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Melvin F. Janowitz

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TOLERANCES, INTERVAL ORDERS, AND SEMIORDERS

M. F. JANOWITZ,* Amherst

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The paper discusses interval orders and semiorders from the viewpoint of tolerance relations on lattices. By concentrating on properties of the associated indifference relations, it is possible to characterize interval orders as meet tolerances and semiorders as lattice tolerances on a chain. Some consideration is also given to partial interval orders and partial semiorders and they are related to certain kinds of poset tolerances.

1. Introduction

Preference theory abounds with instances where an individual may not be able to decide between alternatives x, y or between alternatives y, z, and yet may still be able to decide between x and z. Luce ([11]) defined semiorders and Fishburn ([7]) introduced interval orders to provide mathematical descriptions of such situations. Doignon ([4]) and Fishburn ([7], pp. 19–21) are good sources for references to even earlier work involving these concepts. For example, interval orders are a special case of a scale introduced by Guttman ([8]) and were recognized as an important concept in a paper by Wiener ([16]). The theory of interval orders and semiorders has been largely developed in the context of relational systems. Lattice tolerances on the other hand have mainly been studied as generalizations of congruence relations. Our goal here is to show that there is a natural one-one correspondence between semiorders and lattice tolerances on a chain and to also establish a similar correspondence between interval orders and semilattice tolerances on a chain. Note that some early work relating semiorders on a finite set to tolerances on a finite chain appears in [10].

Interval orders (see Fishburn, [6], p. 144) are often defined to be irreflexive relations $\mathbf{P}$ on a set X that satisfy the *interval order condition*:

$$\text{If } x\mathbf{P}y \text{ and } z\mathbf{P}w \text{ then } x\mathbf{P}w \text{ or } z\mathbf{P}y.$$

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A *semiorder* is taken to be an interval order satisfying the following condition:

$$\text{If } x\mathbf{P}y \text{ and } y\mathbf{P}z \text{ then } x\mathbf{P}w \text{ or } w\mathbf{P}z \quad \text{for each } w \in X.$$

It should be noted that this definition of semiorder is equivalent to the definition originally given by Luce ([11]) when he formally introduced and studied this concept. Fishburn's definition involves only the strict preference $\mathbf{P}$ associated with the semiorder, while Luce uses both $\mathbf{P}$ and the indifference relation $\mathbf{I}$ defined by $x\mathbf{I}y$ if neither $x\mathbf{P}y$ nor $y\mathbf{P}x$ is true. By concentrating primarily on $\mathbf{I}$, we shall establish the desired connection with tolerances on a chain, and show that inteval orders can be profitably studied solely in terms of their associated indifference relations.

To accomplish our goals some preliminary notions are introduced in Section 2, while Section 3 is devoted to an examination of the way that tolerances are induced by certain types of mappings, and in Section 4 these results are extended to a semilattice setting. Having developed this machinery, we can proceed in Section 5 to establish the connection with interval orders and semiorders. Finally, in Sections 6 and 7, consideration is given to partial interval orders and partial semiorders (see [4], [5], [14] and [15]).

2. Preliminaries

A working knowledge of lattice theory will be assumed throughout the paper. The reader is urged to consult any standard text ([3], for example) on lattice theory for a definition of any unfamiliar term. Let $\mathbf{P}$ be a partially ordered set (poset). For each $x \in \mathbf{P}$, let $J_x = \{t \in L : t \leqslant x\}$ denote the principal ideal generated by x, and F_x the principal filter it generates. $\mathcal{O}\mathcal{I}(\mathbf{P})$ will denote the order ideals of $\mathbf{P}$ partially ordered by set inclusion, and $\mathcal{O}\mathcal{F}(\mathbf{P})$ the order filters of $\mathbf{P}$ partially ordered by the converse of set inclusion. For $\mathbf{P}$ a lattice, $\mathcal{I}(\mathbf{P})$ will denote the (lattice) ideals of $\mathbf{P}$, and $\mathcal{F}(\mathbf{P})$ its (lattice) filters, with both sets ordered as expected.

We shall also need the notion of a *residuated mapping* from a poset $\mathbf{P}$ into a poset $\mathbf{Q}$. We say that $f : \mathbf{P} \to \mathbf{Q}$ is *residuated* if f is isotone and there exists an isotone mapping $h : \mathbf{Q} \to \mathbf{P}$ such that:

$$\text{(R1)} \quad p \leqslant h(f(p)) \quad \text{for all } p \in \mathbf{P}, \text{ and}$$

$$\text{(R2)} \quad q \geqslant f(h(q)) \quad \text{for all } q \in \mathbf{Q}.$$

The mapping h is called the *residual mapping* associated with f, and it is easy to see that h uniquely determines and is uniquely determined by f. For that reason the pair (f, h) is sometimes called a *residuated-residual pair*.

If $\mathbf{P} = \mathbf{Q}$, the residuated mapping f is called *decreasing* in case $f(p) \leqslant p$ for all $p \in \mathbf{P}$. This is equivalent to h being *increasing* in the sense that $h(p) \geqslant p$

for all $p \in \mathbf{P}$. It will be convenient to let $\mathrm{Res}(\mathbf{P}, \mathbf{Q})$ denote the set of all residuated mappings from $\mathbf{P}$ into $\mathbf{Q}$, with $\mathrm{Res}^+(\mathbf{Q}, \mathbf{P})$ denoting the corresponding set of residual mappings from $\mathbf{Q}$ into $\mathbf{P}$. When $\mathbf{P} = \mathbf{Q}$, this notation will be shortened to $\mathrm{Res}(\mathbf{P})$, and $\mathrm{Res}^+(\mathbf{P})$.

There is another type of mapping that turns out to be relevant to the upcoming discussion. For join semilattices, we shall say that f is a *partial join homomorphism* in case $f(p \vee p') \leqslant f(p) \vee f(p')$ for all $p, p' \in \mathbf{P}$; the dual notion of a *partial meet homomorphism* on a (meet) semilattice requires that $f(p \wedge p') \geqslant f(p) \wedge f(p')$. As expected, a *partial homomorphism* on a lattice is taken to be a mapping that is both a partial join and a partial meet homomorphism. It should be noted that partial homomorphisms need *not* be isotone. Partial join homomorphisms are characterized by the requirement that for each $q \in \mathbf{Q}$, $f^{-1}(J_q)$ shall be a join subsemilattice of $\mathbf{P}$, while partial meet homomorphisms require the dual condition that $f^{-1}(F_q)$ be a meet subsemilattice of $\mathbf{P}$.

The notion of a *tolerance* dates back at least to Zeeman ([17]). Tolerances on algebraic structures were introduced by Zelinka ([18]) and have been studied by many authors. Our immediate interest lies with a (lattice) tolerance $\mathbf{T}$ on a lattice $\mathbf{L}$. This is a reflexive symmetric relation $\mathbf{T}$ on $\mathbf{L}$ that is *compatible* in the following sense:

$$a\mathbf{T}b,\ c\mathbf{T}d \text{ implies that } a \vee c\mathbf{T}b \vee d \text{ and } a \wedge c\mathbf{T}b \wedge d \quad \text{for all } a, b, c, d \in \mathbf{L}.$$

It will be convenient to simply use the phrase "tolerance relation" on a lattice to refer to a lattice tolerance relation.

