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SOME PROPERTIES OF α-IDEALS AND GENERALIZED α-IDEALS, n-SEMIGROUPS AND n-GROUPS

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The authors of the papers [1], [3] and [7] considered some basic properties of α-ideals and generalized α-ideals in semigroups. In this paper we deal with some further properties of these notions and their connection with the theory of n-semigroups and n-groups.

Let S be a semigroup. The family $P(S)$ of all non-empty subsets of S is a semigroup under complex product. Put $P^0(S) = P(S) \cup \{\emptyset\}$. Let X be a non-empty set. The symbol X^* denotes the free semigroup over the alphabet X. The number of terms of a word $\alpha \in X^*$ is called the length of the word α and denoted by $l(\alpha)$.

Suppose that $F = \{0,1\}^* \setminus \{1\}^*$. Let $\alpha \in F$ be a word of the form $\alpha = \alpha_1\alpha_2\ldots\alpha_n$. We define a mapping $f_\alpha^S : P(S) \to P(S)$ by the formula

$$f_\alpha^S(X) = X_1 X_2 \ldots X_n$$

for every $X \in P(S)$, where

$$X_i = \begin{cases} X & \text{for } \alpha_i = 1, \\ S & \text{for } \alpha_i = 0 \end{cases}$$

and $i = 1, 2, \ldots, n$.

If we do not introduce additional assumptions, we will denote by α any word from F such that $l(\alpha) = n$. We will write f_α instead of f_α^S when no confusion can arise. Unless otherwise stated we assume that S denotes a semigroup.

Definition 1 (cf. [3]). A non-empty subset A of a semigroup S is said to be a generalized α-ideal of S if $f_\alpha(A) \subset A$.

A generalized α-ideal of S is called an α-ideal of S if A is a subsemigroup of S.

The symbol $Ig_\alpha(S)$ $[I_\alpha(S)]$ denotes the family of all generalized α-ideals [α-ideals, respectively] of the semigroup S. Put $I^0g_\alpha(S) = Ig_\alpha(S) \cup \{\emptyset\}$ and $I^0_\alpha(S) = I_\alpha(S) \cup \{\emptyset\}$.

Proposition 1. *If $A_t \in I^0g_\alpha(S)$ for $t \in T$, then*

$$\bigcap(A_t : t \in T) \in I^0g_\alpha(S).$$

Definition 2. Let X be a non-empty subset of a semigroup S. The generalized α-ideal

$$\langle X\rangle_\alpha = \bigcap(A \in Ig_\alpha(S) : X \subset A)$$

is called the generalized α-ideal generated by X in the semigroup S.

From Theorem 1.7 (cf. [3]) we get

Corollary 1. *Let X be a non-empty subset of a semigroup S. Then*

$$\langle X\rangle_\alpha = X \cup f_\alpha(X).$$

Let us define a mapping $G_\alpha : P^0(S) \to P^0(S)$ by the formula

$$G_\alpha(X) = \begin{cases} \langle X\rangle_\alpha & \text{for } X \neq \emptyset, \\ \emptyset & \text{for } X = \emptyset. \end{cases}$$

The mapping G_α is a closure operator on S. Therefore, we have

Proposition 2. *The set $I^0g_\alpha(S)$ is a complete lattice, and for an arbitrary family $(A_t \in I^0g_\alpha(S) : t \in T)$ the following conditions hold:*

(i) $\bigwedge(A_t : t \in T) = \bigcap(A_t : t \in T)$,

(ii) $\bigvee(A_t : t \in T) = G_\alpha\big(\bigcup(A_t : t \in T)\big)$.

Proposition 3. *Assume that $A_t \in I^0_\alpha(S)$ for $t \in T$. Then*

$$\bigcap(A_t : t \in T) \in I^0_\alpha(S).$$

Definition 3. Let X be a non-empty subset of a semigroup S. The α-ideal

$$(X)_\alpha = \bigcap(A \in I_\alpha(S) : X \subset A)$$

is called the α-ideal generated by X in the semigroup S.

