Czechoslovak Mathematical Journal

Michael R. Darnel

Cyclic extensions of the Medvedev ordered groups

Czechoslovak Mathematical Journal, Vol. 43 (1993), No. 2, 193–204

Persistent URL: http://dml.cz/dmlcz/128399

Czechoslovak Mathematical Journal, 43 (118) 1993, Praha

CYCLIC EXTENSIONS OF THE MEDVEDEV ORDERED GROUPS

MICHAEL R. DARNEL,* South Bend

(Received April 30, 1991)

SECTION ONE: INTRODUCTION AND BACKGROUND

An *ℓ-variety* is a class of lattice-ordered groups defined by a set of equations. Any ℓ-group law can be expressed in the form "$w(\vec{x}) = e$," where $w(\vec{x})$ is an element of the free ℓ-group on a countable set X of free generators; $w(\vec{x})$ has, then, a (nonunique) standard form $w(\vec{x}) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \prod_{k=1}^{p} x_{ijk}^{\varepsilon_{ijk}}$, where $\varepsilon_{ijk} = \pm 1$ and $x_{ijk} \in X \cup \{e\}$. An ℓ-group G *satisfies* "$w(\vec{x}) = e$" if for any mapping of X into G, letting g_{ijk} be the image of x_{ijk}, $w(\vec{g}) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \prod_{k=1}^{p} g_{ijk}^{\varepsilon_{ijk}} = e$.

Weinberg [W] showed that the ℓ-variety $\mathscr{A}$ of abelian ℓ-groups is the smallest nontrivial ℓ-variety. Since $\mathscr{A}$ is finitely based, any ℓ-variety properly containing $\mathscr{A}$ contains an ℓ-variety minimal with respect to properly containing $\mathscr{A}$, called a *cover* of $\mathscr{A}$. Scrimger [Sc] proved the existence of countably infinitely many solvable covers of $\mathscr{A}$, one for each prime integer p, known now as the Scrimger covers $\mathscr{S}_p$. These ℓ-varieties were generated by ℓ-groups that are not *representable*: i.e., not representable as subdirect products of totally ordered groups. Subsequently, Gurchenkov–Kopytov [GK], Reilly [R1], and Darnel [D1] showed that the Scrimger covers were the only nonrepresentable covers of $\mathscr{A}$. Medvedev [M] proved the existence of three solvable representable covers of $\mathscr{A}$. Of these, one, herein denoted $\mathscr{M}^0$, is generated by the free nil-2 group on two generators a and b, where if $c = [a, b]$, any element is of the (unique) form $a^k b^m c^n$, ordered lexicographically from the left by k, m, and n.

Describing the other Medvedev covers requires more explanation. Let A and B be totally ordered groups. The restricted wreath product $A \text{ wr } B$ can be ordered in

---

* This research was done while the author was a Visiting Lecturer at the Department of Mathematics and Statistics at Bowling Green State University, Bowling Green, Ohio, 1990-91. The author thanks the Department for it warm welcome and hospitality.

two ways. For $g \in A \text{ wr } B$, $g = (\hat{f}, b)$, where $\hat{f}\colon B \to A$ has finite support. Define $(\hat{f}, b) > (e, e)$ if $b > e$ or $b = e$ and $\hat{f}(b_0) > e$ for $b_0 = \max(\text{supp}(\hat{f}))$. This gives an o-group denoted by $A \overleftarrow{\text{wr}} B$. $A \overrightarrow{\text{wr}} B$ is defined analogously, except $(\hat{f}, b) > (e, e)$ if $b > e$ or if $b = e$ and $\hat{f}(b_1) > e$ for $b_1 = \min(\text{supp}(\hat{f}))$. One Medvedev cover, denoted $\mathscr{M}^+$, is generated by $\mathbf{Z} \overleftarrow{\text{wr}} \mathbf{Z}$, where $\mathbf{Z}$ is the group of integers with the usual order, and the order by $\mathbf{Z} \overrightarrow{\text{wr}} \mathbf{Z}$.

Two other representable covers of $\mathscr{A}$ are known at this time, based on orderings of the free group of rank two. Bergman [B] and Kopytov [K] independently proved the existence of one of these, $\mathscr{B}^+$, and by reversing the order, Kopytov [K] obtained the other, $\mathscr{B}^-$.

The lattice of all ℓ-varieties is distributive, and thus if $\mathscr{U}$ and $\mathscr{V}$ are covers of $\mathscr{A}$, then $\mathscr{U} \vee \mathscr{V}$ covers both $\mathscr{U}$ and $\mathscr{V}$. Gurchenkov [Gu1] proved that all ℓ-varieties have covers. Presently much more is known about the ℓ-varieties containing the Scrimger covers that those containing the Medvedev covers. Indeed, Holland and Reilly [HR] and Gurchenkov [Gu2] independently described all ℓ-metabelian ℓ-varieties whose intersections with the ℓ-variety $\mathscr{R}$ of representable ℓ-groups is the abelian ℓ-variety $\mathscr{A}$. (An ℓ-group G is *ℓ-metabelian* if there exists a convex ℓ-subgroup $A \triangleleft G$ such that A and G/A are abelian. In this case, as indeed for all ℓ-groups, there exists a unique largest abelian convex ℓ-subgroup [H] called the *abelian radical* and which is denoted by $\mathscr{A}(G)$. G is thus ℓ-metabelian if and only if $G/\mathscr{A}(G) \in \mathscr{A}$.)

Darnel [D2] showed that $\mathscr{M}^+$ is contained in the ℓ-variety $\mathscr{T}^+$ generated by all ℓ-metabelian o-groups G having the *positive infinite shifting property*: for any $e < h \in \mathscr{A}(G)$ and $e < g \in G \setminus \mathscr{A}(G)$, $g^{-1}hg \gg h$; and produced laws for $\mathscr{T}^+$. From these laws, results due to Huss [Hu] and Reilly [R2] that $\mathscr{M}^+$ is not closed with respect to lex extensions by the ordered group of integers will be proven in Section Three.

A *convex ℓ-subgroup* C of an ℓ-group G is a sublattice and a subgroup with the property that if $e \leqslant x \leqslant c \in C$, then $x \in C$. The lattice order of G induces a lattice order on the set of right cosets $\mathscr{R}_G(C)$ of C by $Cx \vee Cy = C(x \vee y)$. A convex ℓ-subgroup is *prime* if $x \wedge y \in C$ implies $x \in C$ or $y \in C$; this is equivalent to $\mathscr{R}_G(C)$ being totally ordered. Note a convex ℓ-subgroup P is prime if and only if for convex ℓ-subgroups A and B, $P \subset A$ and $P \subset B$ implies $P \subset A \cap B$.

