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Petr Holický

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ČECH ANALYTIC AND ALMOST K-DESCRIPTIVE SPACES

PETR HOLICKÝ, Praha

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The notion of almost K-descriptive spaces was introduced by R.W. Hansell in [H_1] in connection with the study of Banach spaces and their weak topologies. The notion of Čech analytic spaces was introduced by D. Fremlin [F] (for basic facts see [JNR], [K]), and the connection with Banach spaces was investigated in [JNR]. We add here some observations. We show in Section 1 that all Čech analytic spaces are almost K-descriptive (Theorem 1), which enables us to use the technique of the descriptive set theory to get some properties of them, a separation principle, results on measurable images etc. In Section 2 we give some results concerning the class of all almost K-descriptive spaces. It is closed with respect to the Suslin operation and scattered unions (Propositions 2). We get a characterization of completely regular almost K-descriptive spaces in Theorem 3. We notice the possibility of extending a result of [JNR] from the case of Čech analytic Banach spaces to the case of almost K-descriptive spaces, which gives the equivalence of the classes of almost descriptive and almost K-descriptive spaces within the framework of Banach spaces and their weak topology. Hence, it follows from [JNR] and [H_1] that ℓ_∞ is not almost K-descriptive.

Throughout the paper all topological spaces are supposed to be Hausdorff.

1. Čech complete and Čech analytic spaces

We first recall that a topological space X is *Čech complete* if it is a G_δ subspace of a compact space. Thus X is automatically Hausdorff and completely regular. By [Fr_1] the space X is Čech complete if and only if it is completely regular and there is a *complete sequence of* open *covers* $\mathcal{U}_n$, $n = 0, 1, \ldots$, in X, i.e. any filter $\mathcal{F}$ in X which has non-empty intersections with $\mathcal{U}_n$ for each $n = 0, 1, \ldots$ (we say that $\mathcal{F}$ is a *Cauchy filter* with respect to $\mathcal{U}_n$), has an accumulation point.

A topological space X is called *Čech analytic* if it is in $\mathscr{S}(\mathscr{F} \cup \mathscr{G}) = \mathscr{S}(\mathscr{B})$ in some compact space, i.e. it is the result of the Suslin operation on sets from $\mathscr{F}$ (closed sets) and from $\mathscr{G}$ (open sets) or, equivalently, on Borel sets denoted by $\mathscr{B}$ here. It is shown in [JNR] that X is Čech analytic if and only if it is completely regular and there is a separable metric space M and a Čech complete space $Y \subset X \times M$ such that X is the projection of Y along M.

A family $\mathscr{U}$ of subsets of a topological space X is said to be a *scattered family* if it is disjoint and its elements can be well ordered by $<$ so that the union $\bigcup\{V \mid V < U\}$ is open in $\bigcup \mathscr{U}$ for every $U \in \mathscr{U}$. We notice that the complete sequence of open covers $\mathscr{U}_n$ can be refined to a sequence of scattered covers by making differences in some well-ordering of any $\mathscr{U}_n$, and that the refinements of complete sequence of covers form always a complete sequence again. We also see that a *set is scattered* if its cover by singletons is a scattered family. Countable unions of scattered sets (σ-scattered sets) in metric spaces coincide with σ-discrete (both in the metric and in the topology) sets (this follows e.g. from Lemma 3 below), while all subsets of the space $[0, \kappa]$ of ordinals $\alpha \geqslant 0$ and $\alpha \leqslant \kappa$ with its "order topology" are scattered, so e.g. $[0, \omega_1]$ is a scattered space which is not σ-discrete (topologically).

Now we refine the complete sequence of open covers in a regular topological space slightly more carefully.

Lemma 1. *Let X be a regular topological space and let $\mathscr{U}_n$ form a complete sequence of open covers. Then there is a sequence of disjoint covers $\mathscr{V}_n$ such that*

(a) *$\mathscr{V}_n$ refines $\mathscr{U}_n$ (thus $\mathscr{V}_n$ is complete);*

(b) *for $V_{n+1} \in \mathscr{V}_{n+1}$ there is a $V_n \in \mathscr{V}_n$ with $\overline{V}_{n+1} \subset V_n$ (thus $\mathscr{V}_{n+1}$ refines $\mathscr{V}_n$ and, moreover, the accumulation points of every filter which contains $V_n \in \mathscr{V}_n$, $n = 0, 1, \ldots$ are contained in $\bigcap_{n=0}^{\infty} V_n$);*

(c) *the elements of $\mathscr{V}_n$ may be well ordered by $<^{(n)}$ so that $\bigcup\{V \in \mathscr{V}_n \mid V <^{(n)} W\}$ is open (in X) for each $W \in \mathscr{V}_n$ (thus $\mathscr{V}_n$ are scattered).*

Proof. We put $\mathscr{V}_0 = \{X\}$. Let $\mathscr{V}_m$ and $<^{(m)}$ fulfilling (a), (b), (c) for $m \leqslant n$ be already established. Let $\prec^{(n)}$ be a well-ordering of X such that $x \in \bigcup\{V \in \mathscr{V}_n \mid V <^{(n)} W\}$ and $y \prec^{(n)} x$ imply that $y \in \bigcup\{V \in \mathscr{V}_n \mid V <^{(n)} W\}$ for any $W \in \mathscr{V}_n$. We now define $\mathscr{V}_{n+1}$ and $<^{(n+1)}$ by transfinite induction over $(X, \prec^{(n)})$. Let the sets $V_{n+1}(y)$ for $y \prec^{(n)} x$ be defined. We find $U_{n+1} \in \mathscr{U}_{n+1}$ such that $x \in U_{n+1}$ and (the only) $V_n(x) \in \mathscr{V}_n$ with $x \in V_n(x)$. We choose an open neighbourhood $W_{n+1}(x)$ of x such that $\overline{W_{n+1}(x)} \subset U_{n+1} \cap \bigcup\{V \in \mathscr{V}_n \mid V \leqslant^{(n)} V_n(x)\}$ and $V_{n+1}(x) = W_{n+1}(x) \setminus \bigcup\{V_{n+1}(z) \mid z \prec^{(n)} x\}$. We put $\mathscr{V}_{n+1} = \{V_{n+1}(x) \mid x \subset X\}$ and write $V <^{(n+1)} W$ whenever $V = V_{n+1}(x)$ and $W = V_{n+1}(y)$ for some $x, y \in X$ with $x \prec^{(n)}$

y. Obviously, the families $\mathscr{V}_0$, $\mathscr{V}_1$, ..., $\mathscr{V}_{n+1}$ (with the orderings $<^{(1)}, \dots, <^{(n+1)}$) fulfill (a), (b), (c). □

Since every Čech complete space admits a complete sequence of covers and is even completely regular, we may apply Lemma 1 to it. Our aim is to get some parametrization of Čech complete spaces. It is suitable to introduce some further notions concerning families of subsets of topological spaces.

