Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 44 (1994), No. 2, 231–250

Persistent URL: `http://dml.cz/dmlcz/128465`

Czechoslovak Mathematical Journal, 44 (119) 1994, Praha

DIRECT LIMITS OF CYCLICALLY ORDERED GROUPS

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(Received March 13, 1992)

Direct and inverse limits of groups have been dealt with by G. Higman and A. H. Stone [8] and by P. D. Hill [9], [10], [11].

Basic results on cyclically ordered groups are due to L. Rieger [18] and S. Swierczkowski [19]. For further references concerning cyclically ordered groups cf., e.g., [14]. In a recent paper, D. Gluschankof [5] describes interesting connections between cyclically ordered groups and MV-algebras.

Let us remark that the notion of cyclically ordered group can be considered a generalization of the notion of linearly ordered group.

Classes of cyclically ordered groups which are closed with respect to certain constructions (radical classes) have been studied in [15]; the analogous notion for linearly ordered groups has been introduced by Chehata and Wiegandt [2]; cf. also [4] and [12].

We denote by $\mathscr{C}$ the collection of all nonempty classes of cyclically ordered groups which are closed with respect to direct limits; this collection is partially ordered by inclusion.

In the present paper we investigate the properties of the partially ordered collection $\mathscr{C}$.

1. PRELIMINARIES

For cyclically ordered groups we apply the same definitions and notation as in [14].

Recall that a mapping f of a cyclically ordered group G into a cyclically ordered group G' is said to be a homomorphism if the following conditions are satisfied:

(i) f is a homomorphism with respect to the group operation;

(ii) whenever x, y and z are elements of G such that $f(x)$, $f(y)$ and $f(z)$ are distinct and $[x, y, z]$ holds, then $[f(x), f(y), f(z)]$.

For the sake completeness and for fixing the notation we also recall the notion of direct limit: in fact, we apply it for the case of cyclically ordered groups (cf., e.g., Kurosh [17] (p. 437–438) for the case of groups, and Grätzer [6] for the case of universal algebras).

Let I be a directed set. For each $\alpha \in I$ let G_α be a cyclically ordered group. Suppose that for each pair of elements α and β in I with $\alpha < \beta$ we define a homomorphism $\varphi_{\alpha\beta}$ of G_α into G_β such that $\alpha < \beta < \gamma$ implies that

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \cdot \varphi_{\beta\gamma}.$$

For each $\alpha \in I$ let $\varphi_{\alpha\alpha}$ be the identity on G_α.

Let α and β be elements of I and let $x \in G_\alpha, y \in G_\beta$. We put $x \equiv y$ if there exists $\gamma \in I$ with $\gamma \geqslant \alpha$, $\gamma \geqslant \beta$ such that $\varphi_{\alpha\gamma}(x) = \varphi_{\beta\gamma}(y)$. For each $z \in \bigcup\limits_{\alpha\in I} G\alpha$ put

$$\bar{z} = \Big\{t \in \bigcup_{\alpha\in I} G_\alpha : z \equiv t\Big\}.$$

Let $\overline{G} = \{\bar{z} : z \in \bigcup\limits_{\alpha\in I} G_\alpha\}$.

If z_i and $v_i (i = 1,2)$ are elements of $\bigcup\limits_{\alpha\in I} G\alpha$ such that $\bar{z}_1 = \bar{z}_2$ and $\bar{v}_1 = \bar{v}_2$, then clearly $\overline{z_1 + v_1} = \overline{z_2 + v_2}$. Hence if we put $\bar{z}_1 + \bar{v}_1 = \overline{z_1 + v_1}$, then the operation $+$ on $\overline{G}$ is correctly defined and with respect to this operation $\overline{G}$ is a group.

Next, we define a ternary relation $[\ ,\ ,\ ]$ on $\overline{G}$ as follows. For $\bar{x}$, $\bar{y}$ and $\bar{z}$ in $\overline{G}$ we set $[\bar{x}, \bar{y}, \bar{z}]$ if the following conditions are satisfied:

(i) $\bar{x}$, $\bar{y}$ and $\bar{z}$ are distinct;

(ii) there are $\alpha \in I$, $x_1 \in \bar{x}$, $y_1 \in y$ and $z_1 \in \bar{z}$ such that x_1, y_1 and z_1 belong to G_α and the relation $[x_1, y_1, z_1]$ is valid in G_α.

It is easy to verify that this ternary relation on $\overline{G}$ satisfies the conditions (I)–(IV) from [12]; hence $\overline{G}$ turns out to be a cyclically ordered group. It is said to be the direct limit of the indexed system $\{G_\alpha\}_{\alpha\in I}$. We express this situation by writing

$$\{G_\alpha\}_{\alpha\in I} \longrightarrow \overline{G}. \tag{1}$$

We denote by $\mathscr{C}$ the collection of all nonempty classes C of cyclically ordered groups which are closed with respect to direct limits, i.e., which satisfy the following condition; whenever (1) holds and $G_\alpha \in C$ for each $\alpha \in I$, then each cyclically ordered group isomorphic to $\overline{G}$ belongs to C.

The collection $\mathscr{C}$ is partially ordered by inclusion. The greatest element of $\mathscr{C}$ is the class C_m of all cyclically ordered groups.

Let $G \in C_m$ and let H be a subgroup of G. Assume that there exists a homomorphism ψ of G onto H such that $\psi(h) = h$ for each $h \in H$. Then H is said to be a retract of G and ψ is called a retract mapping corresponding to H.

Retracts of abelian cyclically ordered groups were investigated in [13].