Section Two: O-groups of ℓ-Varieties Generated by Ordered Groups

The ℓ-variety generated by a class $\mathscr{C}$ of ℓ-groups is the class of all ℓ-groups G that are ℓ-homomorphic images of ℓ-subgroups of cardinal products of numbers of $\mathscr{C}$. Thus if $\mathscr{C}$ is a collection of o-groups, any o-group G in ℓ-Var($\mathscr{C}$) is the ℓ-homomorphic image of an ℓ-subgroup S of a cardinal product $\Pi_\Lambda G_\lambda$ of o-groups $\{G_\lambda\} \subseteq \mathscr{C}$ by a prime subgroup P of S. While P is always the intersection of a prime Q of $\Pi_\Lambda G_\lambda$ with S, in general $\mathscr{S}$ need not be contained in the normalizer $N_\Pi(Q)$ of Q in $\Pi_\Lambda G_\lambda$ and so we can not in general substitute SQ for S and Q for P.

Proposition 2.1. *Let G be a representable ℓ-group, S be an ℓ-group of G, and P be a prime of S. If $P \triangleleft S$, then there exists a prime subgroup Q of G such that $P = S \cap Q$, S is contained in the normalizer $N_G(Q)$, and $Q \triangleleft SQ$. In this case, $SQ/Q \cong S/P$.*

Proof. We start by replicating from [C] that there is always a prime subgroup Q of G such that $S \cap Q = P$.

Let $\mathscr{B} = \{C \in \mathscr{C}(G) \colon C \cap S = P\}$. $\mathscr{B} \neq \emptyset$ as the convex ℓ-subgroup of G generated by P is in $\mathscr{B}$. Let $\mathscr{C}$ be a chain in $\mathscr{B}$. Then $P \subseteq S \cap \bigcup \mathscr{C}$. Suppose there exists $g \in (\bigcup \mathscr{C} \cap S) \setminus P$. Then $g \in C \in \mathscr{C}$ and $g \in S$, implying $g \in C \cap S = P$. Thus $\mathscr{B}$ has maximal elements; let Q be one.

Now suppose $e = a \wedge b$ where $a, b \in G \setminus Q$. Then $Q \subset T = G(Q, a)$ and $Q \subset R = G(Q, b)$. So $P \subset S \cap T$ and $P \subset S \cap R$. But since P is prime in S, $P \subset (S \cap T) \cap (S \cap R) = S \cap (T \cap R) = S \cap Q = P$, an obvious contradiction. So Q is prime in G.

Now suppose that G is representable and that $P \triangleleft S$. Suppose by way of contradiction that $S \not\subseteq N_G(Q)$. Choose $s \in S$ such that $s^{-1}Qs \neq Q$. Since G is representable, either $Q \subset s^{-1}Qs$ or $Q \subset sQs^{-1}$.

Assume $Q \subset s^{-1}Qs$. Then $P \subset S \cap s^{-1}Qs = s^{-1}(S \cap Q)s = s^{-1}Ps = P$ since $P \triangleleft S$. So $S \subseteq N_G(Q)$ which is an ℓ-subgroup of G ([BKW, p. 77], [Mc], [R1], and [D1]). The rest follows from the Second Isomorphism Theorem. □

Proposition 2.1 allows us to consider only o-groups arising from quotients of ℓ-subgroups of a cardinal product of o-groups by prime subgroups of that product. We can further specify those primes.

Since the intersection of prime subgroups is prime, every prime subgroup contains a minimal prime subgroup. Conrad and McAlister [CM] showed that the minimal prime subgroups M of a cardinal product $\Pi_\Lambda G_\lambda$ of o-groups $\{G_\lambda\}$ are in a one-to one correspondence with the set of ultrafilters $\mathscr{U}$ on Λ; the correspondence is, for M, $\mathscr{U}_M = \{\{\lambda \in \Lambda;$ there exists $g \in M$ such that $g_\lambda = e\}\}$, and for $\mathscr{U}$, $M_{\mathscr{U}} = \{g \in \Pi_\Lambda G_\lambda \colon \{\lambda \colon g_\lambda = e\} \in \mathscr{U}\}$. Since minimal primes are normal in representable ℓ-groups, this means that o-groups in ℓ-varieties generated by o-groups arise in a very natural way from quotients of ℓ-subgroups of ultraproducts of the generating o-groups.

In the Introduction, $\mathscr{M}^+$ was mentioned as being contained in the ℓ-variety $\mathscr{T}^+$ generated by all ℓ-metabelian o-groups G having the property that for any $e < h \in \mathscr{A}(G)$ and $e < g \in G \setminus \mathscr{A}(G)$, $g^{-1}hg \gg h$. In [D2], it was shown that laws for $\mathscr{T}^+$ are:

Proposition 3.1. *$\mathscr{T}^+$ is defined by the laws:*
(i) $y^{-1}x_+y \wedge x_- = e$
(ii) $\big[|w_1| \wedge |[x_1, y_1]|, |w_2| \wedge |[x_2, y_2]|\big] = e$
(iii) *for* $e \leqslant u \leqslant v$, $v^{-1}|[a,b]|v \geqslant u^{-1}|[a,b]|u$
(iv)* $|[x_3, y_3, t]| \wedge \big|(|[x_3, y_3]|^{|z|}|[x_3, y_3]|^{-2} \wedge e)\big| \wedge \big|(|z|^{|t|}|z|^{-2} \wedge e)\big| = e$
(v-n) *for* $e \leqslant y \leqslant x$, $x^{-1}(|[x,y]| \wedge |w|)x \geqslant (|[x,y]| \wedge |w|)^n$, $n = 1, 2, 3, \ldots$.

Containment of $\mathscr{M}^+$ in $\mathscr{T}^+$ is obvious since $\mathbf{Z} \overleftarrow{\mathrm{wr}} \mathbf{Z}$ is an ℓ-metabelian o-group hasving the infinite shofting property.

These laws also show that $\mathscr{M}^+$ is not closed with respect to lex extensions involving the group of integers.

Proposition 3.2. a) (Huss [Hu]) $(\mathbf{Z} \overleftarrow{\mathrm{wr}} \mathbf{Z})\overleftarrow{x} Z \notin \mathscr{M}^+$,
b) (Reilly [R2]) $\mathbf{Z}\overleftarrow{x}(\mathbf{Z} \overleftarrow{\mathrm{wr}} \mathbf{Z}) \notin \mathscr{M}^+$.

Proof. (a) for $G = (\mathbf{Z} \overleftarrow{\mathrm{wr}} \mathbf{Z})\overleftarrow{x}\mathbf{Z}$, any element is of the form $(\hat{f}, m, n)$, where $\hat{f}\colon \mathbf{Z} \to \mathbf{Z}$ and $m, n \in \mathbf{Z}$. Then $\mathscr{A}(G) = \{(\hat{f}, m, n)\colon m = n = 0\}$. So if $h = \hat{\chi}_0$, the characteristic function of $\{0\}$, and $g = (\hat{0}, 0, 1)$, then $g^{-1}hg = h$ is not infinitely greater than h.