3. Residuated-residual schemes and tolerances

Our goal in this section is to generalize [9], Theorem 12, p. 114 by establishing a bijection between arbitrary lattice tolerances and a generalization of decreasing residuated mappings. First we need some terminology.

Definition 3.1. A *residuated-residual scheme* on a lattice $\mathbf{L}$ is a quadruple (f, g_1, h, g_2) of mappings such that:

- (S1) For $i = 1, 2$ g_i is an isotone mapping of $\mathbf{L}$ into the lattice $\mathbf{L}_i$.
- (S2) f is a partial join homomorphism of $\mathbf{L}$ into $\mathbf{L}_1$.
- (S3) h is a partial meet homomorphism of $\mathbf{L}$ into $\mathbf{L}_2$.
- (S4) $f(x) \leqslant g_1(y)$ in $\mathbf{L}_1 \Leftrightarrow g_2(x) \leqslant h(y)$ in $\mathbf{L}_2$.

In the above definition, note in particular that $f(x) \leqslant g_1(x) \Leftrightarrow h(x) \geqslant g_2(x)$. The scheme (f, g_1, h, g_2) is called *decreasing* if these inequalities hold for all $x \in \mathbf{L}$. When

$\mathbf{L} = \mathbf{L}_1 = \mathbf{L}_2$ and $g_1 = g_2 =$ the identity map, then a residuated-residual scheme becomes a residuated-residual pair. Our goal is to establish a connection between lattice tolerances and decreasing residuated-residual schemes.

Remark 3.2. It is easy to show that any decreasing residuated-residual scheme (f, g_1, h, g_2) defines a tolerance $\mathbf{T}$ on $\mathbf{L}$ by the rule $x\mathbf{T}y$ if and only if $g_2(x) \leqslant h(y)$ and $g_2(y) \leqslant h(x)$.

Definition 3.3. The residuated-residual scheme (f, g_1, h, g_2) will be called a *standard scheme* in case:

$$g_1(x) = F_x \text{ and } g_2(x) = J_x \text{ for all } x \in \mathbf{L},$$
$$f\colon \mathbf{L} \to \mathscr{F}(\mathbf{L}) \text{ is a join homomorphism,}$$
$$h\colon \mathbf{L} \to \mathscr{I}(\mathbf{L}) \text{ is a meet homomorphism, and}$$
$$(f, g_1, h, g_2) \text{ is decreasing.}$$

It is of some interest to note that if every ideal and every filter of $\mathbf{L}$ is principal, then standard schemes may be identified with residuated-residual pairs on $\mathbf{L}$. We summarize the connection between tolerances and decreasing residuated-residual schemes with the following theorem. A related result establishing a bijection between tolerances on $\mathbf{L}$ and certain galois connections between $\mathscr{I}(\mathbf{L})$ and $\mathscr{F}(\mathbf{L})$ was established earlier in [2].

Theorem 3.4. *Let* $\mathbf{L}$ *be a lattice. There is a natural one-one correspondence between tolerances on* $\mathbf{L}$ *and standard schemes.*

Proof. Let $\mathbf{T}$ be a tolerance on $\mathbf{L}$. For each $x \in \mathbf{L}$, define $g_1(x) = F_x$ and $g_2(x) = J_x$. Next define $h\colon \mathbf{L} \to \mathscr{I}(\mathbf{L})$ by

$$h(x) = \{w \in \mathbf{L}\colon w \leqslant v \text{ for some } v \text{ such that } v\mathbf{T}x\}.$$

Similarly, define $f\colon \mathbf{L} \to \mathscr{F}(\mathbf{L})$ by

$$f(x) = \{w \in \mathbf{L}\colon w \geqslant v \text{ for some } v \text{ such that } v\mathbf{T}x\}.$$

Thus $h(x)$ is the ideal generated by $\{w \in \mathbf{L}\colon w\mathbf{T}x\}$, and $f(x)$ is the filter generated by this set.

Evidently g_1, g_2 are both homomorphisms, so we next show that h is a meet homomorphism. Hence let $x, x' \in \mathbf{L}$ and note that $w \in h(x \wedge x')$ implies that $w \leqslant v$ for some v such that $v\mathbf{T}x \wedge x'$. But then $v \vee x\mathbf{T}x$, $v \vee x'\mathbf{T}x'$, and this shows $w \in h(x) \wedge h(x')$. On the other hand, $w \in h(x) \wedge h(x')$ forces the existence of elements v, v' where $w \leqslant v \wedge v'$, $v\mathbf{T}x$ and $v'\mathbf{T}x'$. It follows that $w \in h(x \wedge x')$ since $(v \wedge v')\mathbf{T}(x \wedge x')$. A dual argument establishes that f is a join homomorphism.

Suppose now that $f(x) \leqslant g_1(y)$ in $\mathscr{F}(\mathbf{L})$. This says that $y \in f(x)$, so $y \geqslant v$ for some v such that $v\mathbf{T}x$. Then $v \wedge x\mathbf{T}x$, and $x \geqslant x \wedge y \geqslant v \wedge x$ implies $x\mathbf{T}x \wedge y$. But joining both sides of this equation with y tells us that $x \vee y\mathbf{T}y$, whence $x \in h(y)$ or in other words, $g_2(x) \leqslant h(y)$. A similar argument produces the reverse implication. Evidently $g_2(x) \leqslant h(x)$ and $g_1(x) \geqslant f(x)$ for all $x \in \mathbf{L}$.

At this point we have established that (f, g_1, h, g_2) is a standard scheme. We must still show that it induces the original tolerance $\mathbf{T}$. It will clearly suffice to establish that $a\mathbf{T}b$ is equivalent to the assertion that $g_2(a) \leqslant h(b)$ and $g_2(b) \leqslant h(a)$. If $a\mathbf{T}b$, then clearly $b \in h(a)$ and $a \in h(b)$. Suppose conversely that $a \in h(b)$ and $b \in h(a)$. There then exist elements v, w such that $a \leqslant v\mathbf{T}b$ and $b \leqslant w\mathbf{T}a$. Then $a = a \wedge v\mathbf{T}w \wedge b = b$, thus completing the proof. We leave to the reader the routine proof that $\mathbf{T}$ can be induced by at most one standard scheme (f, g_1, h, g_2). □

We show next that tolerances may always be defined by a suitable *pair* of mappings. Let g be a join homomorphism and h a meet homomorphism of the lattice $\mathbf{L}$ into the lattice $\mathbf{M}$ such that $g(x) \leqslant h(x)$ for all $x \in \mathbf{L}$. Define $\mathbf{T}$ by the requirement that $a\mathbf{T}b$ when $g(a) \leqslant h(b)$ and $g(b) \leqslant h(a)$. Then $\mathbf{T}$ is easily shown to be a tolerance. For if $a\mathbf{T}b$ and $c\mathbf{T}d$, then

$$g(a \vee c) = g(a) \vee g(c) \leqslant h(b) \vee h(d) \leqslant h(b \vee d), \text{ and}$$
$$g(a \wedge c) \leqslant g(a) \wedge g(c) \leqslant h(b) \wedge h(d) = h(b \wedge d).$$

Similarly, $g(c \vee d) \leqslant h(a \vee b)$ and $g(c \wedge d) \leqslant h(a \wedge b)$, whence $a \vee c\mathbf{T}b \vee d$ and $a \wedge c\mathbf{T}b \wedge d$. We shall denote this by saying that the tolerance $\mathbf{T}$ is *induced* by the *L-pair* (h, g).