From Theorem 3 (cf. [1]) we get

Corollary 2. *Let X be a non-empty subset of a semigroup S. Then*

$$(X)_\alpha = X \cup X^2 \cup \ldots \cup X^{l(\alpha)-1} \cup f_\alpha(X).$$

Let us define a mapping $E_\alpha \colon P^0(S) \to P^0(S)$ by the formula

$$E_\alpha(X) = \begin{cases} (X)_\alpha & \text{for } X \neq \emptyset, \\ \emptyset & \text{for } X = \emptyset. \end{cases}$$

The mapping E_α is a closure operator on S. Therefore, we have

Proposition 4. *The set $I^0_\alpha(S)$ is a complete lattice, and for an arbitrary family $(A_t \in I^0_\alpha(S) \colon t \in T)$ the following conditions hold:*
(i) $\bigwedge(A_t \colon t \in T) = \bigcap(A_t \colon t \in T)$,
(ii) $\bigvee(A_t \colon t \in T) = E_\alpha\big(\bigcup(A_t \colon t \in T)\big)$.

Proposition 5. *If $X, Y \in P^0(S)$, then*
(i) $G_\alpha(X) \cup G_\alpha(Y) \subset G_\alpha(X \cup Y)$,
(ii) $G_\alpha(X \cap Y) \subset G_\alpha(X) \cap G_\alpha(Y)$,
(iii) $E_\alpha(X) \cup E_\alpha(Y) \subset E_\alpha(X \cup Y)$,
(iv) $E_\alpha(X \cap Y) \subset E_\alpha(X) \cap E_\alpha(Y)$.

The proof is straightforward.

Corollary 3. *If $A, B \in I^0 g_\alpha(S)$, then*
(i) $G_\alpha(A) \cup G_\alpha(B) \subset G_\alpha(A \cup B)$,
(ii) $G_\alpha(A \cap B) = G_\alpha(A) \cap G_\alpha(B)$.

Corollary 4. *If $A, B \in I^0_\alpha(S)$, then*
(i) $E_\alpha(A) \cup E_\alpha(B) \subset E_\alpha(A \cup B)$,
(ii) $E_\alpha(A \cap B) = E_\alpha(A) \cap E_\alpha(B)$.

In general, the inclusions in Proposition 5 and Corollaries 3 and 4 cannot be replaced by equalities. Let us consider suitable examples.

Let $(\mathbb{N}, \cdot)$ be the semigroup of the natural numbers under multiplication. We take $\alpha = 110$, $X = \{2\}$, $Y = \{3\}$. Therefore, we have $G_\alpha(X) = \{2\} \cup f_\alpha(2) = \{2\} \cup \{4\} \cdot \mathbb{N} = \{2, 4, 8, 12, 16, \ldots\}$, $G_\alpha(Y) = \{3\} \cup f_\alpha(3) = \{3\} \cup \{9\} \cdot \mathbb{N} = \{3, 9, 18, 27, \ldots\}$. Notice that $G_\alpha(X) = E_\alpha(X)$ and $G_\alpha(Y) = E_\alpha(Y)$. Put $A = G_\alpha(X)$ and $B =$

$G_\alpha(Y)$. Thus, $G_\alpha(A\cup B) = (A\cup B)\cup\big[(A\cup B)\cdot(A\cup B)\cdot\mathbb{N}\big]$. Notice that $6 \in G_\alpha(A\cup B)$, but $6 \notin G_\alpha(A) \cup G_\alpha(B)$. Similarly for the operator E_α.

For the intersection we get $G_\alpha(X \cap Y) = G_\alpha(\emptyset) = \emptyset$. On the other hand, $G_\alpha(X)\cap G_\alpha(Y) \neq \emptyset$, because for example $36 \in G_\alpha(X)\cap G_\alpha(Y)$. Similarly for the operator E_α.

Notice that $A, B \in I_\alpha(\mathbb{N})$, but $A \cup B \notin I_\alpha(\mathbb{N})$.

In general, the lattices $I^0 g_\alpha(S)$ and $I^0_\alpha(S)$ are not distributive.

Indeed, assume that α, A and B have the same meaning as in the above example. Consider $C = G_\alpha(6) = \{6\}\cup\{36\}\cdot\mathbb{N} = \{6, 36, 72, \ldots\}$. Of course $A, B, C \in I^0 g_\alpha(\mathbb{N})$. We have

$$(A \vee B) \wedge C = (A \vee B) \cap C,$$

$$(A \wedge C) \vee (B \wedge C) = (A \cap C) \vee (B \cap C).$$

It is easy to check that $6 \in (A \vee B) \wedge C$ but $6 \notin (A \wedge C) \vee (B \wedge C)$. Since $A, B, C \in I^0_\alpha(\mathbb{N})$, the same reasoning applies to the lattice $I^0_\alpha(\mathbb{N})$.