(b) For $H = \mathbf{Z}\overleftarrow{x}(\mathbf{Z} \overleftarrow{\mathrm{wr}} \mathbf{Z})$, elements of the form $(m, \bar{f}, n)$, where $m, n \in \mathbf{Z}$ and $\bar{f}\colon \mathbf{Z} \to \mathbf{Z}$. $\mathscr{A}(H)$ is then $\{(m, \bar{f}, n)\colon n = 0\}$. If $h = (1, \bar{0}, 0)$ and $g = (0, \bar{0}, 1)$, then $g^{-1}hg = g$. □

There is a third way to totally order $(\mathbf{Z}\,\mathrm{wr}\,\mathbf{Z}) \oplus \mathbf{Z}$. Define $\big((\hat{f}, n), m\big) \in (\mathbf{Z}\,\mathrm{wr}\,\mathbf{Z}) \oplus \mathbf{Z}$ to be positive if $n > 0$, if $n = 0$ and $m > 0$, or if $n = m = 0$ and $\hat{f}(k) > 0$ for $k = \max(\mathrm{supp}(\hat{f}))$. We will denote this o-group as $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}\,(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$, the subscript to denote that the wreath action is done by the upper component of $\mathbf{Z}\overleftarrow{x}\mathbf{Z}$ while the lower component has a trivial action. To be consistent with the order on $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}\,(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$, we will write an element $\big((\hat{f}, n), m\big)$ as $(\hat{f}, m, n)$.

Proposition 3.3. $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}\,(\mathbf{Z}\overleftarrow{x}\mathbf{Z}) \notin \mathscr{M}^+$.

Proof. Let $g = (\hat{0}, 0, 1)$ and $h = (\hat{0}, 1, 0)$. Then h is in the abelian radical while g is not, and $g^{-1}hg = h$. □

---

* Due to Andrew Glass

Remarks. This author originally had a proof based on ultrapowers of $\mathbf{Z}\overleftarrow{\mathrm{wr}}\mathbf{Z}$ that showed $\mathbf{Z}\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z}) \notin \mathscr{M}^+$. A. M. W. Glass then devised law (iv) of Proposition 3.1 that simultaneously excluded $\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})$, $(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})\overleftarrow{x}\mathbf{Z}$, and $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$ from $\mathscr{M}^+$.

The following proposition, whose proof will be left to the reader, describes $\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})$, $(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})\overleftarrow{x}\mathbf{Z}$, and $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$ in terms of generators.

Proposition 3.4. *Let G be an o-group generated by elements a, b, and c.*

a) *If $e < c \ll b \ll a$, $[a,c] = [c,b] = [b^{a^m}, b^{a^n}] = e$ for all integers m and n, and if $b \ll b^a$, then $G \cong \mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})$.*

b) *If $e < c \ll b \ll a$, $[a,c] = [a,b] = [c^{b^m}, c^{b^n}] = e$ for all integers m and n, and if $c \ll c^b$, then $G \cong (\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})\overleftarrow{x}\mathbf{Z}$.*

c) *If $e < c \ll b \ll a$, $[a,b] = [b,c] = [c^{a^m}, c^{a^n}] = e$ for all integers m and n, and if $c \ll c^a$, then $G \cong \mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$.*

Definition. $\mathscr{V}_b^+$ will be the ℓ-variety generated by $\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})$, $\mathscr{V}_m^+$ the ℓ-variety generated by $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$, and $\mathscr{V}_t^+$ the ℓ-variety generated by $(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})\overleftarrow{x}\mathbf{Z}$.

Our goal is now to show that $\mathscr{V}_b^+$, $\mathscr{V}_m^+$, and $\mathscr{V}_t^+$ are distinct ℓ-varieties and to discuss their containments. For the discussion, recall that elements g and h of an ℓ-group are *a-equivalent* if there exist positive integers m and n such that $|h| \leqslant |g|^m$ and $|g| \leqslant |h|^n$.

Proposition 3.5. $\mathscr{V}_m^+ \subset \mathscr{V}_t^+$.

Proof. In $(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})\overleftarrow{x}\mathbf{Z}$, let $a = (\hat{0},1,0)$, $b = (\hat{0},0,1)$, and $c = (\hat{\chi}_0,0,0)$. Then $b^{-1}cb = c$, while $a^{-1}ca(\hat{\chi}_1,0,0)$.

Note $(ab)^{-1}c(ab) = a^{-1}ca$ and ab is a-equivalent to b.

Consider the elements $\hat{a} = (a, b, (a,b)^2, (a,b)^3, \ldots)$, $\hat{b} = (b,b,b,\ldots)$, and $\hat{c} = (c,c,c,\ldots)$ of $\prod_{n=0}^{\infty}[(\mathbf{Z}\overleftarrow{\mathrm{wr}}\mathbf{Z})\overleftarrow{x}\mathbf{Z}]$. For any nonprincipal ultrafilter $\mathscr{U}$ on $\omega = \{0,1,2,\ldots\}$, the images $\bar{a}$, $\bar{b}$, and $\bar{c}$ of $\hat{a}$, $\hat{b}$, and $\hat{c}$, respectively, in $[(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\mathbf{Z})\overleftarrow{x}\mathbf{Z}]^{\omega}/\mathscr{U}$, have the properties that $\bar{a} \gg \bar{b} \gg \bar{c}$, $(\bar{a})^{-1}\bar{c}(\bar{a}) \gg \bar{c}$, $(\bar{b})^{-1}\bar{c}(\bar{b}) = \bar{c}$, and $\bar{a}\bar{b} = \bar{b}\bar{a}$. So the ℓ-subgroup generated by $\bar{a}$, $\bar{b}$, and $\bar{c}$ is o-isomorphic to $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$. Thus $\mathscr{V}_m^+ \subseteq \mathscr{V}_t^+$.

To show $\mathscr{V}_{m^+} \subset \mathscr{V}_{t^+}$, it suffices to note that $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$ satisfies (iii) of Proposition 3.1 while $(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})\overleftarrow{x}\mathbf{Z}$ does not. □

Proposition 3.6. *$\mathscr{V}_b^+$ is incomparable to $\mathscr{V}_m^+$.*

Proof. For any integer n, it is easy to verify that $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$ satisfies (v-n) of Proposition 3.1 while $\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})$ does not. So $\mathscr{V}_b^+ \not\subseteq \mathscr{V}_m^+$.

The following law showing $\mathcal{V}_m^+ \not\subseteq \mathcal{V}_b^+$ is due to A. M. W. Glass and again replaces a proof by this author that used ultraproducts.