Lemma 1.1. *Let $X \in \mathscr{C}$ and $G \in X$. Let H be a retract of G. Then $H \in X$.*

Proof. Let I be the set of all positive integers with the natural linear order. For each $\alpha \in I$ put $G_\alpha = G$. Let ψ be a retract mapping corresponding to H. For $\alpha, \beta \in I$ with $\alpha < \beta$ and for each $g \in G$ we put $\varphi_{\alpha\beta}(g) = \psi(g)$. Then (1) is valid, where $\overline{G}$ is isomorphic to H. Hence $H \in X$. □

It is obvious that the above Lemma is not specific for cyclically ordered groups, it is valid for any direct limits of any type of algebraic systems.

The class of all one-element cyclically ordered groups will be denoted by C_0.

Lemma 1.2. *Let $X \in C$. Then $C_0 \subseteq X$.*

Proof. Since $\{0\}$ is a retract of each cyclically ordered group, the assertion follows by 1.1. □

Corollary 1.3. *C_0 is the least element of $\mathscr{C}$.*

Let J be a nonempty class and for each $j \in J$ let C_j be an element of $\mathscr{C}$. Put $C = \bigcap_{j \in J} C_j$. Then C belongs to $\mathscr{C}$; hence C is the greatest lower bound of the collection $\{C_j\}_{j \in J}$. Thus by applying the usual lattice theoretic notation we can write

$$C = \bigwedge_{j \in J} C_j.$$

If D is the intersection of all D_k in $\mathscr{C}$ such that $D_k \geqslant C_j$ for each $j \in J$, then under analogous notation we have

$$D = \bigvee_{j \in J} C_j.$$

Lemma 1.4. *Let (1) be valid. Let $\alpha \in I$ and let f_α be the mapping of G_α into $\overline{G}$ such that $f_\alpha(x) = \bar{x}$ for each $x \in G_\alpha$. The f_α is a homomorphism of G_α into $\overline{G}$.*

This is an immediate consequence of the definition of the relation (1). □

The elements of $\mathscr{C}$ will be called direct limit classes. For $H \in C_m$ let $P(H)$ be the least element of $\mathscr{C}$ which contains H; i.e. $P(H)$ is the direct limit class which is generated by H.

For each cyclically ordered group G we denote by G_ℓ the largest linearly ordered convex subgroup of G (cf., e.g., [14]).

The symbols C^ℓ and C^0 will denote the class of all $G \in C_m$ such that $G_\ell = G$ or $G_\ell = \{0\}$, respectively.

Thus, in fact, C^ℓ is the class of all linearly ordered groups. Also,

$$C^\ell \cap C^0 = C_0.$$

Next, we denote by C^{01} the class

$$C_0 \cup (C_m \setminus C^{\ell}).$$

An important example of cyclically ordered groups is the set K of reals x with $0 \leqslant x < 1$; the group operation in K is the addition mod 1 and for $x, y, z \in K$ the relation $[x, y, z]$ is defined to be valid if some of the following inequalities holds:

$$x < y < z; \quad y < z < x; \quad z < x < y.$$

The importance of K is emphasized by Swierczkowski's Representation Theorem.

Let $\varphi\colon G \longrightarrow K_1 \otimes L$ be a representation of a cyclically ordered group G in the sense of [14], Definition 2.4. Under this notation we have

Lemma 1.5. *G is linearly ordered if and only if $K_1 = \{0\}$.*

Proof. If $K_1 = \{0\}$, then G is isomorphic to L, thus G is linearly ordered. Conversely, suppose that G is linearly ordered. Then Lemma 3.5 of [14] implies that $K_1 = \{0\}$. □

In [14] the notion of *c*-convexity was introduced. In the present paper we shall use the term "convexity" instead of "*c*-convexity".

Let us remark that Lemmas 1.2 and 1.3 [15] (given there for the case of abelian cyclically ordered groups) remain valid also for the non abelian case if we add the assumption that the subgroup H under consideration is a normal subgroup of the group G.

Then from 1.5 and from [15] (1.1, 1.2 and 1.3) we obtain

Lemma 1.6. *Let G' be a homomorphic image a cyclically ordered group G. Assume that G is not linearly ordered and that $G' \neq \{0\}$. Then G' is not linearly ordered.*

It will be proved below that if H is a subgroup of K (with the inherited cyclic order) and if f is a homomorphism of H into K with $f(H) \neq \{0\}$, then $f(x) = x$ for each $x \in H$. This result will be applied for investigating direct limit classes X with $C_0 \subseteq X \subset C^0$.

Let A be a nonempty class of cyclically ordered groups and let us denote by T the class of all ordinals. Put $A_1 = A$. Suppose that $\tau \in T$, $\tau > 1$ and that we have defined a subclass $A_{\tau 1}$ of C_m for each $\tau(1) < \tau$. We denote by A_τ the class of all cyclically ordered groups $\overline{G}$ having the property that there exists an indexed system $\{G_\alpha\}_{\alpha \in I}$ such that

$$\{G_\alpha\}_{\alpha \in I} \subseteq \bigcup_{\tau(1) < \tau} A_{\tau(1)} \quad \text{and} \quad \{G_\alpha\}_{\alpha \in I} \longrightarrow \overline{G}.$$

Put $A^* = \bigcup_{\tau \in T} A_\tau$. Then we obviously have

Lemma 1.5. *A^* is a direct limit class. If B is a direct limit class and $A \subseteq B$, then $A^* \subseteq B$.*

2. The classes C^ℓ and C^0

Let K be as above.