Remark 3.5. If (h, g) is *an L-pair on the lattice* $\mathbf{L}$, and $\mathbf{T}$ is its induced tolerance, it is easily shown that the following conditions are equivalent: (i) $g(a) \leqslant h(b)$; (ii) $a \leqslant w$ *for some* $w\mathbf{T}b$; (iii) $b \geqslant v$ *for some* $v\mathbf{T}a$.

Definition 3.6. Proceeding as in Definition 3.3, we take a lattice $\mathbf{L}$ and define $g(x) = J_x$ and assume that $h \colon \mathbf{L} \longrightarrow \mathscr{I}(\mathbf{L})$ is a meet homomorphism such that $g(x) \leqslant h(x)$ for all $x \in \mathbf{L}$. Such a pair of mappings will be called a *standard L-pair*. A related construction appears in [1] (Proposition 1.3, p. 372) and has been generalized in [13], pp. 142–143.

Theorem 3.7. *There is a bijection between tolerances on the lattice* $\mathbf{L}$ *and standard L-pairs.*

P r o o f. By the remarks preceding Lemma 3.5, every standard L-pair induces a tolerance in $\mathbf{L}$. The remainder of the proof is left for the reader. □

4. Semilattice tolerances

In this section it will be assumed that **L**, **M** denote semilattices. The pair (h, g) of mappings of **L** into **M** will be called an *SL-pair* in case:

g is isotone,

h is a partial meet homomorphism, and

$g(x) \leqslant h(x)$ for all $x \in \mathbf{L}$.

Associated with such a pair (h, g) there is the tolerance relation **T** defined by $a\mathbf{T}b$ if $g(a) \leqslant h(b)$ and $g(b) \leqslant h(a)$. In fact **T** is a meet tolerance on **L**. For if $a\mathbf{T}b$ and $c\mathbf{T}d$, then

$$g(a \wedge c) \leqslant g(a) \wedge g(c) \leqslant h(b) \wedge h(d) \leqslant h(b \wedge d),$$

and a similar argument shows that $g(b \wedge d) \leqslant h(a \wedge c)$, whence $a \wedge c\mathbf{T}b \wedge d$. There are dual notions regarding join semilattices and join tolerances that we shall not bother to state.

Definition 4.1. A *standard SL-pair* is taken to be an SL-pair (h, g) of mappings of **L** into $\mathcal{O}\mathcal{J}(\mathbf{L})$ with $g(x) = J_x$.

Theorem 4.2. (i) *Every SL-pair on the semilattice* **L** *induces a meet tolerance on* **L**.

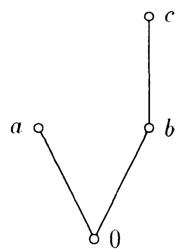
(ii) *Every meet tolerance on a semilattice* **L** *is induced by an SL-pair.*

Proof. The opening remarks of the section established (i), so we need only consider (ii). Taking $g(x) = J_x$ and $h(x)$ the order ideal generated by $\{w\colon w\mathbf{T}x\}$, it is clear that (h, g) is an SL-pair of mappings from **L** into $\mathcal{O}\mathcal{J}(\mathbf{L})$. Evidently,

$$x\mathbf{T}y \Rightarrow g(x) \leqslant h(y) \text{ and } g(y) \leqslant h(x).$$

Suppose conversely that $g(x) \leqslant h(y)$ and $g(y) \leqslant h(x)$. There then exist elements $v, w \in \mathbf{L}$ such that $x \leqslant v$, $y \leqslant w$, $v\mathbf{T}y$ and $w\mathbf{T}x$. But then $x = x \wedge v\mathbf{T}w \wedge y = y$ shows that $x\mathbf{T}y$, thus establishing that **T** is induced by the pair (h, g). □

Remark 4.3. There is no natural one-one correspondence between meet tolerances on **L** and standard SL-pairs of mappings. To see this, consider the semilattice **L** indicated below:



Consider the meet congruence $\mathbf{T}$ on $\mathbf{L}$ defined by $x\mathbf{T}y$ if and only if $x \wedge b = y \wedge b$. This congruence has the classes $\{0, a\}$ and $\{b, c\}$. Letting

$$\begin{aligned} g(x) &= g'(x) = J_x, \text{ with} \\ h(a) &= h'(a) = h(0) = h'(0) = \{0, a\}, \\ h(b) &= h(c) = \{0, b, c\}, \text{ and} \\ h'(b) &= h'(c) = \{0, a, b, c\}, \end{aligned}$$

we have that (h, g) and (h', g') are distinct standard SL-pairs that each induce $\mathbf{T}$. The difficulty arises from the fact that if a standard SL-pair (h'', g'') induces a tolerance $\mathbf{T}''$, then $g''(x) \leqslant h''(y)$ need not imply the existence of an element w such that $x \leqslant w$ with $w\mathbf{T}y$. Corollary 4.4 does however contain a special case that will be of interest in the next section.

Corollary 4.4. *Let* $\mathbf{L}$ *be a chain. There is then a natural one-one correspondence between meet tolerances and standard* SL*-pairs.*

Proof. We need only notice that if $\mathbf{T}$ is induced by the standard SL-pair (h, g), and if $g(y) \leqslant h(x)$ then

$$\begin{aligned} &y \leqslant x \text{ implies } y \leqslant x \text{ with } x\mathbf{T}x, \text{ while} \\ &x \leqslant y \text{ implies } x\mathbf{T}y, \text{ so } y \leqslant y \text{ with } y\mathbf{T}x. \end{aligned}$$

Thus $h(x) = \{y \colon y \leqslant w \text{ for some } w\mathbf{T}x\}$, and this completely specifies the mapping h.
□