An indexed family $(U_a \mid a \in A)$ in X is called *$\mathscr{P}\sigma$-decomposable* if there are $U_{a,n}$, $n = 0, 1, \dots$, such that $U_a = \bigcup\{U_{a,n} \mid n \in \omega\}$ and that the families $\{U_{a,n} \mid a \in A\}$ have the property $\mathscr{P}$. We use this notion for $\mathscr{P}$ being the class of all scattered families in the topological space X (s-σ-decomposable) or, especially, for $\mathscr{P}$ being the class of all disjoint families $\mathscr{D}$ with a scattered *base* $\mathscr{B}$ *in* X, which means that $\mathscr{B}$ is a scattered family in X which refines $\mathscr{D}$ and each element of $\mathscr{D}$ is the union of some elements of $\mathscr{B}$. We write sb_d $-\sigma$-decomposable in the latter case.

The property to be sb_d $-\sigma$-decomposable is equivalent to a notion used in [H_1] as the following lemma says.

Lemma 2. *The family $(X_a \mid a \in A)$ of subsets of a topological space X is sb_d $-\sigma$-decomposable if and only if it is point-countable, and there is a σ-scattered base for it.*

Proof. It is obvious that every sb_d $-\sigma$-decomposable family is point-countable and has a σ-scattered base. Now let $(X_a \mid a \in A)$ be point-countable with a σ-scattered base $\mathscr{B} = \bigcup_n \mathscr{B}_n$ with $\mathscr{B}_n$ scattered. Assign to every nonempty $B \in \mathscr{B}$ and $a \in A$ with $B \subset X_a$ a natural number $m(B, a)$ such that $m(B, a) \neq m(B, b)$ for $a \neq b$. Put $X_{anm} = \bigcup\{B \in \mathscr{B}_n \mid m(B, a) = m\}$. The family $\{X_{anm} \mid a \in A\}$ is disjoint and has a scattered base $\{B \in B_n \mid m(B, a) = m, a \in A\}$. □

Example from Section 2 of [H_1] shows that a disjoint family with scattered base need not be scattered σ-decomposable! However, we notice that the notions of sb_d $-\sigma$-decomposable and scattered σ-decomposable families coincide with metrically discrete σ-decomposable families in metric spaces.

Lemma 3. *Every sb_d $-\sigma$-decomposable indexed family $\mathscr{D}$ in the metric space X is metrically discrete σ-decomposable.*

Proof. It is enough to show it for a disjoint family $\mathscr{D}$ with a scattered base $\mathscr{B}$. Thus there is a well-ordering $<$ of $\mathscr{B}$ and there are open sets $G(B)$ for $B \in \mathscr{B}$ such that $B \subset G(B) \setminus \bigcup\{G(C) \mid C < B\}$. Put $B_n = \{x \in B \mid \varrho(x, X \setminus G(B)) \geqslant 1/n\}$ for $B \in \mathscr{B}$ and $D_n = \bigcup\{B_n \mid B_n \subset D\}$ for $D \in \mathscr{D}$. It follows that $\{D_n \mid D \in \mathscr{D}\}$ are $1/n$-discrete and obviously $\bigcup\{D_n \mid n \in \omega\} = D$. □

We say that the mapping (also multivalued) $f\colon X \to Y$ *preserves the indexed families from* $\mathcal{P}$, or that it is $\mathcal{P}$*-preserving*, if $(f(X_a) \mid a \in A)$ is from $\mathcal{P}$ in Y whenever $(X_a \mid a \in A)$ is in $\mathcal{P}$ in X. Lemma 3 implies that it is sufficient to verify that $(f(X_a) \mid a \in A)$ is $\mathrm{sb}_d - \sigma$-decomposable for discrete $(X_a \mid a \in A)$ only, whenever X is metric, to get that f is $\mathrm{sb}_d - \sigma$-decomposable preserving.

We also recall that a set valued map $f\colon X \to Y$ is called *upper semi-continuous* (usc) if $\{x \in X \mid f(x) \cap F \neq \emptyset\} \equiv f^{-1}(F)$ is closed for every closed $F \subset Y$. A set-valued map is called disjoint if $f(x) \cap f(y) = \emptyset$ for $x \neq y$.

Now we are already able to describe Čech complete spaces by a meaningful parametrization.

Lemma 4. *Let a topological space X admit a complete sequence of disjoint covers $\mathcal{V}_n$ fulfilling* (b), (c) *of Lemma* 1. *Then there is a complete metric space and a disjoint usc compact-valued (usc-K) map $f\colon M \to X$ such that $f(M) = X$ and f preserves $\mathrm{sb}_d - \sigma$-decomposable families (f is an almost K-descriptive parametrization).*

Remark. Due to Lemma 1 and the above Lemma 4 we get that every Čech complete space has the parametrization from Lemma 4. In fact, we get this for all regular "cover-complete" spaces as defined in [H_2]. Let us recall that Z. Frolík proved in [Fr_2] that paracompact Čech complete spaces are exactly those completely regular spaces which admit a perfect map onto a complete metric space, and notice that the inverse map to a perfect one is usc-K.

Proof of Lemma 4. We first choose a cardinal κ with cardinality of $\mathcal{V}_n$ less or equal to κ for $n = 0, 1, \ldots$. We consider a complete metric on the topological product space $M = \kappa^\omega$, with $\kappa = \{\alpha \mid \alpha < \kappa\}$ taken with the discrete topology, such that the diameter of $U_k(\alpha) = \{\beta \in M \mid (\beta_0, \ldots, \beta_k) = (\alpha_0, \ldots, \alpha_k)\}$ is less than ε_k with ε_k tending to zero. We may index each $\mathcal{V}_n$ by elements of κ so that $\mathcal{V}_n = \{V_n(\alpha) \mid \alpha < \kappa\}$ and put $V_n(\alpha) = \emptyset$ if necessary. We define a map $f\colon M \to X$ by $f(\alpha_0, \alpha_1, \ldots) = \bigcap_{n=0}^{\infty} V_n(\alpha_n)$.