Lemma 2.1. *Let $g \in K$ be such that $ng \neq 0$ for each positive integer n. Then there exists a positive integer m such that $[0, mg, g]$ is valid.*

The proof is simple; it will be omitted.

Lemma 2.2. *Let g be an element of a cyclically ordered group G such that $g \notin G_\ell$ and $ng \neq 0$ for each positive integer n. Then there is a positive integer m such that $[0, mg, g]$ is valid.*

Proof. This is a consequence of 2.1 and of Swierczkowski's Representation Theorem (cf. [19]; cf. also [15], Theorem 1.1). □

Theorem 2.3. *$C^\ell \in \mathscr{C}$.*

Proof. In the way of contradiction, assume that C^ℓ does not belong to $\mathscr{C}$. Hence there exists an indexed system $\{G_\alpha\}_{\alpha \in I}$ such that (1) is valid, $G_\alpha \in C^\ell$ for each $\alpha \in I$ and $\overline{G} \notin C^\ell$. Thus there is $\bar{g} \in \overline{G}$ with $\bar{g} \notin \overline{G}_\ell$.

There are $\alpha \in I$ and $g_1 \in G_\alpha$ such that $\bar{g} = \bar{g}_1$. The relation $-\bar{g} \notin G_\ell$ is also valid; hence without loss of generality we can suppose that $g_1 > 0$ holds. Consider the homomorphism f_α from 1.4. We distinguish two cases.

a) Assume that there exists a positive integer n such that $n\bar{g} = 0$. Hence $\overline{ng_1} = 0$. Put $f_\alpha^{-1}(0) = H$. Then H is a convex subgroup of the linearly ordered group G_α and $ng_1 \in H$. Thus g_1 belongs to H as well and therefore $f_\alpha(g_1) = 0$, which is contradiction.

b) Now suppose that $n\bar{g} \neq 0$ for each positive integer n. In view of 2.2 there exists a positive integer m such that $[0, m\bar{g}, \bar{g}]$ is valid. Clearly $m > 1$. Because of $g_1 > 0$ we obtain that $[0, g_1, mg_1]$ and hence $[0, \bar{g}_1, m\bar{g}_1]$, i.e. $[0, \bar{g}, m\bar{g}]$, which is a contradiction. □

In view of [14], Theorem 2.5 we have

Lemma 2.4. *Let $G \in C^0$. Then G is isomorphic to a subgroup of K.*

Theorem 2.5. $C^0 \in \mathscr{C}$.

Proof. By way of contradiction, suppose that there is an indexed system $\{G_\alpha\}_{\alpha\in I}$ such that (1) holds, $G_\alpha \in C^0$ for each $\alpha \in I$ and $\overline{G} \notin C^0$. Hence $G_t \neq \{0\}$ and thus there exists a strictly positive element $\bar{g}$ in $\overline{G}_t$. Therefore $[0, \bar{g}, n\bar{g}]$ is valid for each positive integer $n > 1$.

There are $\alpha \in I$ and $g_1 \in G_\alpha$ such that $\bar{g} = \bar{g}_1$. Since $G_\alpha \in C^0$, in view of 2.4 we infer that G_α is isomorphic to a subgroup of K. Hence according to 2.1 either

a) there is a positive integer m with $mg_1 = 0$,

or

b) there is a positive integer m such that $[0, mg_1, g_1]$ holds.

Each of the conditions a), b) leads to the conclusion that the relation $[0, \bar{g}, m\bar{g}]$ cannot hold and so we have arrived at a contradiction. □

Lemma 2.6. *Let G_1 be a finite cyclically ordered group, $G_1 \neq \{0\}$. Put $A = P(G_1)$. Then for each $G_2 \in A$ either $G_2 = \{0\}$ or G_2 is isomorphic to G_1.*

Proof. Let A_1 be the class of all $G_2 \in C_m$ such that either $G_2 = \{0\}$ or G_2 is isomorphic to G_1. Clearly $A_1 \subseteq A$ and $G_1 \in A_1$. Hence it suffices to verify that A_1 belongs to C.

Assume that (1) is valid, where each G_α belongs to A_1 and $\overline{G} \neq \{0\}$. Then $G_\alpha \in C^0$ for each $\alpha \in I$. Thus in view of 2.5 we obtain that $\overline{G} \in C^0$.

Let $I(1)$ be the set of all $\alpha \in I$ such that $f_\alpha(G_\alpha) \neq \{0\}$ (where f_α is as in 1.4). If $I(1)$ is not confinal with I, then $\overline{G} = \{0\}$, which is a contradiction. Thus $I(1)$ is confinal with I and hence without loss of generality we can suppose that $I(1) = I$.

Let $\alpha \in I$. According to 1.4, f_α is a homomorphism of G_α into $f_\alpha(G_\alpha)$. Put $H = f_\alpha^{-1}(0)$. Then H is a convex subgroup of G_α. Since G_α is finite it must be isomorphic to a subgroup of K. Thus, if $H \neq \{0\}$, then there is $x \in H$ such that x generates the group G_α; this yields that $H = G_\alpha$ and then $f_\alpha(G_\alpha) = \{0\}$, which is a contradiction. Therefore $H = \{0\}$ and hence f_α is an isomorphism of G_α onto $f_\alpha(G_\alpha)$.

Let $\gamma \in I$, $\alpha \leqslant \gamma$. Since $\operatorname{card} G_\alpha = \operatorname{card} G_\gamma$ and since this cardinal is finite, the mapping $\varphi_{\alpha\gamma}$ is onto and the kernel of $\varphi_{\alpha\gamma}$ is $\{0\}$; thus $\varphi_{\alpha,\gamma}$ is an isomorphism.

