polynomial symbol γ which is a shortening of both β_1 and β_2 and for which $\gamma = \beta_1$ and $\gamma = \beta_2$. According to Lemma 1 there exists a unique minimal polynomial symbol $\mu = \mu_1 = \mu_2$ for which $\mu = \beta_1$ and $\mu = \beta_2$, consequently for α there exists a unique minimal polynomial symbol μ for which $\mu = \alpha$. □

Corollary. *The word problem for the variety V_n is solvable.*

References

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