The set $\{\alpha \mid f(\alpha) = \emptyset\} = \bigcup_{k=0}^{\infty} \{\alpha \mid \bigcap_{n=0}^{k} V_n(\alpha_n) = \emptyset\}$ is open in M. If $f(\alpha) \neq \emptyset$ then $f(\alpha) = \bigcap_{n=0}^{\infty} \overline{V_n(\alpha_n)}$, thus $f(\alpha)$ is closed, equals $\bigcap_{n=0}^{\infty} V_n(\alpha_n)$ and every filter in $\bigcap_{n=0}^{\infty} V_n(\alpha_n)$ has an accumulation point. Thus $f(\alpha)$ is compact. Let $f(\alpha) \subset G$ for some open G. For any neighbourhood $U_k(\alpha)$ of α let us have $f(U_k(\alpha)) \cap G^c \neq \emptyset$. Then $W = V_0(\alpha_0) \cap \ldots \cap V_k(\alpha_k) \cap G^c \neq \emptyset$ for every $k \in \omega$. W_k is a filter base of a filter which intersects every $\mathcal{V}_n$ and thus has an accumulation point $x \in \bigcap_{k=0}^{\infty} \overline{W}_k \subset f(\alpha)$

which is in the closed set G^c. This is a contradiction, and f is usc. Obviously, f is disjoint.

A σ-discrete base for open sets in M is formed by the sets $U_k(\alpha)$, $k = 0, 1, \ldots, \alpha \in M$. The family $(f(U_k(\alpha)) \mid k \in \omega, \alpha \in M)$ is obviously σ-scattered. Let $(M_a \mid a \in A)$ be a metrically discrete family in M, i.e. there is an n such that every $U_n(\alpha)$ intersects at most one element of $\{M_a \mid a \in A\}$. The family $(f(M_a) \mid a \in A)$ thus refines some disjoint unions of elements from $\{f(U_n(\alpha))\}$. It has therefore a scattered base formed by $f(U_n(\alpha) \cap M_a)$. □

To get a parametrization of Čech analytic spaces, we recall the characterization by a projection along a separable metric space and add a lemma which generalizes Lemma 7.1 from [H₁]. We shall need the generalization for the proof of Theorem 4 later.

Lemma 5. *Let Y have a σ-scattered network, i.e. a σ-scattered family $\mathcal{V}$ such that $\bigcup\{V \in \mathcal{V} \mid V \subset G\} = G$ for every open subset of Y. Let $f\colon Y \to X$ be a set-valued map (to subsets of X) which preserves $sb_d - \sigma$-decomposable families. Then the projection $\pi\colon X \times Y \to X$ restricted to the graph G of f preserves $sb_d - \sigma$-decomposable families.*

We may notice immediately that the assertion of Lemma 7.1 of [H₁] follows by putting $f(y) = X$ for $y \in Y$:

Corollary. *Let Y have a countable network. Then the projection $\pi\colon X \times Y \to X$ preserves $sb_d - \sigma$-decomposable families.*

In fact, Lemma 7.1 of [H₁] concerns also relatively topologically discrete σ-decomposable families. We omitted the assertion for them, because we are not going to use it here. However it follows similarly as the one stated above.

Proof of Lemma 5. Let $\mathcal{V} = \bigcup \mathcal{V}_n$ where $\mathcal{V}_n$ are scattered. Let $\{S_\alpha \mid \alpha \in A\}$ be a scattered family in G (it is enough to investigate images of such a family to get the assertion for all $\mathrm{sb}_d - \sigma$-decomposable families). Let G_α be open subsets of $X \times Y$ such that $S_\alpha \subset G_\alpha \setminus \bigcup\{G_\beta \mid \beta < \alpha\}$ for $\alpha \in A$ with a suitable well-ordering $<$ of A. Put $S_{\alpha V_n} = \{s \in S_\alpha \mid s \in U \times V_n \subset G_\alpha$ for some open $U \subset X\}$. Then $\{\pi(S_{\alpha V_n}) \mid \alpha \in A\}$ is scattered in $f(V_n)$ for $V_n \in \mathcal{V}_n, n \in \omega$. Let $\{W(V_n, k) \mid V_n \in \mathcal{V}_n, k \in \omega\}$ be an $\mathrm{sb}_d - \sigma$-decomposition of $(f(V_n) \mid V_n \in \mathcal{V}_n)$, i.e. $\{W(V_n, k) \mid V_n \in \mathcal{V}_n\}$ are disjoint with a scattered base and $f(V_n) = \bigcup_{k=0}^{\infty} W(V_n, k)$. Then $\{\pi(S_{\alpha V_n}) \cap W(V_n, k) \mid \alpha, V_n \in \mathcal{V}_n\}$ is disjoint with a scattered base in X for n, k fixed. The sets $T_{\alpha n k} = \bigcup_{V_n \in \mathcal{V}_n} \pi(S_{\alpha V_n}) \cap W(V_n, k)$ are disjoint with a scattered base for n, k fixed and since $\pi(S_\alpha) = \bigcup_{n,k} T_{\alpha n k}$, they form the desired decomposition. □

We may now summarize our observations using the following definition. It coincides with Hansell's one due to Lemma 2.

Definition 1. We say that a topological space X is *almost K-descriptive* (or "scattered-K-analytic") if there is a complete metric space and an usc-K map $f: M \to X$ such that $f(M) = X$ and f preserves the $\mathrm{sb}_d - \sigma$-decomposable families, i.e. if there is an almost K-descriptive parametrization as defined above. If f may be chosen a continuous map, X is called *almost descriptive.*

Theorem 1. *Every Čech analytic space is almost K descriptive. Moreover, any Čech complete space can be parametrized by a disjoint almost K-descriptive parametrization (it is "scattered-K-Luzin").*

Proof. It follows immediately from the recalled characterization of Čech complete spaces by complete sequences of open covers and from Lemma 1 and Lemma 4 that a Čech complete space admits an almost K-descriptive parametrization which is disjoint. The assertion on Čech analytic spaces follows from the characterization of a Čech analytic space as a projection of a Čech complete space along a separable metric space, from Lemma 5, and from the just proved property of Čech complete spaces. □

Remark. We may notice that the K-almost descriptive parametrization with a complete metric domain cannot be always formed in such a way that it preserves the scattered σ-decomposable families. Namely, let $f: M \to [0, \omega_1)$ be such a parametrization of the Čech complete (scattered) space $[0, \omega_1)$. Let us choose $y_\alpha < \omega_1$ and $x_\alpha \in M$ for $\alpha < \omega_1$ so that y_α is the last element of $f(x_\alpha), y_\alpha < y_\beta$ if $\alpha < \beta$. This is possible because each compact set in $[0, \omega_1]$ is countable. Let $L = \{x_\alpha \mid \alpha < \omega_1\}, X = \{y_\alpha \mid \alpha < \omega_1\}$. The space X is homeomorphic to $[0, \omega_1)$. We may thus suppose that we have a $1-1$ parametrization of $[0, \omega_1)$ which is a continuous function from the metric space L. But $(\{x_\alpha\})$ is a scattered, and thus (Lemma 3) a σ-discrete, family. Thus all disjoint families in L and so in $[0, \omega_1)$ are scattered σ-decomposable. The example from [H_1] mentioned after the proof of Lemma 2 contradicts the last property of X.