Let $\alpha, \beta \in I$; there is $\gamma \in I$ with $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$. Let $h \in f_\beta(G_\beta)$. Hence there is $g_\beta \in G_\beta$ with $h = \bar{g}_\beta$. Thus $h = \bar{g}_\gamma$, where $g_\gamma = f_{\beta\gamma}(g_\beta)$. Put $g_\alpha = \varphi_{\alpha\gamma}^{-1}(g_\gamma)$. Then $g_\alpha \in h$ and hence $h \in f_\alpha(G_\alpha)$. Therefore $f_\alpha(G_\alpha) = \overline{G}$. Hence $\overline{G}$ is isomorphic to G_α and thus $A_1 \in C$. □

Corollary 2.7. *Let G_1 and A be as in 2.6. Then A is an atom of $\mathscr{C}$ and $A \leqslant C^0$.*

It is obvious that for each positive integer n there is (up to isomorphism) exactly one cyclically ordered group $G^{(n)}$ with $\operatorname{card} G^{(n)} = n$.

If G_1 and G_2 are finite cyclically ordered groups with $\operatorname{card} G_1 \neq \operatorname{card} G_2$, then in view of 2.6 we obtain that $P(G_1) \neq P(G_2)$. Thus according to 2.7 we have

Corollary 2.8. *The number of atoms A of $\mathscr{C}$ with $A < C^0$ is infinite.*

This result will be sharpened below by performing a more detailed investigation (cf. 4.5).

3. Homomorphisms of subgroups of K into K

In this section the following result on subgroups of K will be proved (it will be applied in subsequent sections):

Theorem 3.1. *Let φ be a homomorphism of a subgroup G of K into K, $\varphi(G) \neq \{0\}$. Then $\varphi(x) = x$ for each $x \in G$.*

We need some Lemmas. We assume that φ and G are as in 3.1.

Lemma 3.2. *φ is an isomorphism of G into K.*

Proof. The set $S = \{g \in G : \varphi(g) = 0\}$ is a convex normal subgroup of G; since the only such subgroups are $\{0\}$ and G, we infer that $S = \{0\}$ and thus φ is an isomorphism of G onto $\varphi(g)$. □

Lemma 3.3. *Let G be finite. Then $\varphi(x) = x$ for each $x \in G$.*

Proof. There exists x_1 in G which satisfies the following condition:

(*) $x_1 \neq 0$ and whenever $x_2 \in G$, then the relation $[0, x_2, x_1]$ does not hold. Moreover, x_1 generates the group G.

It is obvious that $nx = 0$, where $n = \operatorname{card} G$. In view of 3.2, φ is an isomorphism, $\varphi(x_1)$ also satisfies the condition (*) (where G is replaced by $\varphi(G)$). Thus $\varphi(x_1) = \frac{1}{n} = x_1$. Let $x \in G$. There is a positive integer m with $x = mx_1$. Therefore $\varphi(x) = m\varphi(x_1) = \frac{m}{n} = x$. □

Lemma 3.4. *Let $0 \neq x_1 \in G$ and suppose that x_1 is a rational number. Then $\varphi(x_1) = x_1$.*

Proof. Let $x_1 = \frac{m}{n}$, where m and n are positive integers. Then $nx_1 = 0$. Thus the subgroup G' of G which is generated by x_1 is finite. The mapping φ_1 defined by $\varphi_1(z) = \varphi(z)$ for each $z \in G'$ is a homomorphism of G' into K. Thus according to 3.3, the relation $\varphi(x_1) = \varphi_1(x_1) = x_1$ is valid. □

Lemma 3.5. *Let $x \in G$ be irrational. Then $\varphi(x)$ is irrational as well.*

Proof. By way of contradiction, suppose that $\varphi(x) = y$ is rational. In view of 3.2, there exists a positive integer $n > 1$ such that $ny = 0$. Thus neither $[0, y, ny]$ nor $[0, ny, y]$ holds. On the other hand, the elements $0, x$ and nx are distinct. Therefore either $[0, x, nx]$ or $[0, nx, x]$ is valid. Hence φ cannot be an isomorphism, which is a contradiction (cf. again, 3.2). □

Let $\mathbb{N}$ be the set of all positive integers. For irrational numbers x_1 and x_2 in K and for $k \in \mathbb{N}$ we put $x_1 \equiv x_2(R_k)$ if there is $m \in \{0, 1, 2, \dots, k-1\}$ such that both x_1 and x_2 belong to the interval

$$[m/2^{-k}, (m+1)/2^{-k}].$$

Lemma 3.6. *Let $k \in \mathbb{N}$. For each irrational number $x \in G$ the relation $x \equiv y(R_k)$ is valid, where $y = \varphi(x)$.*

Proof. We proceed by induction on k. Let $k = 1$. Suppose that the relation $x \equiv y(R_1)$ does not hold for some irrational $x \in G$. Then in view of 3.3 we have either

(a) $$0 < x < \frac{1}{2} < y < 1,$$

or

$$0 < y < \frac{1}{2} < x < 1.$$

If (a) is valid, then $[0, x, 2x]$ and $[0, 2y, y]$, hence φ fails to be an isomorphism, which is a contradiction (cf. 3.2). In the case (b) the situation is analogous. Hence the assertion holds for $n = 1$.

