Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 44 (1994), No. 1, 81–90

Persistent URL: http://dml.cz/dmlcz/128442

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Czechoslovak Mathematical Journal, 44 (119) 1994, Praha

MINIMAL PRIME IDEALS IN AUTOMETRIZED ALGEBRAS

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(Received April 23, 1992)

1. INTRODUCTION

Swamy [5] introduced the concept of an autometrized algebra which is a generalization of, for example, abelian lattice ordered groups and Brouwerian algebras. Ideals of autometrized algebras were studied by Swamy and Rao [7] and this work has been continued by Rachůnek [2, 3, 4] who has studied prime ideals, polars and regular ideals in autometrized algebras. In this paper minimal prime ideals are studied both for autometrized algebras, and for representable dually residuated lattice ordered semigroups.

A system $(A, +, \leqslant, *)$ is an *autometrized algebra* if and only if

1) $(A, +)$ is commutative semigroup with 0;

2) $\leqslant$ is a partial ordering on A such that $\forall\, a,\, b,\, c \in A$

$$a \leqslant b \implies a + c \leqslant b + c;$$

3) $*$ is a metric operation on A, that is, $*\colon A \times A \to A$ is a mapping such that for all a, b, $c \in A$,

(i) $a * b \geqslant 0$ and $a * b = 0 \iff a = b$,

(ii) $a * b = b * a$,

(iii) $a * c \leqslant (a*) + (b * c)$.

An autometrized algebra $(A,\, +,\, \leqslant,\, *)$ is called *normal* if and only if for all a, b, c, $d \in A$,

(i) $a \leqslant a * 0$,

(ii) $(a + c) * (b + d) \leqslant (a * b) + (c * d)$,

(iii) $(a * c) * (b * d) \leqslant (a * b) + (c * d)$,

(iv) $a \leqslant b$ implies that there exists $x \geqslant 0$ such that $a + x = b$.

* This research was funded in part by an Indiana University at Kokomo Summer Faculty Fellowship, 1991.

An autometrized algebra $(A, +, \leqslant, *)$ is called *semiregular* if for all $a \in A$, $a \geqslant 0$ implies $a * 0 = a$.

An autometrized algebra is called an *ℓ-algebra* if and only if $\leqslant$ is a lattice order and for all $a, b, c \in A$,

$$a + (b \vee c) = (a + b) \vee (a + c) \quad \text{and} \quad a + (b \wedge c) = (a + b) \wedge (a + c).$$

Note that semiregular normal autometrized ℓ-algebras include all Brouwerian algebras and abelian lattice ordered groups.

If $A = (A, +, \leqslant, *)$ is an autometrized algebra then a non-empty subset I of A is called an *ideal* of A if and only if

(i) $\forall a, b \in I$, $a + b \in I$,

(ii) $\forall a \in I$, $x \in A$, $x * 0 \leqslant a * 0$ implies $x \in I$.

For a normal autometrized algebra A, $\mathscr{J}(A)$, the set of all ideals of A, ordered by set inclusion, is a complete algebraic lattice [7]. For $B \subseteq A$, $I(B)$ is used to denote the ideal of A generated by B. $I(\{a\})$ is written simply as $I(a)$, and $I(a) = \{x \in A \mid x * 0 \leqslant m(a * 0) \text{ for some } m \geqslant 0\}$, [7].

An ideal I of an autometrized algebra A is a *prime ideal* if for all J, K in $\mathscr{J}(A)$, $J \cap K = I$ implies $J = I$ or $K = I$.

For a semiregular normal autometrized ℓ-algebra A, Rachůnek [2] has shown that an ideal I is a prime ideal of A if and only if for all $a, b \in I$, $0 \leqslant a \wedge b \in I$ implies $a \in I$ or $b \in I$, and that every prime ideal contains a minimal prime ideal.

Elements a and b of an autometrized ℓ-algebra A are said to be *orthogonal* (denoted $a \perp b$) if $(a * 0) \wedge (b * 0) = 0$. For any subset B of A the *polar* of B is

$$B^{\perp} = \{x \in A \mid x \perp b \text{ for all } b \in B\}.$$

$\{a^{\perp}\}$ is denoted $a^{\perp}$ and $C \subseteq A$ is called a *polar in* A if $C = B^{\perp}$ for some $B \subseteq A$. The collection of all polars in A is denoted $\mathscr{P}(A)$.