5. Tolerances on a chain

When $\mathbf{L}$ is a chain so are $\mathscr{I}(\mathbf{L})$ and $\mathscr{F}(\mathbf{L})$; it follows that every tolerance $\mathbf{T}$ is induced by an L-pair (h, g) where h, g are each homomorphisms from $\mathbf{L}$ into a chain $\mathbf{M}$. Since every mappping from a chain into a lattice is a partial homomorphism, the corresponding result for meet tolerances is that every such tolerance $\mathbf{T}$ is induced by an SL-pair (h, g) where g is a homomorphism and h a partial homomorphism. Given the meet tolerance relation $\mathbf{T}$ on the chain $\mathbf{L}$, one can define new relations $\mathbf{R}$ and $\mathbf{P}$ by the rules

$$\begin{aligned} &x\mathbf{R}y \text{ if } x < y \text{ or } x\mathbf{T}y, \\ &x\mathbf{P}y \text{ if } x < y \text{ and } x\mathbf{T}y \text{ fails.} \end{aligned}$$

If $\mathbf{T}$ is induced by the SL-pair (h, g) where g is a homomorphism and h a partial homomorphism of $\mathbf{L}$ into a chain, one then has:

$$\begin{aligned} &x\mathbf{R}y \text{ if and only if } g(x) \leqslant h(y). \\ &x\mathbf{P}y \text{ if and only if } h(x) < g(y). \end{aligned}$$

This observation has some rather interesting implications, since by [5] (Proposition 1, p. 7) this implies that $\mathbf{P}$ is an *interval order*; if h is in fact also a homomorphism (so that $\mathbf{T}$ is necessarily a lattice tolerance), it even implies that $\mathbf{P}$ is a *semiorder*. To see this, note that by using the standard scheme for $\mathbf{T}$, one can assume that:

$$(*) \qquad g(x) \leqslant g(y) \text{ implies } h(x) \leqslant h(y).$$

Suppose that $x\mathbf{P}y$, $y\mathbf{P}z$ but that $x\mathbf{P}w$ fails, so that $w\mathbf{R}x$. This translates to

$$h(x) < g(y),$$
$$h(y) < g(z),$$
$$g(w) \leqslant h(x).$$

It follows that $g(w) < g(y)$ so by $(*)$, $h(w) \leqslant h(y)$. Hence $h(w) < g(z)$. But this says that $w\mathbf{P}z$, whence $\mathbf{P}$ is in fact a semiorder.

We now supply a converse to the above observations. Thus we assume that $\mathbf{P}$ is an *interval order* on $\mathbf{X}$ and define the relation $\mathbf{R}$ by $x\mathbf{R}y$ if and only if $y\mathbf{P}x$ fails. By [5] (Proposition 1, p. 7) there is a chain $(\mathbf{E}, \leqslant)$ and two mappings h, g from $\mathbf{X}$ to $\mathbf{E}$ such that:

$$x\mathbf{P}y \Leftrightarrow h(x) < g(y)$$

and

$$g(x) \leqslant h(x).$$

It follows that

$$x\mathbf{R}y \Leftrightarrow g(x) \leqslant h(y).$$

If we define $\mathbf{T} = \mathbf{R} \cap \mathbf{R}^{-1}$, we then have that:

$$x\mathbf{T}y \Leftrightarrow g(x) \leqslant h(y) \text{ and } g(y) \leqslant h(x).$$

The idea now is to find a linear order $\leqslant_0$ on $\mathbf{X}$ for which $\mathbf{T}$ is a tolerance and $a\mathbf{P}b$ denotes the fact that $a <_0 b$ with $a\mathbf{T}b$ false. We begin by defining a weak order $\mathbf{W}$ by the agreement that:

$$a\mathbf{W}b \Leftrightarrow g(a) \leqslant g(b).$$

Next we let $\leqslant_0$ be an extension of $\mathbf{W}$ to a linear order, so that $a \leqslant_0 b$ implies $a\mathbf{W}b$. But this is all that needs to be done, as it is clear that g is isotone, so by Theorem 4.2, $\mathbf{T}$ is a meet tolerance on $(\mathbf{X}, \leqslant_0)$. Evidently, $a\mathbf{P}b$ implies $a <_0 b$ with $a\mathbf{T}b$ false. Suppose on the other hand that $a <_0 b$ with $a\mathbf{T}b$ false. Since $a\mathbf{P}b$ implies $a <_0 b$, it follows that $a \leqslant_0 b$ implies $a\mathbf{R}b$. Thus we have $a\mathbf{R}b$ but not $b\mathbf{R}a$, and this says that $a\mathbf{P}b$. If $\mathbf{P}$ is a *semiorder*, we apply [5] (Proposition 2, p. 8) to deduce that g, h can be chosen so that also

$$g(x) \leqslant g(y) \Leftrightarrow h(x) \leqslant h(y).$$

But this says that both g h are homomorphisms from $(\mathbf{X}, \leqslant_0)$ into $(\mathbf{E}, \leqslant)$, so by Theorem 3.7, $\mathbf{T}$ is a tolerance on $(\mathbf{X}, \leqslant_0)$. We summarize these observations in the next Theorem.

Theorem 5.1. (i) *If $\mathbf{T}$ is a meet tolerance on the chain $(\mathbf{X}, \leqslant)$, and if $\mathbf{P}$ is defined by $x\mathbf{P}y$ if and only if $x < y$ with $x\mathbf{T}y$ false, then $\mathbf{P}$ is an interval order on $\mathbf{X}$. If $\mathbf{T}$ is a tolerance then $\mathbf{P}$ is a semiorder on $\mathbf{X}$.*

(ii) *If $\mathbf{P}$ is an interval order on the set $\mathbf{X}$, and if $\mathbf{T}$ is defined by $a\mathbf{T}b \Leftrightarrow a\mathbf{P}b$ and $b\mathbf{P}a$ both fail, then there exists a linear order $\leqslant$ on $\mathbf{X}$ having the property that $\mathbf{T}$ is a meet tolerance on $(\mathbf{X}, \leqslant)$ and $a\mathbf{P}b$ is equivalent to $a < b$ with $a\mathbf{T}b$ false. If $\mathbf{P}$ is a semiorder then $\mathbf{T}$ is a tolerance on $(\mathbf{X}, \leqslant)$.*

Remark 5.2. The portion of Theorem 5.1 (ii) relating to semiorders is essentially contained in the discussion of Roberts ([12], pp. 255–8). The correspondence between semiorders on a finite set and tolerances on a finite chain appears in [10].

6. Partial interval orders

Interval orders involve a pair $(\mathbf{P}, \mathbf{I})$ of relations on $\mathbf{X}$ such that: (i) $\mathbf{P}$ is transitive; (ii) $\mathbf{I}$ is reflexive and symmetric; (iii) $\mathbf{P} \cap \mathbf{I} = \emptyset$; (iv) $a, b \in \mathbf{X}$ implies $a\mathbf{P}b$, $a\mathbf{I}b$, or $b\mathbf{P}a$. Condition (iv) is often not met in practical applications. This is especially true in preference modeling that involves multicriteria decisions. It therefore seems desirable to relax condition (iv) and introduce some sort of partial interval order. This has been done in both [5] and [15]. The connection between the two approaches is explained in [5], p. 10 and p. 16. Results related to those of [5] can be found in [4] and the initial work in this area was done by Roy [14]. In that the definitions in [5] bear more directly on what we have in mind, let us begin by examining them.