Let $k > 1$ and suppose that the assertion holds for $1, 2, \dots, k-1$. Suppose that the relation $x \equiv y(R_k)$ fails to be valid for some irrational $x \in G$. There is $m' \in \{0, 1, 2, \dots, k-2\}$ such that both x and y belong to the interval

$$[m'/2^{1-k}, (m'+1)/2^{1-k}] = [2m'/2^{-k}, (2m'+2)/2^{-k}].$$

Since the relation $x \equiv y(R_k)$ does not hold, either

(a′) $$2m'/2^{-k} < x < (2m'+1)/2^{-k} < y < (2m'+2)/2^{-k}$$

or

(b′) $$2m'/2^{-k} < y < (2m'+1)/2^{-k} < y < (2m'+2)/2^{-k}.$$

Suppose that (a′) is valid (the method for the case (b′) is analogous). Consider the elements $x' = 2x$ and $y' = 2y$ of G. Then $y' = \varphi(x')$. The relation (a′) implies that

$$2x \not\equiv 2y(R_{k-1}),$$

which is a contradiction. □

Lemma 3.7. *Let $x \in G$ be irrational. Then $\varphi(x) = x$.*

Proof. This is an immediate consequence of 3.6. □

Now, 3.4 and 3.7 imply that 3.1 is valid.
The following assertion is an immediate consequence of 3.1.

Corollary 3.8. *Let (1) be valid. Assume that $\overline{G}$ and G_α are subgroups of K for each $\alpha \in I$. Then for each $\alpha, \beta \in I$ the mapping $\varphi_{\alpha,\beta}$ is an embedding of G_α into G_β and $\overline{G} = \bigcup\limits_{\alpha \in I} G_\alpha$.*

4. The interval $[C_0, C^0]$ of $\mathscr{C}$

We will apply the results of the previous section for investigating the interval $[C_0, C^0]$ of the partially ordered collection $\mathscr{C}$.

Lemma 4.1. *Let (1) be valid. Suppose that there is $G \in C^0$. $G \neq \{0\}$ such that for each $\alpha \in I$ either $G_\alpha = G$ or $G_\alpha = \{0\}$. Then $\overline{G}$ is isomorphic to G.*

Proof. Since $\overline{G} \neq \{0\}$, without loss of generality we can assume that $G_\alpha = G$ for each $\alpha \in I$. According to 2.5, $\overline{G} \in C^0$. In view of 2.4 there are subgroups$\overline{G}'$ and $\overline{G}'$ of K such that

(i) G' and $\overline{G}'$ are isomorphic to G or $\overline{G}$ respectively; and
(ii) $\{G'_\alpha\}_{\alpha \in I} \longrightarrow \overline{G}'$, where $G'_\alpha = G'$ for each $\alpha \in I$.

Let $\alpha \in I$ and consider the mapping f_α of G'_α into G' (cf. 1.4). Then according to 3.1, $f_\alpha(G'_\alpha) = G'_\alpha$. Let $g \in \overline{G}'$. There is $\beta \in I$, $\beta \geqslant \alpha$ such that $g \in f_\beta(G'_\beta)$. Since $\varphi_{\alpha,\beta}$ is an isomorphism by 3.1 and $f_\alpha = \varphi_{\alpha,\beta} \cdot f_\beta$, we obtain $g \in f_\alpha(G_\alpha)$.

We have proved that there exists a surjecture homomorphism of G onto $\overline{G}$. Since any $f_{\alpha,\beta}$ is an isomorphism, this homomorphism is an isomorphism as well. Thus $f_\alpha(G'_\alpha) = \overline{G}'$. Therefore $\overline{G}$ is isomorphic to G. □

Theorem 4.2. *Let $G \in C^0, G \neq \{0\}$. Let A be the class of all $G' \in C_m$ such that either $G' = \{0\}$ or G' is isomorphic to G. Then A is a direct limit class; moreover, A is an atom in $\mathscr{C}$.*

Proof. Both assertions follow from 4.1. □

If A and G are as in 4.2, then clearly $A = P(G)$.

Lemma 4.3. *Let G_1 and G_2 be subgroups of K, $G_1 \neq \{0\}$ $(i = 1,2)$, $G_1 \neq G_2$. Then $P(G_1) \neq P(G_2)$.*

Proof. Since $G_1 \neq G_2$, in view of 3.1 we conclude that G_1 and G_2 are not isomorphic. Thus according to 4.2 the relation $P(G_1) \neq P(G_2)$ is valid. □

Lemma 4.4. *Let X be an atom of $\mathscr{C}$ such that $X \leqslant C^0$. Then there is a subgroup $G \neq \{0\}$ of K such that $X = P(G)$.*

Proof. Since X is an atom of $\mathscr{C}$ there is $G_1 \in C_m$ such that $G_1 \neq \{0\}$ and $X = P(G_1)$. Next, from $X \leqslant C^0$ we infer that $G_1 \in C^0$ and hence in view of 2.4 there is a subgroup G of K such that G is isomorphic to G_1. Therefore $X = P(G)$. □

Theorem 4.5. *The mapping $\psi \colon G \longrightarrow P(G)$ is one-to-one correspondence between the system of all nonzero subgroups of K and the system of all atoms X of $\mathscr{C}$ with $X \leqslant C^0$.*

Proof. This is a consequence of 4.2, 4.3 and 4.4. □

We denote by $\mathscr{S}$ the collection of all nonempty systems of subgroups of K containing the one-element group $\{0\}$.

Each system $\mathscr{A}$ in $\mathscr{S}$ will be considered to be partially ordered by inclusion. We shall deal with the following condition for $\mathscr{A}$:

(*) If $\mathscr{A}_1 = \{A_i\}_{i\in I}$ is a nonempty subsystem of $\mathscr{A}$ such that $\mathscr{A}_1$ is directed, then $\bigcup\limits_{i\in I} A_i$ belongs to $\mathscr{A}$.