It is clear that for subsets B and C of A, $B \cap B^{\perp} = \{0\}$ and if $B \subseteq C$ then $B^{\perp} \supseteq C^{\perp}$. Rachůnek [3] has shown that any polar in a semiregular normal autometrized ℓ-algebra A is an ideal of A. Also if $B \subseteq A$, then $B \subseteq B^{\perp\perp}$ and $B^{\perp} = B^{\perp\perp\perp}$.

The following lemma due to Swamy [5] will be needed.

Lemma 1.1. *Let x, y, c be elements of an autometrized ℓ-algebra. If $c \wedge x = c \wedge y = 0$, then $c \wedge (x + y) = 0$.*

More generally the following result holds.

Lemma 1.2. *If a, b, c are elements of an autometrized ℓ-algebra and $a, b, c \geqslant 0$ then*

$$a \wedge (b + c) \leqslant (a \wedge b) + (b \wedge c).$$

P r o o f. If $a,\ b,\ c \geqslant 0$, then $a \wedge b \wedge c \geqslant 0$ and $a \leqslant a + (a \wedge b \wedge c)$. Therefore

$$\begin{aligned} a \wedge (b + c) &\leqslant \left[a + (a \wedge b \wedge c)\right] \wedge (b + c) \\ &= \left[(a + a) \wedge (a + b) \wedge (a + c)\right] \wedge (b + c) \\ &= \left[(a + a) \wedge (a + b)\right] \wedge \left[(c + a) \wedge (c + b)\right] \\ &= \left[a + (a \wedge b)\right] \wedge \left[c + (a \wedge b)\right] \\ &= (a \wedge b) + (a \wedge c). \end{aligned}$$

□

For an autometrized algebra A, the set of positive elements of A if $A^+ = \{x \in A \mid x \geqslant 0\}$.

2. Minimal prime ideals in autometrized algebras

Let $A = (A,\ +,\ \leqslant,\ *)$ be a autometrized ℓ-algebra. A nonempty subset F of A^+ is called a *filter* on A^+ if

(i) $0 \notin F$;

(ii) a and $b \in F$ implies $a \wedge b \in F$;

(iii) $a \in F$ and $b \geqslant a$, implies $b \in F$.

A maximal filter on A^+ is called an ultrafilter, and a filter F on A^+ is called a *prime filter* if, for $x,\ y \in A^+$, $x + y \in F$ implies $x \in F$ or $y \in F$.

Proposition 2.1. *Let A be a normal semiregular autometrized ℓ-algebra. A nonempty subset F of A^+ is a prime filter if and only if $A^+ \setminus F = I^+$ for a prime ideal I of A. Thus the mapping $I \to A^+ \setminus I$ is a one-to-one map of the prime ideals of A onto the prime filters on A^+; the inverse map is $F \to I(A^+ \setminus F)$.*

P r o o f. Let F be a prime filter on A^+ and let $I = I(A^+ \setminus F)$. It must be shown that $I^+ = A^+ \setminus F$ and that I is prime. Clearly $A^+ \setminus F \subseteq I^+$. Suppose $A^+ \setminus F \subsetneqq I^+$, then $\exists\, x \in I^+ \cap F$, and $x \in I^+$ implies that $x*0 \leqslant m_1(a_1*0)+m_2(a_2*0)+\ldots m_k(a_k*0)$ for some positive integers $m_1, \ldots, m_k$ and some $a_1, \ldots, a_k \in A^+ \setminus F$. Then, since F is a prime filter on A^+ and $x \in F$, $a_j * 0 \in F$ for some j. But this gives a contradiction since A is semiregular implies $a_j * 0 = a_j \notin F$. Therefore $A^+ \setminus F = I^+$. Suppose $0 \leqslant a \wedge b \in I$, then $a,\ b \in A^+$, $a \wedge b \notin F$ and since F is a filter on A^+, either $a \notin F$ or $b \notin F$ and therefore, since $A^+ \setminus F = I^+$, either $a \in I$ or $b \in I$ and thus I is a prime ideal of A.