However, we may get "nicer" parametrizations from Čech complete, instead of complete metric, spaces. We show in Lemma 8 that any completely regular almost K-descriptive space is a continuous image of a Čech complete space by a map which preserves $\mathrm{sb}_d - \sigma$-decomposable families.

Recall that a topological space is called analytic (point-analytic in [FH_1]) if it is a continuous image of a complete metric space by a mapping which preserves discretely σ-decomposable families (with discrete in the topological sense; topologically discretely σ-decomposable is equivalent to metrically σ-discretely decomposable in metric spaces). It is well known that analytic is equivalent to K-analytic in the case of metric topologies. In fact a more general sufficient condition for this equivalence can be found in [GP]. Metric K-analytic spaces are exactly those metric spaces which are Suslin ($\mathscr{F}$) in their completion ([H_2]) or in any metrizable embedding (called absolutely analytic in [H_1]), or which are Suslin ($\mathscr{F}$) in every Čech complete space ([Fr_3]). Thus, due to Lemma 3, we have

Proposition 1. *The following properties of a metric space X are equivalent:*

(a) *X is almost K-descriptive;*

(b) *X is analytic;*

(c) *X is a Suslin ($\mathscr{F}$) subset in a complete metric embedding;*

(d) *X is a Suslin ($\mathscr{F}$) subset in every metrizable embedding (absolute Suslin in [Fr_3]);*

(e) *X is Suslin ($\mathscr{Z}$) in a Čech complete space ($\mathscr{Z}$ stands for zero sets);*

(f) *X is Čech analytic.*

Proof. It is well-known that $(b) \Leftrightarrow (c) \Leftrightarrow (d) \cdot (a) \Leftrightarrow (b)$ by Lemma 3 and the above mentioned fact that metric K-analytic spaces are analytic. For $(d) \Rightarrow (e)$ see ([Fr_3], Theorem 4). Since obviously $(e) \Rightarrow (f)$, it remains to use Theorem 1 to show that $(f) \Rightarrow (a)$. □

Now we will study the class of general almost K-descriptive spaces. We recall a result from [H_1] (Theorem 4.1) and add two "absolute descriptive" properties of almost K-descriptive spaces which are partially contained in the proof of that theorem. We say that $X \subset Y$ has the Baire property in the topological space Y if $X = (G \setminus C) \cup (G \cup D)$ with G open and C, D of the first category in Y, and X has the Baire property in the restricted sense if, for each $E \subset Y$, $X \cap E$ has the Baire property in E.

Proposition 2. *Let X be an almost K-descriptive space. Then*

(a) *X has the Baire property in the restricted sense in any topological embedding;*

(b) *$X \in$ Suslin $((\mathscr{F} \wedge \mathscr{G})_s)$ where $(\mathscr{F} \wedge \mathscr{G})_s$ stands for the class of all unions of scattered families of sets from $\mathscr{F} \wedge \mathscr{G} \equiv \{F \cap G \mid F \in \mathscr{F}, G \in \mathscr{G}\}$ in any topological embedding;*

(c) *there are a cardinal* $\kappa \equiv [0, \kappa)$ *and sets* $S_{\sigma|n} \in \mathscr{F} \wedge \mathscr{G}$ *for* $\sigma \mid n = (\sigma_0, \ldots, \sigma_{n-1}) \in \kappa^n$ *such that*

$$X = \bigcup \left\{ \bigcap \{ S_{\sigma|n} \mid n \in \omega \} \mid \sigma \in \kappa^\omega \right\}$$

and $\{S_{\sigma|n} \mid \sigma \mid n \in \kappa^n\}$ *are* σ*-scattered for* $n \in \omega$ *in any topological embedding. We write* $X \in Suslin_s(\mathscr{F} \wedge \mathscr{G})$.

Proof. The assertion (a) is claimed by the above mentioned Theorem 4.1 in [H₁]. The assertion (b) is proved in its proof. We sketch the proof of (c) because it is not contained in the proof of (b) explicitly.

Let $f \colon M \to X$ be an almost K-descriptive parametrization of X and let $X \subset Y$. Let $\mathscr{U}$ be a σ-discrete base for the complete metric space M and $\mathscr{U}_n = \{U \in \mathscr{U} \mid U \neq \emptyset$ and $\operatorname{diam} U < 1/n\}$ for $n \in \omega$. Since $\mathscr{U}_n$ is σ-discrete in M, we can write $f(U) = \bigcup_{m=0}^{\infty} (f(U))_m$ for each $U \in \mathscr{U}_n$ such that $\{f(U))_m \mid U \in \mathscr{U}_n\}$ is disjoint and has a scattered network $\mathscr{B}_m^n = \{B^n(U, m, \alpha) \mid \alpha \in A^n, U \in \mathscr{U}_n\}$ for $m \in \omega$ and sufficiently large A^n (many $B^n(U, m, \alpha)$) may be empty). By Lemma 2.3 from [H₁] we can find scattered collections $\{H(B) \mid B \in \mathscr{B}_m^n\}$ of $\mathscr{F} \wedge \mathscr{G}$ sets in Y such that $B \subset H(B) \subset \overline{B}^Y$ for each $B \in \mathscr{B}_m^n$ and $m \in \omega$. We define sets $H(B^0(U_0, m_0, \alpha_0), \ldots, B^n(U_n, m_n, \alpha_n)) = \bigcap_{i=0}^{n} H(B^i(U_i, m_i, a_i))$ if $\overline{U}_{i+1}^Y \subset U_i, i = 0, \ldots, n-1$, and put $H(B^0(U_0, m_0, \alpha_0), \ldots, B^n(U_n, m_n, \alpha_n)) = \emptyset$ otherwise. We choose a sufficiently large cardinal $\kappa \equiv [0, \kappa)$ and embeddings φ_i of the admissible triples (U_i, m_i, α_i) into κ. We identify $H(B^0(U_0, m_0, \alpha_0), \ldots, B^n, U_n, m_n, \alpha_n))$ with $S_{\sigma|(n+1)}$ if $\sigma_0 = \varphi_0(U_0, m_0, \alpha_0)$, $\ldots$, $\sigma_n = \varphi_n(U_n, m_n, \alpha_n)$, and put $S_{\sigma|(n+1)} = \emptyset$ otherwise. Now we claim that

$$X = \bigcup \left\{ \bigcap \{ S_{\sigma|n} \mid n \in \omega \} \mid \sigma \in \kappa^\omega \right\}.$$