For $\mathscr{A} \in \mathscr{S}$ let $\psi(\mathscr{A})$ be the class of all $G_1 \in C_m$ such that G_1 is isomorphic to an element of $\mathscr{A}$.

Lemma 4.6. *Let $\mathscr{A} \in \mathscr{S}$. Assume that $\mathscr{A}$ satisfies the condition (*). Then $\psi(\mathscr{A})$ belongs to $\mathscr{C}$ and $\psi(\mathscr{A}) \leqslant C^0$.*

Proof. Let (1) be valid and suppose that each G_α belongs to $\mathscr{A}$. First we shall verify that $\overline{G} \in \mathscr{A}$. The case $\overline{G} = \{0\}$ is clear; assume that $\overline{G} \neq \{0\}$. Then without loss of generality we can assume that, whenever $\alpha, \beta \in I$ and $\alpha < \beta$, then $\varphi_{\alpha,\beta}(G_\alpha) \neq \{0\}$. Thus according to 3.1, $G_\alpha \subseteq G_\beta$. Next, $f_\alpha(G_\alpha) \neq \{0\}$ for each $\alpha \in I$. Clearly

$$\overline{G} = \bigcup_{\alpha\in I} G_\alpha.$$

Hence in view of (*) we have $\overline{G} \in \mathscr{A}$. Now by the definition of $\mathscr{C}$ we infer that $\psi(\mathscr{A})$ belongs to $\mathscr{C}$. Let $G \in \psi(\mathscr{A})$. Then G is isomorphic to a subgroup of K, whence $G_\ell = \{0\}$ and so $G \in C^0$. Therefore $\psi(\mathscr{A}) \leqslant C^0$ holds. □

Lemma 4.7. *Let $X \in [C_0, C^0], X \neq C_0$. Next, let $\mathscr{A}$ be the system of all elements of $\mathscr{S}$ which belong to X. Then $\psi(\mathscr{A}) = X$ and $\mathscr{A}$ satisfies the condition (*).*

Proof. Let $A \in \psi(\mathscr{A})$. Then there is $A_1 \in \mathscr{A}$ such that A_1 is isomorphic to A. Hence $A_1 \in X$ and so $\psi(\mathscr{A}) \subseteq X$. Conversely, let $A \in X$. There exists $A_1 \in \mathscr{S}$ with the property that A_1 is isomorphic to A. Then $A_1 \in X$, thus $A_1 \in \mathscr{A}$. Therefore $A \in \psi(\mathscr{A})$ and hence $X \subseteq \psi(\mathscr{A})$.

Now we have to verify that $\mathscr{A}$ satisfies the condition $(*)$. Let $\{A_i\}_{i\in I}$ be as in the assumption of $(*)$. Without loss of generality we can suppose that $A_{i(1)} \neq A_{i(2)}$ whenever $i(1)$ and $i(2)$ are distinct elements of I. For $i(1)$ and $i(2)$ in I we put $i(1) \leqslant i(2)$ if $A_{i(1)} \subseteq A_{i(2)}$. Hence I turns out to be a directed set. Put $A = \bigcup_{i\in I} A_i$. Thus $\{A_i\}_{i\in I} \longrightarrow A$. Since each A_i belongs to X and X is closed with respect to direct limits we obtain that A belongs to X as well. Therefore A belongs to $\mathscr{A}$ and hence $(*)$ is valid for $\mathscr{A}$. □

From the definition of the mapping ψ we immediately obtain

Lemma 4.8. *Let $\mathscr{A}_1, \mathscr{A}_2 \in \mathscr{S}$. Then $\mathscr{A}_1 \subseteq \mathscr{A}_2 \Longleftrightarrow \psi(\mathscr{A}_1) \subseteq \psi(\mathscr{A}_2)$.*

Let $\mathscr{S}_0$ be the collection of all $\mathscr{A} \in \mathscr{S}$ which satisfy the condition $(*)$; $\mathscr{S}_0$ is partially ordered by inclusion.

Theorem 4.9. *The interval $[C_0, C^0]$ of $\mathscr{C}$ is isomorphic to S_0.*

Proof. This is a consequence of 4.6, 4.7 and 4.8. □

Corollary 4.10. *The interval $[C_0, C^0]$ of $\mathscr{C}$ fails to be a proper class.*

Corollary 4.11. *The interval $[C_0, C^0]$ of $\mathscr{C}$ is atomic (in the sense that for each $X \in [C_0, C^0]$ with $X \neq C_0$ there exists an atom Y of $\mathscr{C}$ such that $Y \leqslant X$).*

Proof. This is a consequence of 4.5. □

Below we shall show that the interval $[C_0, C^\ell]$ of $\mathscr{C}$ fails to be atomic.

5. The class C^{01}

In this section we shall prove that the class C^{01} belongs to $\mathscr{C}$. Next we shall investigate the relations between C^{01} and C^ℓ.