$\mathcal{V}_b^+$ satisfies the law:

$$\text{“}|[x,y,t]| \wedge \left|(|z|^{|t|}|z|^{-1} \wedge e)\right| \wedge \left|(|[x,y]|^{|z|}|[x,y]|^{-2} \wedge e)\right| \wedge \left|(|[x,y]|\,|z|^{-1} \wedge e)\right| = e.\text{”}$$

For in $\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\overleftarrow{\mathrm{wr}}\mathbf{Z})$, if $[x,y,t] \neq e$, then $[x,y] \neq e$ and t is of the form $(m_1, \hat{f}_1, n_1)$ where $n_1 \neq 0$. So if z is of the form $(m_2, 0, 0)$, then $\left|(|[x,y]|)|z|^{-1} \wedge e\right| = e$, while if z is of the form $(m_2, \hat{f}_2, 0)$, then $|z|^{|t|} \gg |z|$ and so $\left|(|z|^{|t|}|z|^{-2} \wedge e)\right| = e$. Finally if z is of the form $(m_2, \hat{f}_2, n_2)$, where $n_2 \neq 0$, then $|[x,y]|^{|z|} \gg |[x,y]|$ and so $\left|(|[x,y]|^{|z|}|[x,y]|^{-2} \wedge e)\right| = e$.

Now $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}\,(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$ does not satisfy this law as can be seen by using the substitution $x = (\hat{\chi}_0, 0, 0)$, $y = t = (\hat{0}, 0, 1)$, and $z = (\hat{0}, 1, 0)$. So $\mathcal{V}_m^+ \not\subseteq \mathcal{V}_b^+$. □

Proposition 3.7. $\mathcal{V}_b^+ \not\subseteq \mathcal{V}_t^+$.

Proof. It is easy to verify that $(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})\overleftarrow{x}\mathbf{Z}$ satisfies the law:

$$\text{“for } e \leqslant y \leqslant x, \quad (|[x,y]| \wedge |c|)^{x} \vee (|[x,y]| \wedge |c|)^{x^{-1}} \geqslant (|[x,y]| \wedge |c|)^{2}\text{”}$$

which fails in $\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})$. □

With the aid of two lemmas, we will show that if G is an o-group in $\mathcal{V}_b^+ \setminus \mathcal{M}^+$, then G contains a copy of $\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})$, which will then prove $\mathcal{V}_b^+$ covers $\mathcal{M}^+$.

Lemma 3.8. $\mathcal{V}_b^+ \cap \mathcal{T}^+ = \mathcal{M}^+$.

Proof. Let G be an o-group in $\mathcal{V}_b^+ \cap \mathcal{T}^+$; if $G \in \mathcal{A}$, then $G \in \mathcal{M}^+$. So assume G is not abelian. Since $G \in \mathcal{V}_b^+$, there exists a set Λ, an ℓ-subgroup S of $\Pi_\Lambda(\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z}))$, and prime subgroup P of $\Pi_\Lambda(\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z}))$ such that $G \cong S/P$. Let M be the minimal prime subgroup of $H = \Pi_\Lambda(\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z}))$ contained in P, and let $\mathcal{U}$ be the ultrafilter defined by $\mathcal{M}$.

Suppose $P < Ps \in \mathcal{A}(S/P)$. Then $s_\lambda = (m_\lambda, \hat{f}_\lambda, n_\lambda)$, where $m_\lambda, n_\lambda \in \mathbf{Z}$, and $\hat{f}_\lambda \colon \mathbf{Z} \to \mathbf{Z}$. Since S/P is not abelian, there exists $P < Ps < Pt \notin \mathcal{A}(S/P)$. $S/P \in \mathcal{T}^+$ implies $P[t^{-1}, s^{-1}] < Ps < P[t, s^{-1}]$ and hence $M[t^{-1}, s^{-1}] < Ms < Mt[t, s^{-1}]$, giving us that $\{\lambda \colon [t_\lambda^{-1}, s_\lambda^{-1}] < s_\lambda < [t_\lambda, s_\lambda^{-1}]\} \in \mathcal{U}$. Thus $\{\lambda \colon n_\lambda = 0 \text{ and } \hat{f}_\lambda > 0\} \in \mathcal{U}$. Clearly if $P < Ps \notin \mathcal{A}(S/P)$, $\{\lambda \colon n_\lambda > 0\} \in \mathcal{U}$.

Let $Q = \{g = (m_\lambda, \hat{f}_\lambda, n_\lambda) \in \Pi_\Lambda(\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})) \colon \{\lambda \colon \hat{f}_\lambda = \hat{0} \text{ and } n_\lambda = 0\} \in \mathcal{U}\}$; since $M \subset Q$, Q is prime and so is comparable to P because $M \subseteq P$. Suppose $P \subset Q$. Now from the proof of Proposition 2.1, we can assume that P is maximal with respect to $P \cap S$ being a prime subgroup of S, and so $P = P \cap S \subset Q \cap S$. Let $e < s \in (S \cap Q) \setminus P$. Then if $Ps \in \mathscr{A}(S/P)$, we have seen that $\{\lambda \colon \hat{f}_\lambda > 0$ and $n_\lambda = 0\} \in \mathscr{U}$, while if $Ps \notin \mathscr{A}(S/P)$, $\{\lambda \colon n_\lambda > 0\} \in \mathscr{U}$. So $Q \subseteq P$ and consequently S/P is an o-homomorphic image of S/Q.

Define $\sigma \colon \Pi_\Lambda(\mathbf{Z}\,\overleftarrow{\times}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})) \to \Pi_\Lambda(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z}) \colon (m_\lambda, \hat{f}_\lambda, n_\lambda)\sigma_\lambda = (\hat{f}_\lambda, n_\lambda)$. Then σ is an ℓ-homomorphism. Now for the diagram

$$\begin{array}{ccc} S/Q & & \Pi_\Lambda(\mathbf{Z}\,\mathrm{wr}\,\mathbf{Z})/\mathscr{U} \\ \alpha \big\uparrow & & \big\uparrow \beta \\ S & \xrightarrow[\sigma]{\qquad} & \Pi_\Lambda(\mathbf{Z}\,\mathrm{wr}\,\mathbf{Z}) \end{array}$$

with α and β natural. Clearly $Q\sigma \subseteq \operatorname{Ker}\beta$ and so σ lifts to an ℓ-homomorphism $\bar{\sigma} \colon S/Q \to \Pi_\Lambda(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})/\mathscr{U}$.