Conversely suppose that I is a prime ideal of A and let $F = A^+ \setminus I$. Then

i) $0 \in I \implies 0 \notin F$;

ii) Since $0 \in I$, and I is convex it is clear that $a \in F$ and $b \geqslant a$ implies $b \in F$;

iii) Let a, $b \in F$. Then a, $b \notin I$ and I a prime ideal implies $a \wedge b \notin I$, and thus $a \wedge b \in F$;

iv) Since I is a subsemigroup of A, if x, $y \in A^+$ and $x + y \in F$ then $x \in F$ or $y \in F$.

Therefore F is a prime filter on A^+. $\square$

Proposition 2.2. *Let A be a normal autometrized ℓ-algebra. Every ultrafilter on A^+ is a prime filter.*

Proof. Let $\mathscr{U}$ be an ultrafilter on A^+ and let x, $y \in A^+ \setminus \mathscr{U}$. Since $\mathscr{U}$ is an ultrafilter, $\exists\, a \in \mathscr{U}$ with $a \wedge x = 0$ (otherwise $a \wedge x > 0$ for each $a \in \mathscr{U}$, and $\{x\} \cup \mathscr{U}$ generates a filter on A^+ that properly contains $\mathscr{U}$). Similarly $\exists\, b \in \mathscr{U}$ with $b \wedge y = 0$. Let $c = a \wedge b$, then $c \in \mathscr{U}$ and $c \wedge x = c \wedge y = 0$, so by Lemma 1.1, $c \wedge (x + y) = 0$. Therefore $x + y \notin \mathscr{U}$ and $\mathscr{U}$ is a prime filter. $\square$

The following result is a direct consequence of Propositions 2.1 and 2.2.

Proposition 2.3. *Let A be a normal semiregular autometrized ℓ-algebra. An ideal I of A is a minimal prime ideal if and only if $A^+ \setminus I$ is an ultrafilter on A^+.*

Proposition 2.4. *Let A be a normal semiregular autometrized ℓ-algebra. For a proper prime ideal I of A, the following are equivalent:*

(a) *I is a minimal prime ideal,*

(b) *$I = \bigcup\{g^\perp \mid g \notin I\}$,*

(c) *$\forall x \in I$, $x^\perp \nsubseteq I$.*

Proof. (a) $\Longrightarrow$ (b).

If I is a minimal prime ideal, then by Proposition 2.3, $\mathscr{U} = A^+ \setminus I$ is an ultrafilter. Let $g \in \mathscr{U}$, then $g^\perp = \{x \in A \mid (x * 0) \wedge (g * 0) = 0\}$ and $(g^\perp)^+ = \{x \in A^+ \mid (x * 0) \wedge (g * 0) = 0\} = \{x \in A^+ \mid x \wedge g = 0\}$ (since A is semiregular). Now $x \wedge g = 0$ and $g \in \mathscr{U}$ an ultrafilter, implies $x \notin \mathscr{U} = A^+ \setminus I$. Therefore $(g^\perp)^+ \subset I$ and so $g^\perp \subseteq I$. Let $J = \bigcup\{g^\perp \mid g \notin I\} = \bigcup\{g^\perp \mid g \in \mathscr{U}\} \subseteq I$. It must now be shown that J is a prime ideal of A.

i) Let a, $b \in J$. Then $a \in g^\perp$, $b \in h^\perp$ for some g, $h \in \mathscr{U}$, and thus $a + b \in I(g^\perp \cup h^\perp)$. But $g \wedge h \in \mathscr{U}$ implies $(g \wedge h)^\perp \subseteq J$ and since $g^\perp \cup h^\perp \subseteq (g \wedge h)^\perp$, $a + b \in I(g^\perp \cup h^\perp) \subseteq (g \wedge h)^\perp \subseteq J$.

ii) Let $a \in J$, $b \in A$ and $b * 0 \leqslant a * 0$. Then $a \in g^\perp$ for some $g \in \mathscr{U}$, which implies $b \in g^\perp$ since $g^\perp$ is an ideal of A, and therefore $b \in J$.