Lemma 5.1. *The class C^{01} is closed with respect to homomorphisms.*

Proof. Let $G \in C^{01}$ and let G' be a homomorphic image of G. It suffices to consider the case $G' \neq \{0\}$. Then $G \neq \{0\}$ and G fails to be linearly ordered. Thus in view of 1.6, G' is not linearly ordered. Hence $G' \in C^{01}$. □

Theorem 5.2. *C^{01} is a direct limit class.*

Proof. Let (1) be valid, where $G_\alpha \in C^{01}$ for each $\alpha \in I$. The case $\overline{G} = \{0\}$ is trivial; suppose that $\overline{G} \neq \{0\}$. There is $\alpha \in I$ with $f_\alpha(G_\alpha) \neq \{0\}$. Since f_α is a homomorphism, 5.1 yields that $f_\alpha(G_\alpha)$ fails to be linearly ordered. Thus $\overline{G}$ is not linearly ordered, and hence $\overline{G} \in C^{01}$. □

Remark 5.3. The notion of direct limit class can be, in fact, defined for any type T of algebraic systems which is closed with respect to homomorphism; analogously as above we can consider the collection $\mathscr{C}_T$ of all direct limit classes with respect to the given type T; for some types T it is more convenient to consider the empty set as being an element of $\mathscr{C}_T$ (cf., e.g., [4] for the case when T denotes the variety of all connected monounary algebras). It is easy to verify that in each such partially ordered collection $\mathscr{C}_T$ and any $X, Y, X_i \in \mathscr{C}_T (i \in I)$ we have

$$X \vee Y = X \cup Y, \tag{2.1}$$

$$\bigwedge_{i\in I} X_i = \bigcap_{i \in I} X_i, \tag{2.2}$$

hence

$$Y \vee \left(\bigwedge_{i\in I} X_i\right) = \bigwedge_{i\in I}(Y \vee X_i). \tag{2.3}$$

In particular, $\mathscr{C}_T$ is always a distributive lattice.

We return to the collection $\mathscr{C}$. For each $X \in \mathscr{C}$ we put

$$X_1 = X \cap C^{0,1}, \qquad X_2 = X \cap C^\ell.$$

Lemma 5.4. *Let $X \in \mathscr{C}$. Then X_1 and X_2 belong to $\mathscr{C}$.*

Proof. This is a consequence of (2.2), 5.2 and 2.3. □

Next, (2.1) yields

Lemma 5.5. *Let $X' \in [C_0, C^{01}], X'' \in [C_0, C^\ell]$. Then $X' \cup X'' \in \mathscr{C}$.*

Lemma 5.6. *Let $X \in \mathscr{C}$. Then $X = X_1 \cup X_2$.*

Proof. Since $C^{01} \cup C^\ell = \mathscr{C}$ we obtain $X = X \cap \mathscr{C} = X_1 \cup X_2$. □

Lemma 5.7. *Let $X, Y \in \mathscr{C}$. Then $X \leqslant Y$ iff $X_1 \leqslant Y_1$ and $X_2 \leqslant Y_2$.*

Proof. If $X \leqslant Y$, then clearly $X_1 \leqslant Y_1$ and $X_2 \leqslant Y_2$. The converse implication is a consequence of 5.6. □

For each $X \in \mathscr{C}$ put $f(X) = (X_1, X_2)$. Lemmas 5.5 and 5.4 yield.

Lemma 5.8. *The mapping $f: \mathscr{C} \to [C_0, C^{01}] \times [C_0, C^{\ell}]$ is a surjection.*

Theorem 5.9. *The mapping f is an isomorphism of $\mathscr{C}$ onto $[C_0, C^{01}] \times [C_0, C^{\ell}]$.*

Proof. This is a consequence of 5.7 and 5.8. □

The following example shows that the infinite distributive law dual to (2.3) does not hold in $\mathscr{C}$.

Example 5.10. Put $I = \mathbb{N}$ and for each $n \in I$ let G_n be the subgroup of K generated by the element 2^{-n}. Next, let $\overline{G}$ be the subgroup of K consisting of all elements of the form $x = m \cdot 2^{-n}$, where m is an integer, $n \in \mathbb{N}$ and $0 \leqslant x < 1$. For any $\alpha, \beta \in \mathbb{N}$ with $\alpha \leqslant \beta$ and for each $x \in G_\alpha$ we put $\varphi_{\alpha\beta}(x) = x$. Then $\{G_\alpha\}_{\alpha \in I} \to \overline{G}$. According to 4.5, $P(G_n)$ $(n \in \mathbb{N})$ and $P(\overline{G})$ are atoms of $\mathscr{C}$; moreover, $P(G_n) \neq P(\overline{G})$ for each $n \in \mathbb{N}$. In view of 4.9 the relation $\overline{G} \in \bigvee_{n \in \mathbb{N}} P(G_n)$ is valid, whence $P(\overline{G}) \leqslant \bigvee_{n \in \mathbb{N}} P(G_n)$. Therefore

$$P(\overline{G}) = P(\overline{G}) \wedge \Big(\bigvee_{n \in \mathbb{N}} P(G_n)\Big),$$

$$\bigvee_{n \in \mathbb{N}} (P(\overline{G}) \wedge P(G_n)) = C_0.$$

6. On direct limit classes of linearly ordered groups

In this section we will deal with the interval $[C_0, C^{\ell}]$ of the partially ordered collection $\mathscr{C}$.

Let $G \in C^{\ell}$. For $0 < a \in G$ and $0 < b \in G$ we write $a \ll b$ if $ma < b$ for each $m \in \mathbb{N}$.

We put $\psi(G) = n$ if the following conditions are satisfied:

(i) There exist elements $0 < a_i \in G$ $(i = 1, 2, \dots, n)$ such that $a_1 \ll \dots \ll a_n$.

(ii) If $0 < b_j \in G$ $(j = 1, 2, \dots, m)$ and $b_1 \ll b_2 \ll \dots \ll b_m$, then $m \leqslant n$.