Definition 6.1 ([5], Definition 4, p. 15). A *partial interval order of type* 1 is a pair of relations $(\mathbf{P}, \mathbf{I})$ on $\mathbf{X}$ such that

$$\mathbf{I} \text{ is symmetric and reflexive,}$$

$$\mathbf{P} \cap \mathbf{I} = \emptyset,$$

$$\mathbf{PIP} \subseteq \mathbf{P}.$$

A *partial interval order of type* 2 is defined by adding

$$\mathbf{IP}^{-1}\mathbf{I} \cap \mathbf{IPI} \subseteq \mathbf{I}.$$

It follows from results in [5] that for a given partial interval order $(\mathbf{P}, \mathbf{I})$ there exists a linearly ordered set $(\mathbf{E}, \leqslant)$ and a pair h, g of mappings from $\mathbf{X}$ into $\mathbf{E}$ having the property that

$$g(x) \leqslant h(x) \text{ for all } x,$$
$$x\mathbf{P}y \text{ implies } h(x) < g(y),$$
$$x\mathbf{I}y \text{ implies } g(x) \leqslant h(y) \text{ and } g(y) \leqslant h(x).$$

This representation by the mappings (h, g) is not particulary useful since the implications only go in one direction. This leads us to suggest that another possible model for partial interval orders would involve tolerances on a suitable poset. First the appropriate terminology must be introduced.

Definition 6.2. Let $\mathbf{L}, \mathbf{M}$ be posets. A pair (h, g) of mappings from $\mathbf{L}$ into $\mathbf{M}$ is called a *P-pair* in case

$$g \text{ is isotone, and}$$
$$g(x) \leqslant h(x) \text{ for all } x \in \mathbf{L}.$$

Definition 6.3. Let $\mathbf{L}$ be a poset. A *poset tolerance* on $\mathbf{L}$ is a reflexive symmetric relation $\mathbf{T}$ having the property that

$$a \leqslant v\mathbf{T}b,\ b \leqslant x\mathbf{T}a \Rightarrow a\mathbf{T}b, \text{ and}$$
$$a \leqslant b \leqslant c \text{ with } a\mathbf{T}c \text{ implies } a\mathbf{T}b.$$

Lemma 6.4. *Every P-pair (h, g) of mappings on a poset $\mathbf{L}$ induces a poset tolerance in the usual manner.*

Proof. Left to the reader. □

Lemma 6.5. *Let $\mathbf{T}$ be a poset tolerance on the poset $\mathbf{L}$. There then exists a lattice $\mathbf{M}$ and mappings $g, h \colon \mathbf{L} \to \mathbf{M}$ such that*

$$g \text{ is isotone},$$
$$g(x) \leqslant h(x) \text{ for all } x, \text{ and}$$
$$x\mathbf{T}y \Leftrightarrow g(x) \leqslant h(y) \text{ and } g(y) \leqslant h(x).$$

Proof. If $\mathbf{L}$ does not have a least element, take $\mathbf{M} = \mathcal{O}\mathcal{I}(\mathbf{L}) \cup \{\emptyset\}$; otherwise, set $\mathbf{M} = \mathcal{O}\mathcal{I}(\mathbf{L})$. In either case, $\mathbf{M}$ is a complete distributive lattice with $x \to g(x) = J_x$ isotone. For the poset tolerance $\mathbf{T}$, let $h \colon \mathbf{L} \to \mathbf{M}$ be defined by letting $h(X)$ be the order ideal generated by $\{w \colon w\mathbf{T}x\}$. The assertions of the Lemma are now obvious. □

Theorem 6.6. *Let* $\mathbf{T}$ *be a poset tolerance on the poset* $\mathbf{L}$. *Define* g, h *as in Lemma* 6.5. *If the relation* $\mathbf{P}$ *is specified by*

$$x\mathbf{P}y \Leftrightarrow h(x) < g(y),$$

then $(\mathbf{P}, \mathbf{T})$ *is a partial interval order of type* 2.

Proof. Since $\mathbf{T} \cap \mathbf{P} = \emptyset$ is obvious, we begin by showing that $\mathbf{PTP} \subseteq \mathbf{P}$. So let $a\mathbf{P}b\mathbf{T}c\mathbf{P}d$ and note that

$$h(a) < g(b) \leqslant h(c) < g(d),$$

from which $a\mathbf{P}d$ follows. Next we show that $\mathbf{TP}^{-1}\mathbf{T} \cap \mathbf{TPT} \subseteq \mathbf{T}$. Assume $a\mathbf{T}b\mathbf{P}c\mathbf{T}d$ and note that

$$g(a) \leqslant h(b) < g(c) \leqslant h(d).$$

A symmetric argument shows $a\mathbf{T}b'\mathbf{P}^{-1}c'\mathbf{T}d$ implies $g(d) \leqslant h(a)$, so $a\mathbf{T}d$. Thus the pair $(\mathbf{P}, \mathbf{T})$ is a partial interval order of type 2. □

Remark 6.7. The relation $\mathbf{P}$ specified in the statement of Theorem 6.6 will be called the *strict partial preference associated with* (h, g). A word of caution is in order. If (h, g) and (h', g') are distinct P-pairs that induce the same tolerance $\mathbf{T}$ on the poset $\mathbf{L}$, then the relations $\mathbf{P}$, $\mathbf{P}'$ defined by

$$a\mathbf{P}b \Leftrightarrow h(a) < g(b),$$
$$a\mathbf{P}'b \Leftrightarrow h'(a) < g'(b)$$

need not coincide. This can be concretely illustrated by the example of Remark 4.3. With (h, g) defined as in Remark 4.3, the relation $\mathbf{P}$ is empty, while if $\mathbf{T}$ is induced by the pair (h', g'), we have $x\mathbf{P}'y$ for $x \in \{0, a\}$ and $y \in \{b, c\}$.

The next step is to establish a converse to the above results. Suppose $(\mathbf{P}, \mathbf{I})$ is a partial interval order of type 2. By [5], Proposition 6, p. 14 there exist interval order extensions $(\mathbf{Q}_k, \mathbf{J}_k)$ of $(\mathbf{P}, \mathbf{I})$ such that $\mathbf{P} = \bigcap\limits_k \mathbf{Q}_k$ and $\mathbf{I} = \bigcap\limits_k \mathbf{J}_k$. By Theorem 5.2, there exist linear orders $\leqslant_k$ on $\mathbf{X}$ such that $\mathbf{J}_k$ is a meet tolerance on $(\mathbf{X}, \leqslant_k)$. Thus there exist chains $(\mathbf{E}_k, \leqslant'_k)$ and mappings g_k, h_k from $(\mathbf{X}, \leqslant_k)$ into $\mathbf{E}_k$ such that

g_k is one-one and isotone,

$g_k(x) \leqslant'_k h_k(x)$ for all x, and

$x\mathbf{J}_k y \Leftrightarrow g_k(x) \leqslant'_k h_k(y)$ and $g_k(y) \leqslant'_k h_k(x)$.