The inclusion $\subset$ is obvious. Let $x \in \bigcap_{n=0}^{\infty} S_{\sigma|n} = \bigcap_{n=0}^{\infty} H(B^0(U_0, m_0, \alpha_0), \ldots, B^n(U_n, m_n, \alpha_n))$. Thus $\overline{U}_{i+1}^Y \subset U_i$ for $i = 0, \ldots, n-1$ and we know that $\overline{H(B^i(U_i{}^Y, m_i, \alpha_i))} \subset B^i(U_i, m_i, \alpha_i)$. Thus $x \in H(B^i(U_i, m_i, \alpha_i)) \subset \overline{B^i(U_i{}^Y, m_i, \alpha_i)} \subset \overline{(f(U_i))_{m_i}}^Y \subset \overline{f(U_i)}^Y$. Now $\{t\} = \bigcap_{n=0}^{\infty} U_n$ and $x \in f(t)$ due to the fact that f is usc-K. □

We further establish the following properties of the class of almost K-descriptive spaces.

Theorem 2. (a) *The union of the* σ*-scattered family of almost* K*-descriptive subspaces is almost* K*-descriptive.*

(b) *Countable products of almost K-descriptive spaces are almost K-descriptive. Countable intersections of almost K-descriptive subspaces are almost K-descriptive.*

(c) *The result of the Suslin operation working on almost K-descriptive subspaces is an almost K-descriptive space.*

(d) *Every closed subspace of an almost K-descriptive space is almost K-descriptive.*

(e) *Every open subspace of an almost K-descriptive completely regular space is almost K-descriptive.*

Proof. (a) The assertion follows by taking the parametrizations $f_a : M_a \to X_a$ for each almost K-descriptive space $X_a \subset X$ from the σ-scattered family and taking the discrete sum $\sum M_a$ of their domains for the domain of the new parametrization f defined by f_a on M_a.

(b) Let $f_n : M_n \to X_n$ be almost K-descriptive parametrizations of X_n. Put $f(m_1, m_2, \ldots) = \prod f_n(m_n)$ for the parametrization of $\prod X_n$ from the complete metric space $M = \prod M_n$. Let $\{N_a\}$ be a discrete family in M. Then there is an open cover $\mathscr{U}$ of M each element of which intersects at most one N_a. We may suppose that the open sets from the cover $\mathscr{U}$ are of the form $G_1 \times G_2 \times \ldots G_k \times \prod\limits_{l>k} M_l$ with $k \geqslant 1$ and the sets G_i from a σ-discrete open base $\mathscr{B}_i$ of M_i. Then $f(\mathscr{B}_1 \times \mathscr{B}_2 \times \ldots \times \mathscr{B}_k \times \prod\limits_{l>k} M_l)$ is an $\mathrm{sb}_d - \sigma$-decomposable family and thus $f(N_a)$ has a countable decomposition to $(f(N_a))_k = f(N_a) \cap \bigcup \{f(B_1 \times \ldots \times B_k \times \prod\limits_{l>k} M_l) \mid \prod\limits_{i \leqslant k} B_i \times \prod\limits_{l>k} M_l \cap N_a \neq \emptyset$ and $B_i \in \mathscr{B}_i\}$ which, for a fixed k, are $\mathrm{sb}_d - \sigma$-decomposable, too.

To get the result for a countable intersection of subspaces we similarly put $f(m_1, m_2, \ldots) = \bigcap\limits_n f_n(m_n)$.

(c) let $S = \bigcup \left\{ \bigcap \{S_{\sigma|n} \mid n \in \omega\} \mid \sigma \in \omega^\omega \right\}$ with $S_{\sigma|n}$ almost K-descriptive subsets of X. We define a subset Y of $X \times \omega^\omega$ by

$$Y = \bigcap \left\{ \bigcup \{(S_{\sigma|n} \times \{\bar\sigma \mid \bar\sigma \mid n = \sigma \mid n\}) \mid \sigma \in \omega^\omega\} \mid n \in \omega \right\}.$$

Then S is the projection of Y along ω^ω and thus it is almost K-descriptive if Y is almost K-descriptive due to Lemma 5. But Y is almost K-descriptive by (a) and (b).

(d) If $f: M \to X$ is an almost K-descriptive parametrization of X and F is closed in X, then $g: M \to F$ defined by $g(m) = f(m) \cap F$ is an almost K-descriptive parametrization of F.

(e) Let G be an open subset of the almost K-descriptive space X that is completely regular. Let K be a compactification of X and H an open subset of K such that $H \cap X = G$. The set H is almost K-descriptive by Theorem 1, X is almost K-descriptive by the assumption and thus G is almost K-descriptive due to (b). □

We may notice the following fact.

Lemma 6. *The σ-scattered unions of Borel subsets of an almost K-descriptive completely regular space are almost K-descriptive together with their complements.*

Proof. Let $B_a \subset G_a \setminus \bigcup\{G_b \mid b < a\}$ be Borel sets and let G_a be open sets for a from some well-ordered A. Then the family $(G_a \setminus \bigcup\{G_b \mid b < a\}) \setminus B_a$, $a \in A$, is a scattered family of Borel and thus almost K-descriptive subsets. Thus their union are almost K-descriptive as well as their union with the closed subset $X \setminus \bigcup G_a$. □

Now we state a characterization of general almost K-descriptive completely regular spaces.