Proposition 6. *If $X \in P(S)$, then $f_\alpha(X) \in I_\alpha(S)$.*

Proof. By Lemma 1.4 (cf. [3]) we have $f_\alpha(X)f_\alpha(X) \subset f_\alpha(X)$, hence $f_\alpha(X)$ is a subsemigroup of S. Applying Lemma 1.5 (cf. [3]) we obtain $f_\alpha(f_\alpha(X)) \subset f_\alpha(X \cup f_\alpha(X)) \subset f_\alpha(X)$, and so $f_\alpha(X) \in I_\alpha(S)$. □

Proposition 7. *If $X \in P(S)$ and $l(\alpha) = n$, then*

$$\forall m \geqslant n\colon X^m \subset f_\alpha(X).$$

Proof. Since $f_\alpha(X) = X_1 \ldots X_n$ and $X \subset X_i$ for $i = 1, \ldots, n$, it follows that $X^n \subset f_\alpha(X)$. By Lemma 1.3 (cf. [3]) we have $X^{n+1} \subset X f_\alpha(X) \subset f_\alpha(X)$. Thus, $X^m \subset f_\alpha(X)$ for $m \geqslant n$. □

Proposition 8. *If $X \in P(S)$ and $l(\alpha) = n$, then*

$$\forall m \geqslant 1\colon E_\alpha(X^m) \subset E_\alpha(X).$$

Proof. We know that $E_\alpha(X) = X \cup X^2 \cup \ldots \cup X^{n-1} \cup f_\alpha(X)$. Observe that according to Proposition 7, X^m for $m \geqslant 1$ is any one of the sets $X, \ldots, X^{n-1}$ or $X^m \subset f_\alpha(X)$. Thus $X^m \subset E_\alpha(X)$, and so $E_\alpha(X^m) \subset E_\alpha(X)$. □

Proposition 9 (cf. [7]). *Let $\varphi\colon S \to S'$ be an epimorphism of a semigroup S onto a semigroup S'. If $A \in Ig_\alpha(S)$ [$A \in I_\alpha(S)$], then $\varphi(A) \in Ig_\alpha(S')$ [$\varphi(A) \in I_\alpha(S')$, respectively].*

Let $\varphi\colon S \to S'$ be an epimorphism of a semigroup S onto a semigroup S'. If $X', Y' \in P^0(S')$, then

$$\varphi^{-1}(X')\varphi^{-1}(Y') \subset \varphi^{-1}(X'Y'). \tag{1}$$

In general, the above inclusion cannot be replaced by equality. For example, it is enough to take the null semigroup S such that $\mathrm{card}(S) > 1$, and for S' to take the one-element semigroup S'.

Proposition 10. *Let $\varphi\colon S \to S'$ be an epimorphism of semigroups S and S'. If $A' \in Ig_\alpha(S')$ [$A' \in I_\alpha(S')$], then $\varphi^{-1}(A') \in Ig_\alpha(S)$ [$\varphi^{-1}(A') \in I_\alpha(S)$, respectively].*

Proof. It is enough to prove that $f_\alpha^S(\varphi^{-1}(A')) \subset \varphi^{-1}(A')$. By the definition we have $f_\alpha^{S'}(A') = A_1' \dots A_n' \subset A'$. Applying (1) we get $f_\alpha^S(\varphi^{-1}(A')) = \varphi^{-1}(A_1') \dots \varphi^{-1}(A_n') \subset \varphi^{-1}(A_1' \dots A_n') = \varphi^{-1}(f_\alpha^{S'}(A')) \subset \varphi^{-1}(A')$. □

By Propositions 9 and 10 we obtain

Corollary 5. *Let σ be a congruence on a semigroup S. A subset $A' \subset S/\sigma$ is a generalized α-ideal [α-ideal] if and only if there exists a generalized α-ideal [α-ideal, respectively] $A \subset S$ such that $A' = \{a/\sigma : a \in A\}$.*

Theorem 1. *Let us suppose that $\alpha = \alpha_1 \dots \alpha_n$ and there exists an $1 \leqslant i \leqslant n$ such that $\alpha_i = 1$. Let A be an α-ideal in a semigroup P. Let P be an α-ideal in a semigroup S. If $AA = A$, then A is an α-ideal in the semigroup S.*

Proof. We shall distinguish the following four cases:

(1) $f_\alpha^S(A) = AA_2 \dots A_{n-1}S = A^{n-1}(AA_2 \dots A_{n-1}S) = A^{n-1}f_\alpha^S(A) \subset A^{n-1}f_\alpha^S(P) \subset A^{n-1}P \subset f_\alpha^P(A) \subset A$;