Let $t \in \operatorname{Ker}\beta$ and $s \in S \cap \{t\}\sigma^{-1}$. Then $\{\lambda \colon t_\lambda = (\hat{0}_\lambda, 0)\} \in \mathscr{U}$ and so $\{\lambda \colon s_\lambda = (m_\lambda, \hat{f}_\lambda, n_\lambda)$ with $\hat{f}_\lambda = \hat{0}_\lambda$ and $n_\lambda = 0\} \in \mathscr{U}$. Thus $s \in Q$ and so $\bar{\sigma}$ is an ℓ-isomorphism. □

Lemma 3.9. *For every positive integer n, $\mathcal{V}_b^+$ satisfies the law:*

$$\text{“}|[r,s,t]| \wedge \left|(|[a,b]|^{|z|}|[a,b]|^{-n} \wedge e)\right| \wedge \left|(|z|^{|t|}|z|^{-n} \wedge e)\right| \\ \wedge \left|(|[a,b]|\,|z|^{-1} \wedge e)\right| = e.\text{”}$$

Proof. Assume $[r,s,t] \neq e \neq [a,b]$ in $\mathbf{Z}\,\overleftarrow{\times}(\mathbf{Z}\,\overleftarrow{\mathrm{wr}}\,\mathbf{Z})$. If $|z| = (m, \hat{0}, 0)$, then $|[a,b]|\,|z|^{-1} > e$. If $|z| = (m, \hat{f}, 0)$, where $\hat{f} > 0$, then for any n, $|z|^{|t|} \gg |z|$. Finally, if $|z| = (m, \hat{f}, k)$, when $k > 0$, $|[a,b]|^{|z|} \gg |[a,b]|$. □

Theorem 3.10. *$\mathcal{V}_b^+$ covers $\mathcal{M}^+$.*

Proof. Let G be an o-group in $\mathcal{V}_b^+ \setminus \mathcal{M}^+$. Then there exist $e < h \in \mathscr{A}(G)$ and $e < g \in G \setminus \mathscr{A}(G)$ such that $g^{-1}hg$ is a-equivalent to h.

We show first that if $[a,b] \neq e$, then $h \ll |[a,b]|$. Suppose $e < |[a,b]| \leqslant h$. Since G is not nil-2, there exist $r,s,t \in G$ such that $[r,s,t] \neq e$. Then $t \in G \setminus \mathscr{A}(G)$; and since $G/\mathscr{A}(G)$ is abelian, $|g|^{|t|}|g|^{-n} < e$ for all $n \geqslant 2$. Likewise, since $g \gg |[a,b]|$, $|[a,b]|g^{-1} < e$ and so for the law of Lemma 3.9 to hold for $z = g$, $|[a,b]|^g > |[a,b]|^n$ for all $n \geqslant 0$. Thus $g^{-1}|[a,b]|g \gg |[a,b]|$.

So $|[a,b,g]| \neq e$. For $n \geqslant 2$ and any integer k, $h^{-k}|[a,b]|h^k|[a,b]|^{-n} < e$, while $g^{-1}(h^k)g$ being a-equivalent to h implies there exists $M > 0$ such that for all $n \geqslant M$, $(g^{-1}h^kg)h^{-n} < e$. So again for Lemma 3.9 to hold with $r = a$, $s = b$, $t = g$, and $z = h$, we must have $|[a,b]| > h^k$ for all k.

In particular, $[g,h]=e$, since $g^{-1}hg$ being a-equivalent to h implies either $[g,h] \ll h$ or $[g,h]$ is a-equivalent to h.

Thus the o-subgroup of G generated by h, $|[a,b]|$, and g is o-isomorphic to $\mathbf{Z} \overleftarrow{x} (\mathbf{Z} \overleftarrow{\mathrm{wr}} \mathbf{Z})$. □

The proof that $\mathscr{V}_m^+$ covers $\mathscr{M}^+$ is much the same except in one key step which will be pointed out later.

Lemma 3.11. $\mathscr{V}_m^+ \cap \mathscr{T}^+ = \mathscr{M}^+$.

Proof. Let G be an o-group in $\mathscr{V}_m^+ \cap \mathscr{T}^+$. If G is abelian, then $G \in \mathscr{M}^+$. Otherwise, for any $e < h \in \mathscr{A}(G)$ and $e < g \in G \setminus \mathscr{A}(G)$, $g^{-1}hg \gg h$.

$G \in \mathscr{V}_m^+$ implies, by Proposition 2.1, that there is a set Λ, an ℓ-subgroup S of $H = \Pi_\Lambda(\mathbf{Z} \overleftarrow{\mathrm{wr}}_{(0\times \mathbf{Z})} (\mathbf{Z} \overleftarrow{x} \mathbf{Z}))$, and a prime subgroup P of H such that $P \lhd S$ and $G \cong S/P$.

Let M be the minimal prime subgroup of H contained in P. Then $M \lhd S$ and since S/P is an o-homomorphic image of S/M, it suffices to show that $S/M \in \mathscr{M}^+$.

We first show $S/M \in \mathscr{T}^+$. To do so, we must show for any $M \leqslant Ms \in \mathscr{A}(S/M)$ and any $M < Mt \notin \mathscr{A}(S \setminus M)$, $Ms \ll Mt^{-1}st$.

Suppose by way of contradiction that $Pt \in \mathscr{A}(S/P)$. Since S/P is nonabelian, there exists $r \in S$ such that $P \leqslant Pt < Pr \notin \mathscr{A}(S/P)$. Since $S/P \in \mathscr{T}^+$, $Pt \ll Pr^{-1}tr$ and so $Mt \ll Mr^{-1}tr$. Now $\mathscr{A}(S/M) = Q/M$ for some prime $Q \lhd S$. Since S/M is ℓ-metabelian, S/Q is abelian and so there is $q \in Q$ such that $qt = r^{-1}tr$. Since $Mt \notin \mathscr{A}(S/M) = Q/M$, $M \leqslant M|q| \ll Mt \ll Mr^{-1}tr$, implying $Mqt \ll Mr^{-1}tr$ [BCD, Prop. 1.4], a contradiction since $qt = r^{-1}tr$. So $Mt \notin \mathscr{A}(S/M)$ implies $Pt \notin \mathscr{A}(S/M)$.

So $P \leqslant Ps \ll Pt$ implies $Ps \ll Pt^{-1}st$, and hence $Ms \ll Mt^{-1}st$.

Let $\mathscr{U}$ be the ultrafilter on Λ such that $M = \{g \in H \colon \{\lambda \colon g_\lambda = e\} \in \mathscr{U}\}$. For $s \in S$, s_λ will be written $(\hat{f}_\lambda, m_\lambda, n_\lambda)$ as before.

$M < Ms \notin \mathscr{A}(S/M)$ implies $\{\lambda \colon n_\lambda > 0\} \in \mathscr{U}$, as if $\{\lambda \colon n_\lambda = 0\} \in \mathscr{U}$, then $Ms \in \mathscr{A}(H/M)$ and so is in $\mathscr{A}(S/M)$.