Theorem 3. *X is a completely regular almost K-descriptive space if and only if one and then necessarily all of the following representations of X in a compactification K of X take place:*

(a) $X = \bigcup\{\bigcap\{S_{\sigma|n} \mid n \in \omega\} \mid \sigma \in \kappa^\omega\}$ *where $S_{\sigma|n}$ are in $\mathscr{F} \wedge \mathscr{G} = \{F \cap G \mid F \in \mathscr{F}, G \in \mathscr{G}\}$ with $\{S_{\sigma|n} \mid \sigma \in \kappa^\omega\}$ σ-scattered. We write $X \in Suslin_s(\mathscr{F} \wedge \mathscr{G})$ in this case.*

(b) $X = \bigcup\{\bigcap\{S_{\tau|n} \mid n \in \omega\} \mid \tau \in \omega^\omega\}$ *where $S_{\tau|n}$ are scattered unions of sets from $\mathscr{F} \wedge \mathscr{G}$. We write $X \in Suslin\ ((\mathscr{F} \wedge \mathscr{G})_s)$ in this case.*

(c) $X = \bigcup\{\bigcap\{S_{\tau|n} \mid n \in \omega\} \mid \tau \in \omega^\omega\}$ *with $S_{\tau|n}$ from the σ-algebra $\mathscr{B}_{\sigma s}$ of σ-scattered unions of Borel subsets of K.*

Proof. Let K be a compactification of the almost K-descriptive space X. Then, according to Proposition 2, X fulfils (a), (b), (c). The space X from (a), (b), or (c), respectively, is almost K-descriptive due to Theorem 2. $\mathscr{B}_{\sigma s}$ is a σ-algebra to Lemma 6. □

Remark. We will show that the class of sets which arise by the Suslin operation from sets which are Borel or scattered (from Σ) or completements of scattered (from Σ^c), i.e. sets from Suslin $(\mathscr{B} \cup \Sigma \cup \Sigma^c)$ in a compact space K, is not the class of all K-descriptive subspaces of K.

Let $K = [0, \omega_1] \times [0, 1]$. Let $S \subset [0, \omega_1]$ be such that S is not Čech analytic. This is possible as seen from the following example. Then $S \times [0, 1]$ cannot be contained in Suslin $(\mathscr{B} \cup \Sigma \cup \Sigma^c)$ in K. If it were then $S \times [0, 1] = \bigcup\bigcap S_{\tau|n}$ with $S_{\tau|n} \in \mathscr{B} \cup \Sigma \cup \Sigma^e$.

If we replace all $S_{\tau|n}$ from Σ by $\bar{S}_{\tau|n} = \emptyset$ and $S_{\tau|n}$ from Σ^c by $[0, \omega_1] \times [0, 1]$, then $\bar{S} = \bigcup\bigcap \bar{S}_{\sigma|n}$ differes from $S \times [0, 1]$ by omitting at most countably many points from each $\{x\} \times [0, 1]$. Here $\bar{S}_{\tau|n} = S_{\tau|n}$ if $S_{\tau|n} \notin \Sigma \cup \Sigma^c$. Thus $\bar{S}$ projects onto S and it follows that S is Čech analytic. This is a contradiction.

The following example was shown to me by J. Pelant.

Example. There is a subset S of $[0,\omega_1]$ which is not Čech analytic.

Proof. Čech analytic spaces in $[0,\omega_1]$ are exactly the sets which arise by the Suslin operation applied to Borel sets. Obviously, the same is true for $[0,\omega_1)$ and we shall show that there is a non-Čech analytic subset of $[0,\omega_1)$. The main point is to prove

Lemma 7. *Every set from $\mathscr{S}(\mathscr{F}\cup\mathscr{G})$ in $[0,\omega_1)$ which is stationary (i.e. intersects each closed unbounded subset of $[0,\omega_1)$) contains a closed unbounded subset.*

Then we use the existence of two disjoint stationary sets in $[0,\omega_1)$, see [Fo], and Lemma 7 shows that they cannot be Čech analytic. □

Proof of Lemma 7. For a closed set the assertion is trivial.

Let G be an open subset of $[0,\omega_1)$ which is stationary. Then $G_0 = G \setminus \{$ isolated ordinals from $[0,\omega_1)\}$ is stationary, too. We may define a map $f\colon G_0 \to [0,\omega_1)$ by $f(\alpha) = \min\{\beta \mid [\beta,\alpha] \subset G\}$. Since G is open and the elements of G_0 are not isolated, we get $f(\alpha) < \alpha$. Again by [Fo] such a map assumes a constant value on a stationary (thus unbounded) $G_{oo} \subset G_0$. Consequently, G contains a closed unbounded set $[\alpha_0,\omega_1)$.

Let $S = \bigcup\{\bigcap\{S_{\tau|n} \mid n \in \omega\} \mid \tau \in \omega^\omega\}$ with each $S_{\tau|n}$ either closed or open. Denote $T_{\tau|n} = \bigcup\{\bigcap\{S_{\sigma|m} \mid m \geqslant n\} \mid \sigma \mid n = \tau \mid n\}$. If $T_{\tau|n}$ is not stationary, then, for some τ_{n+1}, $T_{\tau|n+1}$ is stationary. Thus also $S_{\tau|n+1}$ is staionary and contains a closed unbounded $F_{\tau|n+1}$. In this way we may get a sequence τ and closed unbounded sets $F_{\tau|n+1} \subset S_{\tau|n+1}$. The intersection $\bigcap\limits_n F_{\tau|n}$ is closed unbounded and is contained in S. (We have used the fact that the intersection of countably many closed unbounded sets in $[0,\omega_1)$ is a closed unbounded set and thus also any countable union of non-stationary sets is not stationary.) □

It is proved in [K], [JNR] that every Čech analytic space is either σ-scattered, or contains a non-empty compact perfect subset. Using another method R.W. Hansell showed in $[\mathrm{H}_1]$ the same for continuous images of Čech analytic spaces by maps that take scattered sets to σ-scattered ones. We may show the following.