Thus J is an ideal. To show that J is prime let $0 \leqslant a \wedge b \in J$. Then for some $g \in \mathscr{U}$, $a \wedge b \in g^\perp \Longrightarrow (a \wedge b) \wedge g = 0$ and, since $\mathscr{U}$ is a filter, $a \wedge b \notin \mathscr{U}$ and either $a \notin \mathscr{U}$ or $b \notin \mathscr{U}$. Now if $0 \leqslant x \notin J$, then $x \wedge g > 0$ for all $g \in \mathscr{U}$ and since $\mathscr{U}$ is an ultrafilter on A^+, $x \in \mathscr{U}$. Therefore, since either $a \notin \mathscr{U}$ or $b \notin \mathscr{U}$, it must be the case that either $a \in J$ or $b \in J$, and thus J is a prime ideal. Finally since J is a prime ideal contained in I, and I is a minimal prime ideal, $I = J = \bigcup\{g^\perp \mid g \notin I\}$.

(b) $\Longrightarrow$ (c).

Let $x \in I$, then $\exists g \notin I$ such that $x \in g^\perp$, which implies $(g * 0) \wedge (x * 0) = 0$ and $g \in x^\perp$. Thus $x^\perp \nsubseteq I$.

(c) $\Longrightarrow$ (a).

Suppose J is an ideal properly contained in I, and let $x \in I^+ \setminus J$. Then $x^\perp \nsubseteq I$ implies $\exists y \notin I$ with $x \wedge y = 0$. Then $x \wedge y = 0$ and $x \notin J$, $y \notin J$ implies that J is not a prime ideal. Therefore I is a minimal prime ideal. □

Corollary 2.5. *Let A be a normal semiregular autometrized ℓ-algebra. Each polar P in A is the intersection of all those minimal prime ideals not containing $P^\perp$.*

P r o o f. Let P be a polar in A and suppose M is a minimal prime ideal of A such that $P^\perp$ is not contained in M. Then there exists $x \in (P^\perp)^+ \setminus M$. Since $x \in P^\perp$, $x^\perp \supseteq P^{\perp\perp} = P$; also $x \notin M$, M a minimal prime ideal implies $x^\perp \subseteq M$. Thus $P \subseteq x^\perp \subseteq M$ and the intersection of such minimal prime ideals contains P. If $x \notin P$, then there exists $y \in P^\perp$ with $(x * 0) \wedge (y * 0) > 0$ and therefore there is an ultrafilter $\mathscr{U}$ on A^+ containing $x * 0$ and $y * 0$. Then $M = I(A^+ \setminus \mathscr{U})$ is a minimal prime ideal that does not contain $P^\perp$ (since $y \notin M$) and further $x \notin M$. Thus for each $x \notin P$, x is not in the intersection of all minimal prime ideals not containing $P^\perp$. Therefore the intersection of all such minimal prime ideals is P. □

Proposition 2.6. *Let A be a normal semiregular autometrized ℓ-algebra. Let $a \in A$, let $X = \{x \mid 0 \leqslant x \leqslant a * 0\}$ and let $\mathscr{U}$ be an ultrafilter on X. Then $M = \bigcup\{x^\perp \mid x \in \mathscr{U}\}$ is a minimal prime ideal of A and $a \notin M$. Moreover each minimal prime ideal of A not containing a is obtained in this way.*

P r o o f. Let $\mathscr{V} = \{y \in A \mid y \geqslant x$ for some $x \in \mathscr{U}\}$. Since $0 \leqslant x \leqslant y$ implies $y^\perp \subseteq x^\perp$, $\bigcup\{y^\perp \mid y \in \mathscr{V}\} \subseteq \bigcup\{x^\perp \mid x \in \mathscr{U}\}$. However $\mathscr{U} \subseteq \mathscr{V}$, so the reverse inclusion also holds and $\bigcup\{y^\perp \mid y \in \mathscr{V}\} = \bigcup\{x^\perp \mid x \in \mathscr{U}\}$. Clearly $\mathscr{V}$ is a filter on A^+. Further $\mathscr{V}$ is an ultrafilter on A^+, for if not there exists $0 < b \in A^+ \setminus \mathscr{V}$ such that $b \wedge y > 0$ for all $y \in \mathscr{V}$, and then for each $x \in \mathscr{U} \subseteq \mathscr{V}$, $\big(b \wedge (a * 0)\big) \wedge x = b \wedge \big((a*0) \wedge x\big) = b \wedge x > 0$. Then since $\mathscr{U}$ is an ultrafilter on X, $b \wedge (a*0) \in \mathscr{U} \subseteq \mathscr{V}$, whence $b \geqslant b \wedge (a * 0)$ implies $b \in \mathscr{V}$, a contradiction.