Let $\mathbf{M} = \prod\limits_k \mathbf{E}_k$ be equipped with the product partial order $\leqslant_0$. Define $g, h\colon \mathbf{X} \to \mathbf{M}$ by the agreement that

$$g(x) = \big(g_k(x)\big)_k \text{ and } h(x) = \big(h_k(x)\big)_k.$$

Thus

$$g(x) \leqslant_0 h(y) \Leftrightarrow g_k(x) \leqslant_k h_k(y) \text{ for all } k.$$

It is immediate that

$$g \text{ is one-one,}$$
$$g(x) \leqslant_0 h(x) \text{ for all } x, \text{ and}$$
$$x\mathbf{I}y \Leftrightarrow g(x) \leqslant_0 h(y) \text{ and } g(y) \leqslant_0 h(x).$$

One can now define a partial order $\leqslant_1$ on $\mathbf{X}$ by taking $\leqslant_1 = \bigcap_k \leqslant_k$. If $x \leqslant_1 y$, then for each index k, we have $x \leqslant_k y$ so that $g_k(x) \leqslant g_k(y)$, and consequently $g(x) \leqslant_0 g(y)$ in $\mathbf{M}$. Thus g is an isotone mapping from $(\mathbf{X}, \leqslant_1)$ into $\mathbf{M}$. By Lemma 6.4, $\mathbf{I}$ is a poset tolerance on $(\mathbf{X}, \leqslant_1)$. This is all summarized in the next theorem.

Theorem 6.8. *Let $(\mathbf{P}, \mathbf{I})$ be a partial interval order of type 2 on $\mathbf{X}$. There exists a partial order $\leqslant$ on $\mathbf{X}$ such that $\mathbf{I}$ is a poset tolerance on $(\mathbf{X}, \leqslant)$. Every poset tolerance arises in this manner.*

7. Partial semiorders

In this section we shall extend the definition of poset tolerances and explore their connection to what are called partial semiorders in [5].

Definition 7.1. Let $\mathbf{P}, \mathbf{Q}$ be posets. A pair (h, g) of mappings from $\mathbf{P}$ into $\mathbf{Q}$ is called an *S-pair* in case:

$$g, h \text{ are each isotone, and}$$
$$g(x) \leqslant h(x) \text{ for all } x \in \mathbf{P}.$$

Definition 7.2. Let $\mathbf{L}$ be a poset. A *poset tolerance* $\mathbf{T}$ on $\mathbf{L}$ is said to be of type 0 in case it satisfies the following conditions:

(S1) $a \leqslant v\mathbf{T}b,\ b \leqslant w\mathbf{T}a \Rightarrow a\mathbf{T}b$,

(S2) $a \geqslant v\mathbf{T}b,\ b \geqslant w\mathbf{T}a \Rightarrow a\mathbf{T}b$, and

(S3) $a \leqslant b \leqslant c$ with $a\mathbf{T}c$ implies both $a\mathbf{T}b$ and $b\mathbf{T}c$.

This merely says that $\mathbf{T}$ is a poset tolerance on both $\mathbf{L}$ and its dual. $\mathbf{T}$ is called a *poset tolerance of type* 1 in case it satisfies (S1), (S2), and the stronger condition

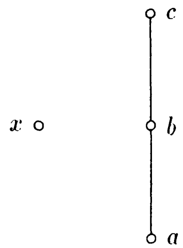
(S3′) $[x] = \{w : w\mathbf{T}x\}$ is convex for each $x \in \mathbf{L}$.

Finally, the pair $(\mathbf{P},\mathbf{T})$ is called a *poset interval order* if there is an S-pair (h,g) such that

$$g(x) \leqslant g(y) \Leftrightarrow h(x) \leqslant h(y),$$
$$x\mathbf{T}y \Leftrightarrow g(x) \leqslant h(y) \text{ and } g(y) \leqslant h(x),$$
$$x\mathbf{P}y \Leftrightarrow h(x) < g(y).$$

We shall call such an S-pair *symmetric* and say that the pair $(\mathbf{P},\mathbf{T})$ is *induced* by the symmetric S-pair (h,g).

Remark 7.3. Evidently every poset interval order yields a poset tolerance of type 1, and every poset tolerance of type 1 is of type 0. Robert C. Powers has provided an example of a type 0 poset tolerance that is not type 1. With $\mathbf{L}$ as in the diagram below, take $\mathbf{T}$ so that $x\mathbf{T}a$, $x\mathbf{T}c$ and $y\mathbf{T}y$ for all y. Note that $\mathbf{T}$ is a poset tolerance on both $\mathbf{L}$ and its dual; yet (S3′) fails since $[x]$ is not convex.



The construction of Lemma 6.5 now produces the following mappings from $\mathbf{L}$ into $\mathcal{O}\mathcal{I}(\mathbf{L}) \cup \{\emptyset\}$:

$$g(y) = J_y \text{ for all } y,$$
$$h(x) = h(c) = \{x,a,b,c\},\ h(b) = \{a,b\},\ h(a) = \{x,a\}.$$

It is curious that the associated strict partial preference $\mathbf{P}$ consists only of the pair (b,c), while if $\mathbf{T}$ is viewed as a tolerance $\mathbf{T}^*$ on the dual of $\mathbf{L}$, the corresponding construction would produce $b\mathbf{P}^*a$. We still need an example of a type 1 partial semiorder that is not type 2. We precede this with a preliminary result.

Lemma 7.4. *Let $(\mathbf{P},\mathbf{T})$ be a poset interval order that is induced by the symmetric S-pair (h,g). Then:*

$$(\alpha)\ \mathbf{PTP} \subseteq \mathbf{P} \qquad (\beta)\ \mathbf{PPT} \subseteq \mathbf{P} \qquad (\gamma)\ \mathbf{TPP} \subseteq \mathbf{P}.$$

P r o o f. To establish these facts, note that $a\mathbf{P}b\mathbf{T}c\mathbf{P}d$ implies that $h(a) < g(b) \leqslant h(c) < g(d)$, $a\mathbf{T}b\mathbf{P}c\mathbf{P}d$ forces $g(a) \leqslant h(b) < g(c)$ so $h(a) \leqslant h(c) < g(d)$; finally, if $a\mathbf{P}b\mathbf{P}c\mathbf{T}d$, then $h(b) < g(c) \leqslant h(d)$ implies $g(b) \leqslant g(d)$ so $h(a) < g(b) \leqslant h(d)$. □