Theorem 4. *Let X be an almost K descriptive completely regular space. Then X is either σ-scattered, or contains a compact perfect set.*

Proof. It follows from Theorem 5.4.(a) of $[\mathrm{H}_1]$ and the following Lemma □

Lemma 8. *Let X be a completely regular almost K-descriptive space. Then there is a Čech complete space Y and a continuous map f from Y onto X which preserves the sb_d $-\sigma$-decomposable families.*

P r o o f. Let $\bar{f}\colon M \to X$ be an almost K-descriptive parametrization of X. Let $\mathcal{U}_n$ be a complete sequence of open covers of M with $\mathcal{U}_n$ consisting of sets with diameter less than $1/n$. Then the families $\mathcal{V}_n = \{G \cap (X \times U_n) \mid U_n \in \mathcal{U}_n\}$ form a complete sequence of open covers of the graph G of $\bar{f}$. Due to Lemma 5, X is the continuous image of the Čech complete space G by the projection which preserves $\mathrm{sb}_d - \sigma$-decomposable families. □

Due to the definition via a parametrization, we could get for almost K-descriptive spaces, and thus for all Čech analytic spaces, results analogous to the first separation principle (separation from a Suslin ($\mathcal{F}$) set), measurability of graph of some maps to metric spaces, ...similarly as e.g. in [FH$_1$], [FH$_2$], ... The essential difference with respect to the theory developed in [FH$_1$] and [FH$_2$] is that almost K-descriptive spaces are not necessarily Suslin ($\mathcal{F}$). Thus we do not get simple characterization of sets which are almost K-descriptive together with their complements from the separation principle. Neither do we know if disjoint families of subsets with all unions almost K-descriptive in a compact space K are necessarily $\mathrm{sb}_d - \sigma$-decomposable (cf. the problem formulated in ([H$_1$], p. 48) for K-descriptive spaces).

3. Almost K-descriptive Banach spaces

This section is a slight strengthening of an interesting result from [JNR]. Together with results from [N] and [H$_1$], it gives a new characterization of σ-fragmented Banach spaces.

Before we state and prove our result, we recall the definitions of some notions needed in the sequel which were not defined above.

A topological space X is said to be *σ-fragmented* by the metric ϱ if for every $\varepsilon > 0$ there are subspaces $X_n, n = 1, 2, \dots$ with $X = \bigcup\{X_n \mid n = 1, 2, \dots\}$, and for any $n \in \mathbf{N}$ and any non-empty subset $Y \subset X_n$ there is a relatively open G in Y which is non-empty and has a ϱ-diameter less than ε (we say also that X_n is *ε-fragmented* and X is *$\sigma - \varepsilon$-fragmented* by ϱ in such a case).

Recall further that a space X is *fragmented* by a metric if it is ε-fragmented for every $\varepsilon > 0$.

A metric ϱ is *lower semi-continuous* on the topological space X if $\{(x, y) \in X \times X \mid \varrho(x, y) > r\}$ is open for every $r \geqslant 0$.

A set $S \subset X$ is called an *ε-tree for the metric ϱ* if $S = \{x(s) \mid s \in \{0, 1\}^\ell, \ell \in \mathbf{N}\}$ and for each $\ell \geqslant 0$ and each sequence $s \in \{0, 1\}^\ell$ the sets

$\mathrm{cl}_X\{x(t) \mid (t \mid (l + 1)) = (s, i)$ and t has lenght $\geqslant t+\}$ with $i = 0$ and $i = 1$ are separated by a ϱ-distance ε.

Now we prove the following lemma repeating almost word by word the proof of Lemma 6.10 from [JNR]. Due to the change of the assumption in [JNR] that Z is Čech-analytic we have to do only minor adaptations.

Lemma 9. *If a completely regular almost K-descriptive space X is not σ-fragmented by a lower semi-continuous metric ϱ, then there is an $\varepsilon > 0$ and a relatively compact ε-tree for ϱ in X.*

We introduce the proof of this lemma at the end of this paper since the reader may find it more convenient than to indicate the necessary changes only.

First we state some conclusions which follow from it. The proof of Theorem 6.1 in [JNR] uses the assumption that the space is Čech analytic only to prove the statement of the above lemma. Thus we may restate all Theorem 6.1 from [JNR] for almost K-descriptive spaces. We formulate only one equivalence which we are mainly interested in.

Theorem 5. *Let ϱ be a lower semi-continuous metric on completely almost K-descriptive space. Then X is σ-fragmented by ϱ if and only if each compact subset of X is fragmented by ϱ.*

The following theorem follows from Theorem 1.6 of [H_1] and Theorem 1.2 of [N].

Theorem 6. *Let X be a Banach space. Then the following properties of a subspace Y of X with its weak topology are equivalent:*

(a) *Y is σ-fragmented by the norm;*

(b) *Y is almost descriptive;*

(c) *Y is almost K-descriptive.*

Remark. Due to Example 8.5 of [JNR], ℓ_∞ is a Banach space which is not almost K-descriptive in the weak topology.

Proof of Lemma 9. Due to Theorem 3(a) the almost K-descriptive completely regular space X is in $\mathrm{Suslin}_s(\mathcal{F} \wedge \mathcal{G})$ in some compactification $\hat{X}$. Thus

$$X = \bigcup\{\bigcap\{S_{\sigma|n} \mid n \in \omega\} \mid \sigma \in \kappa^\omega\}$$

with each set $S_{\sigma|n} \in \mathcal{F} \wedge \mathcal{G}$ and with $\{S_{\sigma|n} \mid \sigma_{n-1} \in \kappa\}$ being σ-scattered. Write

$$X_{\sigma|n} = \bigcup\{\bigcap\{S_{\tau|m} \mid m \in \omega\} \mid \tau \in \kappa^\omega, \sigma \mid n = \tau \mid n\}$$

for each finite sequence $\sigma \mid n = (\sigma_0, \ldots, \sigma_{n-1})$ of elements of $\kappa \equiv [0, \kappa)$. Put $X_{\sigma|o} = X$ for any σ. Since X is not σ-fragmented by ϱ, we can choose $\varepsilon >$

0 such that X is not the countable union of subspaces X_n each of which is ε-fragmented (i.e. X is not $\sigma-\varepsilon$-fragmented). We describe an inductive choice of families $\{B_{\gamma|r}\}$ of subsets of X and $\{s_{\gamma|r}\}$ of finite sequences of positive integers, each indexed by a finite sequence $\gamma \mid r = (\gamma_0, \dots, \gamma_{r-1})$ with $\gamma_i = 0$ or 1 for $0 \leqslant i \leqslant t-\tau$, statisfying the following conditions ($r \geqslant 1$):

(i) $B_{\sigma|o} = X_{\sigma|o}$;

(ii) $B_{\gamma|r}$ is relatively closed subset of $X_{s_{\gamma|r}}$ which is contained in $B_{\gamma|(r-1)}$:

(iii) $B_{\gamma|r}$ cannot be written as a countable union of ε-fragmented sets;

(iv) $s_{\gamma|r}$ has length r;

(v) $s_{\gamma|(r+1)} \mid r = s_{\gamma|r}$;

(vi) $\emptyset \neq B_{\gamma|r} \subset \overline{B_{\gamma|r}}^{\hat{X}} \subset S_{s_{\gamma|r}}$;

(vii) the sets $\overline{B_{\gamma|r,0}}^{\hat{X}}$ and $\overline{B_{\gamma|r,1}}^{\hat{X}}$ are disjoint and their intersections with X are separated by a ϱ-distance greater than $\frac{1}{2}\varepsilon$.