Now since $\mathscr{V}$ is an ultrafilter on A^+, there is a minimal prime ideal M of A with $A^+ \setminus M = \mathscr{V}$, and then $a \notin M = \bigcup\{y^\perp \mid y \notin M\} = \bigcup\{y^\perp \mid y \in A^+ \setminus M\} = \bigcup\{y^\perp \mid y \in \mathscr{V}\} = \bigcup\{x^\perp \mid x \in \mathscr{U}\}$.

Conversely suppose that M is a minimal prime ideal of A and $a \notin M$. Then $\mathscr{V} = A^+ \setminus M$ is an ultrafilter on A^+, $a * 0 \in \mathscr{V}$, and $M = \bigcup\{x^\perp \mid x \in \mathscr{V}\}$. Let $X = \{x \mid 0 \leqslant x \leqslant a * 0\}$ and let $\mathscr{U} = \mathscr{V} \cap X$. Clearly $\mathscr{U}$ is a filter on X. If $0 \leqslant b \leqslant a * 0$ and $b \notin \mathscr{U}$, then $b \notin \mathscr{V}$ and so $b \wedge y = 0$ for some $y \in \mathscr{V}$. Let

$x \in \mathscr{U}$, then $0 < x \wedge y \in \mathscr{V}$ and $0 < x \wedge y \leqslant x \leqslant a * 0$ so that $x \wedge y \in \mathscr{U}$. Also $b \wedge (x \wedge y) = x \wedge (b \wedge y) = 0$, and thus if $b \in X \setminus \mathscr{U}$ then $b \wedge c = 0$ for some $c \in \mathscr{U}$ and therefore $\mathscr{U}$ is an ultrafilter on X.

Then from above $M' = \bigcup\{x^{\perp} \mid x \in \mathscr{U}\}$ is a minimal prime ideal of A. Further $\mathscr{U} \subseteq \mathscr{V}$ implies $M' \subseteq \bigcup\{y^{\perp} \mid y \in \mathscr{V}\} = M$, and since M is a minimal prime ideal $M = M' = \bigcup\{x^{\perp} \mid x \in \mathscr{U}\}$ as required. □

An argument similar to that above gives the following result.

Proposition 2.7. *Let A be a normal semiregular autometrized ℓ-algebra. Let I be an ideal of A and $\mathscr{U}$ an ultrafilter on I^+. Then $M = \bigcup\{x^{\perp} \mid x \in \mathscr{U}\}$ is a minimal prime ideal of A not containing I. Moreover each minimal prime ideal of A not containing I is obtained in this way.*

3. Representative DRℓ-semigroups

The notation of a dually residuated lattice ordered semigroup (DRℓ-semigroup) was introduced by Swamy [6].

A system $A = (A, +, \leqslant, -)$ is a DRℓ-semigroup if

(i) $(A, +, \leqslant)$ is a commutative lattice ordered semigroup with a zero element 0;

(ii) for each $a, b \in A$ there exists a least element $x \in A$ such that $b + x \geqslant a$, and this element x is denoted by $a - b$;

(iii) for all $a, b \in A$, $(a - b) \vee 0 + b \leqslant a \vee b$;

(iv) for all $a \in A$, $a - a \geqslant 0$.

If we define $a * b = (a - b) \wedge (b - a)$ for each $a, b \in A$, then $(A, +, \leqslant, *)$ is a semiregular normal autometrized ℓ-algebra.

A DRℓ-semigroup is called *representable* if $\forall a, b \in A$, $(a - b) \wedge (b - a) \leqslant 0$. Examples of representable DRℓ-semigroups include abelian ℓ-groups and Boolean algebras.

Proposition 3.1. *An ideal I of a DRℓ-semigroup A is a prime ideal if and only if for all $a, b \in A$, $a \wedge b = 0$ implies $a \in I$ or $b \in I$.*

Proof. Clearly if I is a prime ideal of A, then $a \wedge b = 0$ implies $a \wedge b \in I$ and therefore $a \in I$ or $b \in I$.