On the other hand, $M < Ms \in \mathscr{A}(S/M)$ implies that for any $M < Mt \in (S/M) \setminus \mathscr{A}(S/M)$, $Ms \ll Mt^{-1}st$; so $Ms < M[t, s^{-1}]$ and thus $\{\lambda \colon s_\lambda < [t_\lambda, s_\lambda^{-1}]\} \in \mathscr{U}$. Thus $M \leqslant Ms \in \mathscr{A}(S/M)$ implies $\{\lambda ; m_\lambda = n_\lambda = 0\} \in \mathscr{U}$.

Define σ; $\Pi_\Lambda(\mathbf{Z} \overleftarrow{\mathrm{wr}}_{(0\times \mathbf{Z})} (\mathbf{Z} \overleftarrow{x} \mathbf{Z})) \to \Pi_\Lambda(\mathbf{Z} \overleftarrow{\mathrm{wr}} \mathbf{Z})$ by $(\hat{f}_\lambda, m_\lambda, n_\lambda)\sigma_\lambda = (\hat{f}_\lambda, n_\lambda)$. σ is a group homomorphism but, even on S, need not be an ℓ-homomorphism. Consider, though, the diagram:

$$\begin{array}{ccc} S/M & & \Pi_\Lambda(\mathbf{Z}\ \mathrm{wr}\ \mathbf{Z})/\mathcal{U} \\ \alpha\Big\uparrow & & \Big\uparrow\beta \\ S & \xrightarrow{\qquad\qquad} & \Pi_\Lambda(\mathbf{Z}\ \mathrm{wr}\ \mathbf{Z}) \\ & \sigma & \end{array}$$

where α and β are natural. Then $M\sigma \subseteq \operatorname{Ker}\beta$ and so σ lifts to a homomorphism $\bar{\sigma}$ from S/M into $(\mathbf{Z}\ \overleftarrow{\mathrm{wr}}\ \mathbf{Z})^\Lambda/\mathcal{U}$.

Now $t = (\ldots,(\hat{f}_\lambda, n_\lambda),\ldots) \in \operatorname{Ker}\beta$ implies $\{\lambda\colon (\hat{f}_\lambda, n_\lambda) = (\hat{0},0)\} \in \mathcal{U}$. So $s = (\ldots,(\hat{f}_\lambda, m_\lambda, n_\lambda),\ldots) \in t\bar{\sigma}^{-1}$ implies $Ms \in \mathscr{A}(S/M)$. Since then $\{\lambda\colon m_\lambda = 0\} \cap \{\lambda\colon \hat{f} = \hat{0}$ and $n_\lambda = 0\} \in \mathcal{U}$, $Ms = M$ and thus $\bar{\sigma}$ is an isomorphism.

$Ms > M$ implies, if $Ms \notin \mathscr{A}(S/M)$, $\{\lambda\colon n_\lambda > 0\} \in \mathcal{U}$ and so $(Ms)\bar{\sigma}$ is positive in $(\mathbf{Z}\ \overleftarrow{\mathrm{wr}}\ \mathbf{Z})^\Lambda/\mathcal{U}$ or, if $Ms \in \mathscr{A}(S/M)$, $\{\lambda\colon n_\lambda = m_\lambda = 0$ and $\hat{f}_\lambda \geqslant \hat{0}\} \in \mathcal{U}$ and again $(Ms)\bar{\sigma}$ is positive. So $\bar{\sigma}$ is an o-isomorphism and thus $S/M \in \mathscr{M}^+$. □

Lemma 3.12. *For every positive integer n, $\mathscr{V}_m^+$ satisfies*

$$\text{“}|[r,s,t]| \wedge \big|(|[x,y]|^{|t|}|[x,y]|^{-n} \wedge e)\big| = e.\text{”}$$

P r o o f. Suppose $[r,s]$, $[r,s,t]$, and $[x,y]$ are all nonidentity elements of $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$. Then $t = (\hat{f}_1, m, n)$ where $n \neq 0$ since $[r,s,t] \neq e$ and $|[x,y]| = (\hat{f}_2, 0, 0)$. Then $|[x,y]|^{|t|} \gg |[x,y]|$ and so the law is true. □

(The above laws were proposed by Reilly [R2] as part of a set of laws that might define $\mathscr{M}^+$.)

In showing that $\mathscr{V}_b^+$ covers $\mathscr{M}^+$, we showed that any o-group $G \in \mathscr{V}_b^+ \setminus \mathscr{M}^+$ contains a copy of $\mathbf{Z}\overleftarrow{x}(\mathbf{Z}\ \overleftarrow{\mathrm{wr}}\ \mathbf{Z})$. Unfortunately, it is not true that every o-group $K \in \mathscr{V}_m^+ \setminus \mathscr{M}^+$ contains a copy of $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$. Indeed, let H be the o-subgroup of $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$ generated by $(\hat{0},0,1)$ and $(-\hat{\chi}_0,1,0)$. Then H does not contain a copy of $\mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$. We will, however, show that every o-group $K \in \mathscr{V}_m^+ \setminus \mathscr{M}^+$ contains a copy of H and so ℓ-Var(H) does cover $\mathscr{M}^+$. The next lemma shows that ℓ-Var$(H) = \mathscr{V}_m^+$.

Lemma 3.13. *Let $G = \mathbf{Z}\,\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$ and H be the ℓ-subgroup of G generated by $\bar{a} = (\hat{0},0,1)$ and $\bar{b} = (-\hat{\chi}_0,1,0)$. Then ℓ-Var$(H) = \mathscr{V}_m^+$.*

P r o o f. It suffices to show that if G does not satisfy an ℓ-group law “$w(\vec{x}) = e$,” neither does H.

So suppose $\vec{g} = \{g_{ijk}\}$ is a substitution for $w(\vec{x}) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \prod_{k=1}^{p} x_{ijk}^{\epsilon_{ijk}}$ into G such that $w(\vec{g}) \neq e$. Since G is totally ordered, we can, by using $w^{-1}(\vec{x})$ if necessary, assume $w(\vec{x}) > e$. Furthermore, if "$w(\vec{x}) = e$" is not a law for abelian ℓ-groups, it obviously is not a law for ℓ-Var(H) [W] and so we can assume $w(\vec{g}) = (\hat{f}, 0, 0)$.

Note that $g_{ijk} = (\hat{f}_{ijk}, m_{ijk}, n_{ijk})$.

For H, $[H, H]$ is freely generated as an abelian o-group by $\{[\bar{a}, \bar{b}]^{(\bar{a})^n} : n \in \mathbf{Z}\}$ and $[H, H]$ is convex in H. Thus any element of H can be written uniquely in the form $(\bar{f}, \bar{b}^m, \bar{a}^n)$, where $\bar{f} \in [H, H]$.