We first take $B_\emptyset = X$ and suppose that for some $r \geqslant 0$ we have chosen $B_{\gamma|t}$, $s_{\gamma|t}$, $0 \leqslant t \leqslant r$ so as to satisfy the conditions (ii) to (vi).

Now, removing from $B_{\gamma|r}$ all relatively open subsets which are $\sigma-\varepsilon$-fragmented, we get $C_{\gamma|r} \subset B_{\gamma|r}$ which is non-empty relatively closed in $B_{\gamma|r}$, and no non-empty relatively open subset of $C_{\gamma|r}$ is $\sigma-\varepsilon$-fragmented. Thus $C_{\gamma|r}$ has diameter at least ε and we can choose points $c_{\gamma|r,0}, c_{\gamma|r,1}$ in $C_{\gamma|r}$ that are at ϱ-distance greater than $\frac{1}{2}\varepsilon$. (Here we have used Theorem 4.1 of [JNR] or Lemma 10 below.)

Since ϱ is lower semi-continuous on X we can choose open sets $U_{\gamma|r,0}$ and $U_{\gamma|r,1}$ in $\hat{X}$ containing $c_{\gamma|r,0}$ and $c_{\gamma|r,1}$, respectively, such that $U_{\gamma|r,0} \cap X$ and $U_{\gamma|r,1} \cap X$ are separated by ϱ-distance greater than $\frac{1}{2}\varepsilon$. Let $V_{\gamma|r,1}$ and $V_{\gamma|r,1}$ be closed subsets of $U_{\gamma|r,0}$ and $U_{\gamma|r,1}$, respectively, which are neighbourhoods of $c_{\gamma|r,0}$ and $c_{\gamma|r,1}$. The sets $D_{\gamma|r,0} = C_{\gamma|r} \cap V_{\gamma|r,0}$ and $D_{\gamma|r,1} = C_{\gamma|r} \cap V_{\gamma|r,1}$ are not $\sigma-\varepsilon$-fragmented. Since $D_{\gamma|(r+1)} = \bigcup_{\sigma_{r+1}<\kappa} (D_{\gamma(r+1)} \cap X_{s_{\gamma|r},\sigma_{r+1}})$ and the sets $X_{s_{\gamma|r},\sigma_{r+1}}, \sigma_{r+1} \in \kappa$, form a σ-scattered family, we can choose, again due to Lemma 10 below, a $\sigma_{r+1} \equiv (\sigma_{r+1})_{\gamma|(r+1)}$ such that $E_{\gamma|(r+1)} \equiv D_{\gamma|(r+1)} \cap X_{s_{\gamma|r},\sigma_{r+1}}$ is not $\sigma-\varepsilon$-fragmented. Put $s_{\gamma|(r+1)} = (s_{\gamma|r}, \sigma_{r+1})$ for $\gamma_{r+1} = 0$ or 1.

Now $E_{\gamma|(r+1)} \subset S_{s_{\gamma|(r+1)}} \in \mathscr{F} \wedge \mathscr{G}$, say $S_{s_{\gamma|(r+1)}} = F_{\gamma|(r+1)} \cap G_{\gamma|(r+1)}$ with $F_{\gamma|(r+1)} \in \mathscr{F}$ and $G_{\gamma|(r+1)} \in \mathscr{G}$ in X. Find a non-empty relatively closed subset $H_{\gamma(r+1)}$ of $E_{\gamma|(r+1)}$ such that no non-empty relatively open subset of $H_{\gamma|(r+1)}$ is $\sigma-\varepsilon$-fragmented.

We may choose arbitrarily a closed neighbourhood $P_{\gamma|(r+1)}$ of an element of $H_{\gamma(r+1)}$ contained in $G_{\gamma(r+1)}$. Then $B_{\gamma|(r+1)} = H_{\gamma|(r+1)} \cap P_{\gamma|(r+1)}$ is not $\sigma-\varepsilon$-fragmented and $\overline{B}^{\hat{X}}_{\gamma|(r+1)} \subset S_{s_{\gamma|(r+1)}}$.

Now for $s \in \{0,1\}^r$ choose any

$$x(s) \in \bigcup \{ \bigcap \{ \overline{B}^{\hat{X}}_{\gamma|\ell} \mid \ell = 0, \dots \} \mid \gamma \in \{0,1\}^\omega \text{ and } \gamma \mid r = s \}.$$

The family $\{x(s)\}$ is a relatively compact $\frac{1}{2}\varepsilon$-tree for ϱ. □

Lemma 10. *Let $S(\alpha), \alpha < \kappa$, be a σ-scattered (or scattered σ-decomposable) cover of a non-$\sigma - \varepsilon$-fragmented S. Then there is $\alpha_0 < \kappa$ such that $S(\alpha_0)$ is not $\sigma - \varepsilon$-fragmented.*

Proof. Let us suppose that $S(\alpha)$ is $\sigma - \varepsilon$-fragmented for every $\alpha < \kappa$. Thus there are $S(\alpha, n)$ which are ε-fragmented and such that $S(\alpha) = \bigcup \{S(\alpha, n) \mid n \in \mathbf{N}\}$ for $\alpha < \kappa$. Moreover, we may suppose that $\{S(\alpha, n) \mid \alpha < \kappa\}$ are scattered. The union S_n of the scattered family $\{S(\alpha, n) \mid \alpha < \kappa\}$ is ε-fragmented. (Let $T \subset S_n$ be non-empty. Let $<_n$ be a well-ordering of κ such that $\bigcup_{\alpha < \beta} S(\alpha, n)$ are relatively open in S_n. We find the first α_0 in $[0, \kappa)$ with respect to $<_n$ such that $T \cap S(\alpha_0, n) \neq \emptyset$, and find a relatively open non-empty subset $G \subset T \cap S(\alpha_0, n)$ of diameter less than ε. It is relatively open in S_n due to our choice of $<_n$.

Thus $S = \bigcup S_n$ is $\sigma - \varepsilon$-fragmented, which is a contradiction. □

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Author's address: Petr Holický, MFF UK Sokolovská 83, 186 00 Praha 8, Czech Republic.