Remark 7.5. To obtain an example of a type 1 poset tolerance $\mathbf{T}$ such that $(\mathbf{P},\mathbf{T})$ is not a poset *interval* order, consider the example of Remark 7.3 and define $g', h' \colon \mathbf{L} \to \mathcal{O}\mathcal{I}(\mathbf{L}) \cup \{\emptyset\}$ by taking

$$\begin{aligned} g'(x) &= \{x\}, & h'(x) &= \{x,a,b,c\},\\ g'(c) &= \{x,a,b,c\}, & h'(c) &= \{x,a,b,c\},\\ g'(b) &= \{x,a,b\}, & h'(b) &= \{x,a,b\},\\ g'(a) &= \{x,a\}, & h'(a) &= \{x,a\}. \end{aligned}$$

The pair (h',g') is then an S-pair in the sense of Definition 7.7 and the poset semiorder $\mathbf{T}'$ that it induces has the classes

$$[x] = \{x,a,b,c\} \qquad [y] = \{y,x\} \text{ for } y = a, b \text{ or } c.$$

If $\mathbf{P}'$ is the strict partial preference associated with (h',g'), we then have $a\mathbf{P}'b$, $b\mathbf{P}'c$ and $c\mathbf{T}'x$. In that $a\mathbf{P}'x$ fails, we have that $\mathbf{T}'$ is type 1 but $(\mathbf{P}',\mathbf{T}')$ is not a poset semiorder.

Lemma 7.6. *Every S-pair (h,g) of mappings on a poset $\mathbf{L}$ induces a poset tolerance of type 1 in the usual manner.*

Proof. This is left to the reader. □

Lemma 7.7. *Every poset tolerance $\mathbf{T}$ of type 1 on the poset $\mathbf{L}$ is induced by an S-pair.*

Proof. Taking $\mathbf{M}$ as in the proof of Lemma 6.5, we define mappings $g, h \colon \mathbf{L} \to \mathbf{M}$ by taking $g(x) = J_x$ and $h(x)$ to be the order ideal generated by $\{w \colon w\mathbf{T}x_1$ for some $x_1 \leqslant x\}$. Then g, h are isotone and $g(x) \leqslant h(x)$ for all $x \in \mathbf{L}$. The trick now is to show that $\mathbf{T}$ is the poset tolerance induced by (h,g).

Trivially, if $x\mathbf{T}y$ then $x \in h(y)$ since $x\mathbf{T}y$ with $y \leqslant y$, so $g(x) \leqslant h(y)$ and similarly $g(y) \leqslant h(x)$. So let us assume conversely that $g(x) \leqslant h(y)$ and $g(y) \leqslant h(x)$ and try to establish $x\mathbf{T}y$. We know that there exist elements $v, w, x_1, y_1 \in \mathbf{L}$ such that

$$x \leqslant w\mathbf{T}y_1 \leqslant y \qquad \text{and} \qquad y \leqslant v\mathbf{T}x_1 \leqslant x.$$

It follows that $x_1 \leqslant w\mathbf{T}y_1$ and $y_1 \leqslant v\mathbf{T}x_1$ so by (S1), $x_1\mathbf{T}y_1$. Using (S3), we now see that $x_1\mathbf{T}y_1$, $x_1\mathbf{T}v$ with $y_1 \leqslant y \leqslant v$ implies $x_1\mathbf{T}y$. Similarly by (S2), $v\mathbf{T}x_1 \leqslant w$, $w\mathbf{T}y_1 \leqslant v$ together imply $v\mathbf{T}w$; the combination of $v\mathbf{T}w$ and $v\mathbf{T}x_1$ now forces $v\mathbf{T}x$. Using the fact that $x_1\mathbf{T}y$ and $v\mathbf{T}x$, we now have

$$x \geqslant x_1\mathbf{T}y \qquad \text{and} \qquad y \geqslant v\mathbf{T}x.$$

A second application of (S2) now forces $x\mathbf{T}y$. □

Our next goal is to establish the analog of Theorem 6.8 for poset interval orders. Before this task can be undertaken we shall need some machinery from [5]. A *cycle of* a relation $\mathbf{R}$ is any sequence of pairs in $\mathbf{R}$ of the form $x_0x_1, x_1x_2, \ldots, x_nx_0$. Given two relations $\mathbf{V}$ and $\mathbf{W}$ on $\mathbf{X}$, we follow [5], p. 16 and denote by $\mathbf{V} \bigtriangledown \mathbf{W}$ the relation formed by all pairs xy for which there exist $x_0, x_1, \ldots, x_n \in \mathbf{X}$, with xx_0, $x_0x_1, \ldots, x_ny$ members of $\mathbf{V} \cup \mathbf{W}$, with the number of pairs in $\mathbf{V}$ strictly greater than the number taken in $\mathbf{W}$. Similarly, $\mathbf{V} \bigtriangleup \mathbf{W}$ is formed by all pairs xy for which there exist $x_0, x_1, \ldots, x_n \in \mathbf{X}$ with $xx_0, x_0x_1, \ldots, x_ny$ belonging to $\mathbf{V} \cup \mathbf{W}$, and for which the number of pairs from $\mathbf{W}$ is strictly less than 2 plus the number of those from $\mathbf{V}$. We are now ready to introduce the semiorder version of Definition 6.1.

Definition 7.8 ([5], Definition 6, p. 17). A *partial semiorder of type* 1 is a pair of relations $(\mathbf{P}, \mathbf{I})$ on $\mathbf{X}$ such that

$\mathbf{I}$ is symmetric and reflexive,

$\mathbf{P} \cap \mathbf{I} = \emptyset$,

each cycle of $\mathbf{P} \cup \mathbf{I}$ has fewer pairs in $\mathbf{P}$ than in $\mathbf{I}$,

$\mathbf{P} \bigtriangledown \mathbf{I} \subseteq \mathbf{P}$.

A *partial semiorder of type* 2 is defined by adding

$$\mathbf{P} \bigtriangleup \mathbf{I} \cap \mathbf{P}^{-1} \bigtriangleup \mathbf{I} \subseteq \mathbf{I}.$$

Doignon et al ([5], Proposition 8, p. 16) prove that if $(\mathbf{P}, \mathbf{I})$ is a partial semiorder of type 2, then there exist semiorder extensions $(\mathbf{Q}_k, \mathbf{J}_k)$ of $(\mathbf{P}, \mathbf{I})$ such that $\mathbf{P} = \bigcap_k \mathbf{Q}_k$ and $\mathbf{I} = \bigcap_k \mathbf{J}_k$. We begin our discussion by constructing a poset interval order from the partial semiorder $(\mathbf{P}, \mathbf{I})$ of type 2. By [5], Proposition 2, p. 8 there exist a family $(\mathbf{E}_k, \leqslant'_k)$ of chains and mappings g_k, h_k of $\mathbf{X}$ into $\mathbf{E}_k$ such that

$g_k(x) \leqslant'_k h_k(x)$ for all x,

$x\mathbf{P}_k y \Leftrightarrow h_k(x) <'_k g_k(y)$,

$x\mathbf{J}_k y \Leftrightarrow g_k(x) \leqslant'_k h_k(y)$ and $g_k(y) \leqslant'_k h_k(x)$, and

$g_k(x) \leqslant'_k g_k(y) \Leftrightarrow h_k(x) \leqslant'_k h_k(y)$.