For $g_{ijk} = (\hat{f}_{ijk}, m_{ijk}, n_{ijk})$, $\hat{f}_{ijk} = \sum_{r \in \text{supp}(\hat{f}_{ijk})} c_{ijkr}\hat{\chi}_r$. Define $\bar{f}_{ijk}$ to equal $\sum_{r \in \text{supp}(\hat{f}_{ijk})} c_{ijk}(\hat{\chi}_r - \hat{\chi}_{r-1})$; then $\bar{f}_{ijk} = [\hat{f}_{ijk}^{-1a}, (\bar{a})^{-1}]$. Moreover, $[\bar{a}, \bar{b}] = \hat{\chi}_1 - \hat{\chi}_0$ and so $\hat{\chi}_r - \hat{\chi}_{r-1} = \bar{a}^{-(r-1)}[\bar{a}, \bar{b}]\bar{a}^{r-1} \in H$; hence $\bar{f}_{ijk} \in H$.

We first show that the substitution $t_{ijk} = (\bar{f}_{ijk}, m_{ijk}, n_{ijk})$ gives $w(\vec{t}) \neq e$. It is easily verified that if $w_{ij}(\vec{g}) = \prod_{k=1}^{p} (\hat{f}_{ijk}, m_{ijk}, n_{ijk})^{\epsilon_{ijk}} = (\hat{g}_{ij}, u_{ij}, v_{ij})$, then $\prod_{k=1}^{p} (\bar{f}_{ijk}, m_{ijk}, n_{ijk})^{\epsilon_{ijk}} = ([\hat{g}_{ij}^{-1}, (\bar{a})^{-1}], u_{ij}, v_{ij}) = (\bar{g}_{ij}, u_{ij}, v_{ij})$. Hence if $v_{ij} \neq 0$ and/or $u_{ij} \neq 0$, $(\bar{g}_{ij}, u_{ij}, v_{ij})$ is positive or negative as $(\hat{g}_{ij}, u_{ij}, v_{ij})$ is. Suppose, though, that $v_{ij} = u_{ij} = 0$ and $\hat{g}_{ij} \neq \hat{0}$; then supp$(\hat{g}_{ij})$ has a maximal element s_{ij}. Now for any s, $\bar{g}_{ij}(s) = \hat{g}_{ij}(s) - \hat{g}_{ij}(s+1)$. So $\bar{g}_{ij}$ is positive or negative as $\hat{g}_{ij}$ is. If $v_{ij} = u_{ij} = 0$ and $\hat{g}_{ij} = \hat{0}$, then $\bar{g}_{ij} = \hat{0}$ as well. So $w(\vec{t}) \neq e$.

Unfortunately, $\vec{t}$ may not be contained in H. A naïve substitution (that almost works) is to substitute for t_{ijk} the element $h_{ijk} = \bar{f}_{ijk}\bar{b}^{m_{ijk}}\bar{a}^{n_{ijk}} = (\bar{f}_{ijk} - m_{ijk}\hat{\chi}_0, m_{ijk}, n_{ijk})$. However, letting $\prod_{k=1}^{p} (m_{ijk}\hat{\chi}_0, m_{ijk}, n_{ijk})^{\epsilon_{ijk}} = (\hat{h}_{ij}, u_{ij}, v_{ij})$, it is easily verified that $\prod_{k=1}^{p} h_{ijk} = (\bar{g}_{ij} - \hat{h}_{ij}, u_{ij}, v_{ij})$, and so, if $s_{ij} = \max(\text{supp}(\bar{g}_{ij}))$, we might easily obtain that $\hat{h}_{ij}(s_{ij}) = \bar{g}_{ij}(s_{ij})$.

But $\hat{h}_{ij} = \sum_{r \in \text{supp}(\hat{h}_{ij})} c_{ijk}\hat{\chi}_r$. Now since $w(\vec{t}) \neq e$, then for any q, $(\bar{a})^{-q}w(\vec{t})(\bar{a})^q \neq e$, $(\bar{a})^{-q}w(\vec{t})\bar{a}^q = \bigvee_I \bigwedge_J ((\bar{a})^{-q}\bar{g}_{ij}(\bar{a})^q, u_{ij}, v_{ij})$, and $(\bar{g}_{ij})^{\bar{a}^q}$ has as its maximal support element $s_{ij}+q$. Thus if we choose q such that for all i and j, $s_{ij}+q > \max(\text{supp}(\hat{h}_{i'j'}))$ for all i' and j' and let $h'_{ijk} = (\bar{a})^{-q}\bar{f}_{ijk}(\bar{a})^q\bar{b}^{m_{ijk}}\bar{a}^{n_{ijk}}$, we obtain that $w(\vec{h'}) \neq e$. □

Theorem 3.14. *$\mathcal{V}_m^+$ covers $\mathcal{M}^+$.*

Proof. Let G be an o-group in $\mathscr{V}_m^+ \setminus \mathscr{M}^+$. Since $G \notin \mathscr{T}^+$ (by Lemma 3.11), there exist $e < h \in \mathscr{A}(G)$ and $e < g \in G \setminus \mathscr{A}(G)$ such that $g^{-1}hg$ is a-equivalent to h.

There exists $e < c < d \leqslant g$ such that $|[c,d]| \neq e$. So $g^{-1}|[c,d]|g \geqslant d^{-1}|[c,d]|d \gg |[c,d]|$. Now for any $e < a < b$ such that $[a,b] \neq e$, by Lemma 3.12, we have $|[c,d,g]| \wedge \big|(|[a,b]|^g|[a,b]|^{-n} \wedge e)\big| = e$, implying $g^{-1}|[a,b]|g \gg |[a,b]|$. We must have, then, that $h \gg |[a,b]|$ for any a and b, since if $h \ll |[a,b]|$ for some $e < a < b$, $b^{-1}hb = b^{-1}(h \wedge |[a,b]|)b \gg h \wedge |[a,b]| = h$, and thus $[b,h^{-1}] \gg h \geqslant [h^{-1},b^{-1}] > e$. So $g^{-1}[b,h^{-1}]g \gg g^{-1}hg \geqslant g^{-1}[h^{-1},b^{-1}]g \gg [h^{-1},b^{-1}]$ which is a-equivalent to h, a contradiction to $g^{-1}hg$ and h being a-equivalent. Hence $h \gg |[g,h]|$.

If $[g,h] = e$, then the o-subgroup of G generated by g, h, and $|[c,d]|$ is o-isomorphic to $\mathbf{Z}\ \overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}\ (\mathbf{Z}\overleftarrow{x}\mathbf{Z})$.

Suppose $[g,h] \neq e$; first assume $[g,h] > e$. Then the o-subgroup of G generated by g and h is o-isomorphic to the o-subgroup of $\mathbf{Z}\overleftarrow{\mathrm{wr}}_{(0\times\mathbf{Z})}(\mathbf{Z}\overleftarrow{x}\mathbf{Z})$ generated by $\bar{a} = (\hat{0},0,1)$ and $\bar{b} = (-\hat{\chi}_0,1,0)$. By Lemma 3.13, this o-subgroup generates $\mathscr{V}_m^+$.