Conversely suppose that I satisfies the condition $a \wedge b = 0$ implies $a \in I$ or $b \in I$ and let $0 \leqslant x \wedge y \in I$. Then by Lemma 6 of Rachůnek [2], $(x-(x \wedge y)) \wedge (y-(x \wedge y)) = 0$ which implies $x - (x \wedge y) \in I$ or $y - (x \wedge y) \in I$. Without loss of generality assume $x - (x \wedge y) \in I$. Then $0 \leqslant x \leqslant (x \wedge y) + (x - (x \wedge y)) \in I$ and by the convexity of I, $x \in I$. Therefore I is a prime ideal. □

Corollary 3.2. *If $I \subseteq J$ are ideals of a* DRℓ*-semigroup A and I is prime, then J is a prime ideal.*

Proof. Since I is a prime ideal, if $a \wedge b = 0$ then $a \in I$ or $b \in I$. But $I \subseteq J$ and thus $a \wedge b = 0$ implies $a \in J$ or $b \in J$; therefore J is a prime ideal of A. $\square$

One consequence of Proposition 3.1 is that for a DRℓ-semigroup A, Proposition 2.4 ca be strengthened in the following way.

Proposition 3.3. *For a proper ideal I of a representable* DRℓ*-semigroup A, the following are equivalent:*

(a) *I is a minimal prime ideal;*
(b) *$A^+ \setminus I$ is an ultrafilter on A^+;*
(c) *$I = \bigcup\{a^\perp \mid a \notin I\}$;*
(d) *I is prime and for all $x \in I$, $x^\perp \not\subseteq I$.*

Proof. From Propositions 2.3 and 2.4, all that remains to be shown is that if $I = \bigcup\{a^\perp \mid a \notin I\}$, then I is prime. Suppose x, $y \in A$ with $x \wedge y = 0$. Then $y \in x^\perp$ and if $x \notin I$, $x^\perp \subseteq I$, therefore $x \in I$ or $y \in I$, and thus I is prime. $\square$

Definition. For each $a \in A$, define the *positive part of a* to be $a^+ = (a-0) \vee 0 = a \vee 0$ and define the *negative part of a* to be $a^- = (0-a) \vee 0$.

The properties of a^+ and a^- are given in the following proposition.

Proposition 3.4. *Let A be a representative* DRℓ*-semigroup and let a, $b \in A$. Then*

(i) $$a = a^+ \iff a \geqslant 0;$$
(ii) $$a^- = 0 \iff a \geqslant 0;$$
(iii) $$a^+ = 0 \iff a \leqslant 0;$$
(iv) $$a^+ \wedge a^- = 0;$$
(v) $$a^+ \vee a^- = a * 0 = a^+ + a^-;$$
(vi) $$a + a^- = a^+;$$
(vii) $$(a+b)^+ \leqslant a^+ + b^+.$$

Proof.

(i) $$a = a^+ \iff a = a \vee 0 \iff a \geqslant 0.$$
(ii) $$a^- = 0 \iff (0-a) \vee 0 \iff 0 - a \leqslant 0 \iff a \geqslant 0.$$
(iii) $$a^+ = 0 \iff a \vee 0 = 0 \iff a \leqslant 0.$$

$$
\begin{aligned}
\text{(iv)} \qquad a^+ \wedge a^- &= \big[(a - 0 \vee 0\big] \wedge \big[(0 - a) \vee 0\big] \\
&= \big[(a - 0) \wedge (0 - a)\big] \vee 0 \\
&= 0 \quad \text{since } A \text{ is a representable DR}\ell\text{-semigroup.} \\
\text{(v)} \qquad a^+ \vee a^- &= \big[(a - 0 \vee 0\big] \vee \big[(0 - a) \vee 0\big] \\
&= \big[(a - 0) \vee (0 - a)\big] \vee 0 \\
&= (a * 0) \vee 0 \\
&= a * 0. \\
a^+ + a^- &= (a^+ \vee a^-) + (a^+ \wedge a^-) \\
&= (a^+ \vee a^-) + 0 \\
&= a^+ \vee a^- \\
&= a * 0. \\
\text{(vi)} \qquad a + a^- &= a + \big[(0 - a) \vee 0\big] \\
&= \big[a + (0 - a)\big] \vee [a + 0] \\
&= \big[a + (0 - a)\big] \vee a \\
&\geqslant 0 \vee a \\
&= a^+ \\
\text{and also } a + a^- &= a + \big[(0 - a) \vee 0\big] \leqslant a \vee 0 = a^+; \\
\text{therefore } a + a^- &= a^+.
\end{aligned}
$$