For each index k, define a weak order $\mathbf{W}_k$ on $\mathbf{X}$ by the rule $x\mathbf{W}_k y$ if and only if $g_k(x) \leqslant'_k g_k(y)$. Now let $\leqslant_k$ be a linear order on $\mathbf{X}$ that *extends* $\mathbf{W}_k$ in the sense that $x \leqslant_k y$ implies $x\mathbf{W}_k y$. It is immediate that g_k, h_k are isotone mappings from $(\mathbf{X}, \leqslant_k)$ into the chain $\mathbf{E}_k$. Taking $\mathbf{M} = \prod_k \mathbf{E}_k$ with the product partial order $\leqslant_0$, we now define g, h as we did in the proof of Theorem 6.7. Letting $\leqslant_1 = \bigcap_k \leqslant_k$, it is clear that

$$g, h \text{ are isotone mappings of } (\mathbf{X}, \leqslant_1) \text{ into } \mathbf{M},$$
$$g(x) \leqslant_0 h(x) \text{ for all } x,$$
$$g(x) \leqslant_0 g(y) \Leftrightarrow h(x) \leqslant_0 h(y), \text{ and}$$
$$x\mathbf{I}y \Leftrightarrow g(x) \leqslant_0 h(y) \text{ and } g(y) \leqslant_0 h(x).$$

It follows that $\mathbf{I}$ is a poset interval order on $(\mathbf{X}, \leqslant_1)$.

Next we assume that $(\mathbf{P}, \mathbf{T})$ is a poset interval order, so that $(\mathbf{P}, \mathbf{T})$ is induced by a symmetric S-pair (h, g). In view of Lemma 7.4, each of the following assertions is true:

$$(\alpha)\ \mathbf{PTP} \subseteq \mathbf{P} \qquad (\beta)\ \mathbf{PPT} \subseteq \mathbf{P} \qquad (\gamma)\ \mathbf{TPP} \subseteq \mathbf{P}.$$

We shall prove that $(\mathbf{P}, \mathbf{T})$ is a partial semiorder of type 2 by proving a number of claims.

Claim 1. *Each cycle of* $\mathbf{P} \cup \mathbf{I}$ *has fewer pairs in* $\mathbf{P}$ *than in* $\mathbf{I}$.

Proof. Suppose there were a cycle $x\mathbf{P}^{j_1}\mathbf{I}^{k_1}\mathbf{P}^{j_2}\mathbf{I}^{k_2}\ldots\mathbf{P}^{j_t}\mathbf{I}^{k_t}x$ that had at least as many pairs from $\mathbf{P}$ as from $\mathbf{I}$. With no loss in generality, we may assume that (α), (β), (γ) have been applied to reduce the length of the cycle as far as possible. Evidently $j_i \leqslant 1$ for all i, and $k_i \geqslant 2$ for $i < t$. It follows that $t \leqslant 2$, so $x\mathbf{PI}x$, $x\mathbf{IP}x$, or $x\mathbf{PIIP}x$, all of which are impossible. □

Claim 2. $\mathbf{P} \bigtriangledown \mathbf{I} \subseteq \mathbf{P}$.

Proof. If $x\mathbf{P} \bigtriangledown \mathbf{I}y$, then $x\mathbf{P}^{j_1}\mathbf{I}^{k_1}\mathbf{P}^{j_2}\mathbf{I}^{k_2}\ldots\mathbf{P}^{j_t}\mathbf{I}^{k_t}y$ with more pairs from $\mathbf{P}$ than from $\mathbf{I}$. Assuming that (α), (β) and (γ) have been applied as many times as possible, it is immediate that $t > 2(t-1)$, whence $t = 1$, and consequently $x\mathbf{P}y$. □

Claim 3. *If* $x\mathbf{P} \bigtriangleup \mathbf{I}y$, *then* $g(x) \leqslant h(y)$.

Proof. If the assertion of Claim 3 were false, then it would fail for some pair xy for which $x\mathbf{P} \bigtriangleup \mathbf{I}y$ with x, y connected by a minimal length sequence of relations from $\mathbf{P} \cup \mathbf{I}$. In view of (α), (β), and (γ), it follows as in the proof of Claim 1 that

$$(**) \qquad x\mathbf{P}^{j_1}\mathbf{I}^{k_1}\mathbf{P}^{j_2}\mathbf{I}^{k_2}\ldots\mathbf{P}^{j_t}\mathbf{I}^{k_t}y$$

with $j_i \leqslant 1$ for all i and $k_i \geqslant 2$ for $i < t$. If the last two relations in $(**)$ were $\mathbf{PP}$, $\mathbf{PI}$, or $\mathbf{IP}$ the choice of $(**)$ as having minimal length would force the existence of elements v, w such that $g(x) \leqslant h(v)$ and (i) $v\mathbf{P}w\mathbf{P}y$, or (ii) $v\mathbf{P}w\mathbf{I}y$, or (iii) $v\mathbf{I}w\mathbf{P}y$. Now (i) clearly forces $g(x) < h(y)$; with (ii), $g(x) \leqslant h(v) < g(w) \leqslant h(y)$; finally, in (iii) we have $g(v) \leqslant h(w) < g(y)$, so $h(v) \leqslant h(y)$ and consequently, $g(x) \leqslant h(v) \leqslant h(y)$. □

The situation is summarized in the next Theorem.

Theorem 7.8. *Let* $\mathbf{P}$, $\mathbf{I}$ *be binary relations on* $\mathbf{X}$. *The following conditions are then equivalent:*

(i) $(\mathbf{P}, \mathbf{I})$ *is a partial semiorder of type 2.*

(ii) *There exists a partial order* $\leqslant$ *on* $\mathbf{X}$ *for which* $(\mathbf{P}, \mathbf{I})$ *is a poset semiorder.*

(iii) *Each of the following is true:*

$$\mathbf{I} \text{ is symmetric and reflexive,}$$

$$\mathbf{P} \cap \mathbf{I} = \emptyset,$$

$$\mathbf{PIP} \cup \mathbf{PPI} \cup \mathbf{IPP} \subseteq \mathbf{P}.$$

8. Conclusion

The original motivation behind the definition of semiorders by Luce [11] and interval orders by Fishburn [6] involved considering measurements to have validity only within some interval of the reals. Abstract versions of both concepts have usually been studied from the aspect of relational systems. Investigations into their ordinal properties seem largely to have concentrated on properties of their associated strict orders $\mathbf{P}$. The focus of Theorem 5.2 lies with the indifference relation $\mathbf{I}$, and comes full circle to show that the abstract version of interval orders and semiorders can still be thought of in terms of measurement that has associated with it a notion of fuzziness. These ideas are formalized by relating them to the theory of tolerances on lattices. This is of course closely related to results such as those in Fishburn [7].

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Author's address: Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003, U.S.A.