If $[g,h] < e$, then

$$\begin{aligned}[g,[g,h]h] &= g^{-1}h^{-1}[g,h]^{-1}g[g,h]h = g^{-1}h^{-1}gg^{-1}[g,h]^{-1}gh[g,h]\\ &= g^{-1}h^{-1}ghg^{-1}g[g,h]^{-1}g[g,h] = [g,h]^2[g,h]^{-g} > e.\end{aligned}$$

So the o-subgroup generated by g and $[g,h]h$ is o-isomorphic to H. □

For the other Medvedev ℓ-variety $\mathscr{M}^-$, we likewise obtain $\mathscr{V}_b^-$ covering $\mathscr{M}^-$, $\mathscr{V}_m^-$ covering $\mathscr{M}^-$, $\mathscr{V}_b^- \neq \mathscr{V}_m^-$, and $\mathscr{V}_m^- \subset \mathscr{V}_t^-$. A surprising result, due to Huss [Hu] but with a proof simpler than hers, is:

Proposition 3.15. $\mathscr{V}_t^+ = \mathscr{V}_t^-$.

Proof. Let $H = \prod_{n=0}^{\infty} [(\mathbf{Z}\ \overleftarrow{\mathrm{wr}}\ \mathbf{Z})\overleftarrow{x}\mathbf{Z}]$ and let a be the element $((\hat{0},0,1),(\hat{0},0,2),(\hat{0},0,3),\ldots)$, b be the element $((\hat{0},0,1),(\hat{0},0,1),(\hat{0},0,1),\ldots)$, c be the element $((\hat{0},1,0),(\hat{0},1,0),(\hat{0},1,0),\ldots)$ and $d = ((\hat{\chi}_0,0,0),(\hat{\chi}_0,0,0),(\hat{\chi}_0,0,0),\ldots)$.

Let $\mathscr{U}$ be any nonprincipal ultrafilter on $\omega = \{0,1,2,\ldots\}$ and $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ be the respective images of a, b, c, and d in $H/\mathscr{U}$. Then $\hat{a} \gg \hat{b} \gg \hat{c} \gg \hat{d} > e$ and $\hat{a}$, $\hat{b}$ are central elements. Then the o-subgroup generated by $\hat{a}$, $\hat{b}(\hat{c})^{-1}$, and $\hat{d}$ is o-isomorphic to $(\mathbf{Z}\ \overleftarrow{\mathrm{wr}}\ \mathbf{Z})\overleftarrow{x}\mathbf{Z}$. □

Finally, we prove:

Proposition 3.16. $\mathscr{M}^0 \not\subseteq \mathscr{V}_t^+$.

Proof. We leave it to the reader to check that $(\mathbb{Z} \overleftarrow{\mathrm{wr}} \mathbb{Z}) \overleftarrow{\times} \mathbb{Z}$ satisfies the law:

$$\text{“for } e \leqslant y \leqslant x, \quad |[x,y]|^{x} \vee |[x,y]|^{x^{-1}} \geqslant |[x,y]|^2.\text{”}$$

Now for the free nil-2 o-group on generators a and b with $a^k b^m [a,b]^n \geqslant e$ if $k > 0$, $k = 0$ and $m > 0$, or $k = m = 0$ and $n \geqslant 0$, it is clear that $[a,b] = [a,b]^a \vee [a,b]^{a^{-1}} < [a,b]^2$. □

References

[B] *Bergman, G.*: Specially ordered Groups, Comm. Alg. *12* (1984), 2315–2333.

[BCD] *Ball, R. N.; Conrad, P. F.; Darnel, M. R.*: Above and below subgroups of a lattice-ordered group, Trans. Amer. Math. Soc. *259* (1980), 357–392.

[BKW] *Bigard, A.; Keimel, K.; Wolfenstein, S.*: Groupes et Anneaux Réticulés, Springer, 1977.

[C] *Conrad, P.*: Torsion radicals of lattice-ordered groups, Symposia Math. *21* (1977), 479–513.

[CM] *Conrad, P.; McAlister, D.*: The completion of a lattice-ordered group, J. Austral. Math. Soc. *9* (1969), 182–209.

[D1] *Darnel, M.*: Special-valued ℓ-groups and abelian covers, Order *4* (1987), 191–194.

[D2] *Darnel, M.*: Metabelian ordered groups with the infinite shifting property, in preparation.

[Gu1] *Gurchenkov, S. A.*: Coverings in the lattice of ℓ-varieties, Mat. Zametki *35* (1984), 677-684.

[Gu2] *Gurchenkov, S. A.*: Theory of varieties of lattice-ordered groups, Alg. i Logika *27(3)* (1988), 249–273.

[GK] *Gurchenkov, S. A.; Kopytov, V. M.*: On covers of the variety of abelian lattice-ordered groups, Siber. Math. J. *28* (1987).

[H] *Holland, W. C.*: Varieties of ℓ-groups are torsion classes, Czech. Math. J. *29(104)*, 11-12.

[HR] *Holland, W. C.; Reilly, N. R.*: Metabelian varieties of ℓ-groups which contain no non-abelian o-groups, Alg. Univ. *24* (1989), 203–204.

[Hu] *Huss, M.*: Varieties of lattice ordered groups, Ph.D. dissertation, Simon Fraser University, 1984.

[K] *Kopytov, V. M.*: Nonabelian varieties of lattice-ordered groups in which every solvable ℓ-group is abelian, Mat. Sb. *126(168)* (1985), 247–266, 287.

[Mc] *McCleary, S. H.*: The lateral completion of an arbitrary lattice-ordered group, Alg. Univ. *13* (1981), 251–263.

[M] *Medvedev, N. Ya.*: Lattices of varieties of lattice-ordered groups and Lie groups, Alg. i Logika *16* (1977), 40–45, 123.

[R1] *Reilly, N. R.*: Varieties of lattice ordered groups that contain no non-abelian o-groups are solvable, Order *3* (1986), 287–297.

[R2] *Reilly, N. R.*: personal communication to W. C. Holland.

[Sc] *Scrimger, E. B.*: A large class of small varieties of lattice-ordered groups, Proc. Amer. Math. Soc. *51* (1975), 301–306.

[W] *Weinberg, E.*: Free lattice-ordered abelian groups, II, Math. Ann. *154* (1965), 217–222.

Authors' address: Visiting (1990–1991): Department of Mathematics and Statistic, Bowling Green State University, Bowling Green, OH 43403-0221; Permanent: Department of Mathematics and Computer Science, Indiana University, South Bend, IN 46634.