(vii) $a \leqslant a \vee 0 = a^+$, $b \leqslant b \vee 0 = b^+$ and thus $a + b \leqslant a^+ + b^+$.

Therefore $(a + b)^+ = (a + b) \vee 0 \leqslant (a^+ + b^+) \vee 0 = a^+ + b^+$.

□

Let $0 \neq a \in A$. An ideal I of A is a *value* of the element a if I is maximal with respect to not containing a. The set of all values of a is denoted by $\mathrm{val}(a)$. Rachůnek [4] has shown that if I is an ideal of A and $a \notin I$ then the is a value of a containing I, and also that every ideal which is the value of some element $a \in A$, is a prime ideal.

Proposition 3.5. *Let A be a representable* DRℓ*-semigroup. Then for each $0 \neq a \in A$, $\mathrm{val}(a) = \mathrm{val}(a * 0) = \mathrm{val}(a^+) \cup \mathrm{val}(a^-)$ and $\mathrm{val}(a^+) \cap \mathrm{val}(a^-) = \emptyset$.*

Proof. For each $a \in A$, and each ideal I of A, $a \in I \iff a * 0 \in I$, therefore $\mathrm{val}(a) = \mathrm{val}(a * 0)$. Let $I \in \mathrm{val}(a^+)$. Then $a^+ \notin I$, $a^+ \wedge a^- = 0$ and, since I is a prime ideal, $a^- \in I$, therefore $I \notin \mathrm{val}(a^-)$ and $\mathrm{val}(a^+) \cap \mathrm{val}(a^-) = \emptyset$.

If $I \in \mathrm{val}(a^+)$ then $a^+ \notin I$, $a^- \in I$ and since $a^+ = a + a^-$, $a \notin I$. Therefore there is a value I' of a such that $I \subseteq I'$. If $I \subsetneqq I'$ then $a^+ \in I'$, $a^- \in I$ and since $a^+ = a + a^-$, by Lemma 2 of [1], $a \in I'$ contradicting that I' is a value of a. Therefore

$I' = I$ and $\mathrm{val}(a^+) \subseteq \mathrm{val}(a)$. A similar argument shows that $\mathrm{val}(a^-) \subseteq \mathrm{val}(a)$ and therefore $\mathrm{val}(a^+) \cup \mathrm{val}(a^-) \subseteq \mathrm{val}(a)$. To show the reverse inclusion let $I \in \mathrm{val}(a)$. Since $a^+ = a + a^-$ and $a \notin I$, by Lemma 2 of [1], either $a^+ \notin I$ or $a^- \notin I$. If $a^+ \notin I$, there is a value I' of a^+ with $I' \supseteq I$, and if $I' \supsetneqq I$, then $a^- \in I'$, $a \in I'$ and thus $a^+ = a + a^- \in I'$ contradicting that $I \in \mathrm{val}(a^+)$. Therefore $I' = I$ and $I \in \mathrm{val}(a^+)$. Similarly if $a^- \notin I$ it can be shown that $I \in \mathrm{val}(a^-)$, and thus $\mathrm{val}(a) \subseteq \mathrm{val}(a^+) \cup \mathrm{val}(a^-)$, as required. □

An element of a normal autometrized algebra, A is called *special* if it has a unique value.