(2) $f_\alpha^S(A) = SA_2 \dots A_{n-1}A = (SA_2 \dots A_{n-1}A)A^{n-1} = f_\alpha^S(A)A^{n-1} \subset f_\alpha^S(P)A^{n-1} \subset PA^{n-1} \subset f_\alpha^P(A) \subset A$;

(3) $f_\alpha^S(A) = AA_2 \dots A_{k-1}SA_{k+1} \dots A_{n-1}A = A^{k-1}(AA_2 \dots A_{k-1}SA_{k+1} \dots A_{n-1}A)A^{n-k} = A^{k-1}f_\alpha^S(A)A^{n-k} \subset A^{k-1}f_\alpha^S(P)A^{n-k} \subset A^{k-1}PA^{n-k} \subset f_\alpha^P(A) \subset A$;

(4) $f_\alpha^S(A) = SA_2 \dots A_{k-1}AA_{k+1} \dots A_{n-1}S = (SA_2 \dots A_{k-1}A^{n-k+1})A^{n-2} \cdot (A^kA_{k+1} \dots A_{n-1}S) \subset f_\alpha^S(A)A^{n-2}f_\alpha^S(A) \subset f_\alpha^S(P)A^{n-2}f_\alpha^S(P) \subset PA^{n-2}P \subset f_\alpha^P(A) \subset A$.

The proof is complete. □

The assumptions of Theorem 1 that (i) $AA = A$ and (ii) there exists an i, $1 \leqslant i \leqslant n$ such that $\alpha_i = 1$ cannot be omitted. To this end, consider the semigroup $S = \{a, b, c, 0\}$ under multiplication such that the product of two elements is equal 0, except for the case $ab = c$. To check that the assumption (i) is essential, it is enough to take $S = \{a, b, c, 0\}$, $P = \{b, c, 0\}$, $A = \{b, 0\}$. Notice that A is a left ideal in P and P is a left ideal in S, but A is not a left ideal in S.

For the assumption (ii), it is enough to take $S = \{a, b, c, 0\}$, $P = \{c, 0\}$, $A = \{0\}$. Put $\alpha = 00$. Therefore $AA = A$, $PP \subset A$, $SS \subset P$, and $SS \not\subset A$. Thus, A is an α-ideal in P and P is an α-ideal in S, but A is not an α-ideal in S. In addition, observe that $A = \{0\}$ is not an α-ideal of the semigroup S with zero.

Now we will investigate some relationships between the generalized α-ideals and α-ideals in semigroups, and the theory of n-semigroups and n-groups (cf. [4], [5], [6]). For simplicity of notation, it will be convenient to abbreviate $x_l, \ldots, x_k$ as x_l^k for $l \leqslant k$. If $l > k$, then x_l^k is an empty symbol. If $x_1 = x_2 = \ldots = x_k = x$, then we write x^k.

Let S be a semigroup. Define a mapping $g \colon S^n \to S$ ($n \geqslant 2$) by

$$g(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n \tag{1}$$

for all $x_1, x_2, \ldots, x_n \in S$.

The algebraic structure (S, g) is an n-semigroup.

Assume that $A \in Ig_\alpha(S)$ and $l(\alpha) \geqslant 2$. Notice that (A, g) with g given by (1) is an n-semigroup.

Since every α-ideal $A \in I_\alpha(S)$ with $l(\alpha) = n \geqslant 2$ is an n-semigroup (A, g) and a subsemigroup of S, our further considerations will imply some consequences for α-ideals.

Theorem 2. *Let A be a subsemigroup of a semigroup S. The n-semigroup (A, g) is an n-group if and only if A is a subgroup of the semigroup S.*

Proof. If A is a subgroup of the semigroup S, then the proof is immediate. Let (A, g) be an n-group. Let $p \in A$ be a fixed element. Therefore, the grupoid $(A, \circ)$ endowed with the operation

$$x \circ y = xp^{n-2}y \quad \text{for } x, y \in A,$$

is a group (cf. [6]). Put $q = p^{n-2}$. Hence $x \circ y = xqy$ for $x, y \in A$. Let us take $h(y) = qy$ for $y \in A$. Notice that h is an injection. Indeed, if $h(y_1) = h(y_2)$ for $y_1, y_2 \in A$, then $x \circ y_1 = x(qy_1) = x(qy_2) = x \circ y_2$ for any fixed $x \in A$. Hence $y_1 = y_2$. Since $q \circ y = q(qy) = h(h(y))$, it follows that h is a bijection. Since $x \circ y = xh(y)$ for $x, y \in A$, the semigroup A and the group $(A, \circ)$ are isotopic, and so they are isomorphic (cf. [2]). □

Definition 4. Let (S, f) be an n-semigroup. An element $a \in S$ is said to be a k-divisor of an element $b \in S$ if there exist elements $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n \in S$ such that $f(x_1^{k-1}, a, x_{k+1}^n) = b$. An element $a \in S$ is called a k-divisor in the n-semigroup (S, f) if a is a k-divisor for every $b \in S$.