Corollary 3.6. *Each special element of a DRℓ-semigroup A is positive or negative.*

Proof. If a is special element of A, then by Proposition 3.5, $\mathrm{val}(a) = \mathrm{val}(a^+) \cup \mathrm{val}(a^-)$ and $\mathrm{val}(a^+) \cap \mathrm{val}(a^-) = \emptyset$, so either $\mathrm{val}(a^+) = \emptyset$ or $\mathrm{val}(a^-) = \emptyset$. Therefore either $a^+ = 0$ or $a^- = 0$ which implies that either $a \leqslant 0$ or $a \geqslant 0$ so that every special element of A is positive or negative. □

Proposition 3.7. *Let A be a representative DRℓ-semigroup. Then A is totally ordered if and only if $x \wedge y = 0$ implies $x = 0$ or $y = 0$.*

Proof. If A is totally ordered the condition is obviously satisfied. Conversely suppose A satisfied $x \wedge y = 0$ implies $x = 0$ or $y = 0$. For each $x \in A$, $x^+ \wedge x^- = 0$ and thus $x^+ = 0$ or $x^- = 0$ and so either $x \leqslant 0$ or $x \geqslant 0$. Therefore every element of A is comparable to 0. Let $a, b \in A$, then either $a - b \leqslant 0$ or $b - a \leqslant 0$, for otherwise $a - b > 0$ and $b - a > 0$ implies $(a-b) \wedge (b-a) > 0$ which contradicts that $(a-b) \wedge (b-a) \leqslant 0$. Therefore $a \leqslant b$ or $b \leqslant a$ and thus A is totally ordered. □

Proposition 3.8. *Let $P \neq \{0\}$ be a polar in A. Then the following are equivalent:*
(a) *P is totally ordered;*
(b) *$P^\perp$ is prime;*
(c) *$P^\perp$ is a minimal prime;*
(d) *$P^\perp$ is a maximal polar;*
(e) *P is a minimal polar.*

Proof. (a) $\Longrightarrow$ (b). If $P^\perp$ is not prime then there exist $x, y \in A^+$ with $0 \leqslant x \wedge y \in P^\perp$ and $x, y \notin P^\perp$. Then $x \notin P^\perp$ implies there is an $a \in P$ with $x \wedge (a * 0) > 0$. Similarly there exists $b \in P$ with $y \wedge (b * 0) > 0$. Then since P is totally ordered, $0 \neq \big(x \wedge (a*0)\big) \wedge \big(y \wedge (b*0)\big) = (x \wedge y) \wedge \big((a*0) \wedge (b*0)\big)$; but $(a*0) \wedge (b*0) \in P$ and thus $x \wedge y \notin P^\perp$, a contradiction. Therefore $P^\perp$ is prime.

(b) $\Longrightarrow$ (c). Since $P^\perp$ is a polar in A, by Corollary 2.5, $P^\perp$ is the intersection of the minimal primes not containing $P^{\perp\perp} = P$. Also since $P^\perp$ is prime, $P^\perp$ contains a minimal prime, M say, and then $P \cap M \subseteq P \cap P^\perp = \{0\}$ so that M does not contain P. Then $M \subseteq P^{\perp} = \bigcap\{\text{minimal primes not containing } P\} \subseteq M$. Therefore $P^{\perp} = M$ and $P^{\perp}$ is a minimal prime.

(c) $\Longrightarrow$ (d). Let Q be a polar in A with $P^{\perp} \subseteq Q$. Since $P^{\perp}$ is a prime ideal, by Corollary 3.2, Q is also a prime ideal; since Q is also a polar, by applying the preceding implication, Q is a minimal prime ideal. Therefore $P^{\perp} = Q$ and $P^{\perp}$ is a maximal polar.

(d) $\Longrightarrow$ (e). For polars P and Q, $Q \subseteq P \iff P^{\perp} \subseteq Q^{\perp}$.Therefore if $P^{\perp}$ is a maximal polar, P must be a minimal polar.

(e) $\Longrightarrow$ (a). Let P be a minimal polar and let $x, y \in P$ with $x \wedge y = 0$. If $x > 0$ then $y \in x^{\perp}$. Also $0 \neq x \in P$ implies that $\{0\} \neq x^{\perp\perp} \subseteq P$ and since P is a minimal polar, $x^{\perp\perp} = P$ and $y \in x^{\perp\perp}$. Therefore $y \in x^{\perp} \cap x^{\perp\perp} = \{0\}$. Thus for $x, y \in P$, $x \wedge y = 0$ implies $x = 0$ or $y = 0$ and by Proposition 3.7, P is totally ordered. $\square$

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