Proposition 11. *An element $a \in S$ is a k-divisor in the n-semigroup (S, f) if and only if $f(S^{k-1}, a, S^{n-k}) = S$.*

Definition 5. An element $a \in S$ is said to be a divisor of an element $b \in S$ in the n-semigroup (S, f) if a is a k-divisor of b for each $k = 1, \dots, n$. An element $a \in S$ is called a divisor in the n-semigroup (S, f) if a is a divisor in the n-semigroup (S, f) for every element $b \in S$.

Proposition 12. *An element $a \in S$ is a divisor in the n-semigroup (S, f) if and only if $f(S^{k-1}, a, S^{n-k}) = S$ for each $k = 1, \dots, n$.*

Proposition 13. *An element $a \in S$ is a divisor in the n-semigroup (S, f) if and only if a is simultaneously the 1-divisor and the n-divisor.*

Proof. Since $f(a, S^{n-1}) = S$, we have $f(S^n) = S$. Therefore, we obtain $S = f(S^n) = f(S^{k-1}, f(a, S^{n-1}), S^{n-k}) = f(S^{k-1}, a, f(S^n), S^{n-k-1}) = f(S^{k-1}, a, S^{n-k})$. □

Let us denote by $D(S)$ the set of all divisors in the n-semigroup (S, f).

Theorem 3. *Let (S, f) be an n-semigroup. If $D(S) \neq \emptyset$, then $D(S)$ is an n-subgroup of the n-semigroup (S, f).*

Proof. Assume that $a_1, \dots, a_n \in D(S)$. Then $f(f(a_1, \dots, a_n), S^{n-1}) = f(a_1, \dots, a_{n-1}, f(a_n, S^{n-1})) = f(a_1, \dots, a_{n-1}, S) = f(a_1, \dots, a_{n-1}, f(S^n)) = f(a_1, \dots, a_{n-2}, f(a_{n-1}, S^{n-1}), S) = f(a_1, \dots, a_{n-2}, S, S) = \dots = f(a_1, S^{n-1}) = S$. Similarly, $f(S^{n-1}, f(a_1, \dots, a_n)) = S$. Therefore, $f(a_1, \dots, a_n) \in D(S)$. Assume that $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_{n+1} \in D(S)$ for a fixed $k = 1, \dots, n$. We will prove that $f(a_1^{k-1}, S, a_{k+1}^n) = S$. Indeed, $f(a_1^{k-1}, S, a_{k+1}^n) = f(a_1^{k-1}, f(S^n), a_{k+1}^n) = f(a_1^{k-2}, f(a_{k-1}, S^{n-1}), S, a_{k+1}^n) = f(a_1^{k-2}, S, S, a_{k+1}^n) = \dots = f(S^{k-1}, S, a_{k+1}^n) = f(S^{k-1}, f(S^n), a_{k+1}^n) = f(S^{k-1}, S, f(S^{n-1}, a_{k+1}), a_{k+2}^n) = f(S^k, S, a_{k+2}^n) = \dots = f(S^{n-1}, a_n) = S$. Consequently, the equation $f(a_1^{k-1}, x, a_{k+1}^n) = a_{n+1}$ has a solution for each $k = 1, \dots, n$. Therefore, $D(S)$ is an n-subgroup of the n-semigroup (S, f). □

Proposition 14. *Let S be a semigroup. Let $A \subset S$ be a subsemigroup of S such that $AA = A$. If $D(A) \neq \emptyset$, then $D(A)$ is a subgroup of the semigroup A.*

Proof. Assume that $a, b \in D(A)$. Thus, $(ab)A^{n-1} = a(bA^{n-1}) = aA = aA^{n-1} = A$. Similarly, $A^{n-1}(ab) = A$. Hence $D(A)$ is a subsemigroup of the semigroup A. By Theorem 3, $D(A)$ is an n-subgroup of the n-semigroup (A, g). Theorem 2 implies that $D(A)$ is a subgroup of the semigroup A. $\square$

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