For $G = \{0\}$ we set $\psi(G) = 0$. If $\psi(G) \neq 0$ and $\psi(G) \neq n$ for each $n \in \mathbb{N}$ is valid, then we put $\psi(G) = \infty$.

Lemma 6.1. *Let $G \in C^\ell$, $n \in \mathbb{N}$, $\psi(G) \leqslant n$. Let G_1 be a homomorphic image of G. Then $\psi(G_1) \leqslant n$.*

Proof. The case $G_1 = \{0\}$ is trivial; suppose that $G_1 \neq \{0\}$. Without loss of generality we can assume that $G_1 = G/H$, where H is an ℓ-ideal of G. For $g \in G$ we denote $\bar{g} = g + H$. Let $g_1, \ldots, g_k \in G$ be such that $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k \in G/H$ and suppose that $\bar{0} < \bar{g} \ll \bar{g}_2 \ll \ldots \ll \bar{g}_k$. Then $0 \vee g_1 \in \bar{g}_1$, thus we can take $0 \vee g_1$ for g_1; next, $g_i > 0$ for $i = 1, 2, 3, \ldots, k$. If $i \in \{1, 2, \ldots, k-1\}$, $t \in \mathbb{N}$ and $tg_i \geqslant g_{i+1}$ then $t\bar{g}_i \geqslant t\bar{g}_{i+1}$. For each $i \in \{1, 2, \ldots, k-1\}$ we have arrived at a contradiction. Hence $g_1 \ll g_2 \ll \ldots \ll g_k$ and thus $k \leqslant n$. Therefore $\psi(G_1) \leqslant n$. □

Proposition 6.2. *Let $n \in \mathbb{N}$. Let X be the class of all linearly ordered groups G such that $\psi(G) \leqslant n$. Then X is a direct limit class.*

Proof. Let (1) be valid and suppose that each G_α $(\alpha \in I)$ belongs to X. It suffices to consider the case $\overline{G} \neq \{0\}$. Without loss of generality we can suppose that $f_\alpha(G_\alpha) \neq \{0\}$ for each $\alpha \in I$. Let $k \in \mathbb{N}$ and assume that $\bar{0} < \bar{g}_1 \ll \bar{g}_2 \ll \ldots \ll \bar{g}_k$ for some elements $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k$ in $\overline{G}$. There exists $\alpha \in I$ with the property that there are $h_i \in G_\alpha\{i = 1, 2, \ldots, k\}$ such that $f_\alpha(h_i) = \bar{g}_i$ for each $i = 1, 2, \ldots, k$. Then according to 1.4 and 6.1 the relation $k \leqslant n$ is valid. Therefore $\psi(\overline{G}) \leqslant n$. □

A linearly ordered group G is archimedean iff $\psi(G) \leqslant 1$; hence we obtain

Corollary 6.3.1. *The class of all archimedean linearly ordered groups is a limit class.*

Corollary 6.3.2. *The interval $[C_0, C^\ell]$ of $\mathscr{C}$ is infinite.*

A stronger result will be proved below (cf. 6.25).

Let $p > 1$ be a prime and let H_p be the additive group of all reals of the form ap^{-n}, where a is an integer and n is a non-negative integer; we consider the linear order on H_p.

The following Lemma is obvious.

Lemma 6.4. (i) *If $p(1)$ and $p(2)$ are distinct primes, then $H_{p(1)}$ and $H_{p(2)}$ are not isomorphic.*

(ii) *If $H \neq \{0\}$ is a homomorphic image of H_p, then H is isomorphic to H_p.*

Let I be a linearly ordered set and for each $i \in I$ let G_i be a linearly ordered group. The lexicographic product of the indexed system $\{G_i\}_{i \in I}$ will be denoted by $\mathbf{\Gamma}_{i \in I} G_i$. Next, let $\mathbf{\Gamma}^0_{i \in I} G_i$ be the linearly ordered group consisting of those $g \in \mathbf{\Gamma}_{i \in I} G_i$ which satisfy the condition that the set $\{i \in I \colon g(i) \neq 0\}$ is finite.

If $I(1)$ is a nonempty subset of I and $g \in \mathbf{\Gamma}_{i\in I(1)}G_i$, then the element g will be identified with the element g' of $\mathbf{\Gamma}_{i\in I}G_i$ such that $g'(i) = g(i)$ for each $i \in I(1)$ and $g'(i) = 0$ for each $i \in I \setminus I(1)$.

Let P be the set of all positive primes with the linear order which is dual to the natural one.

Put

$$H = \mathbf{\Gamma}^0_{p\in P}H_p.$$

For each $p(1) \in P$ let

$$G_{p(1)} = \mathbf{\Gamma}^0_{p\geqslant p(1)}H_p.$$

Lemma 6.5. (i) *Let H' be a homomorphic image of H (or of $G_{p(1)}$, respectively). Then there is $p(2) \in P$ (with $p(2) \geqslant p(1)$) such that H' is isomorphic to $G_{p(2)}$.*

(ii) *Let $p(1)$ and $p(2)$ be elements of P with $p(1) \leqslant p(2)$. Assume that there exists an isomorphism φ of $G_{p(1)}$ into $G_{p(2)}$. Then $p(1) = p(2)$ and φ is the identity on $G_{p(1)}$.*

The proof is simple; it will be omitted.

Let us denote by X the class of all linearly ordered groups G such that either

(i) $G = \{0\}$ or

(ii) there is a prime $p(1)$ such that G is isomorphic to $G_{p(1)}$.

Suppose that (1) is valid and that $G_\alpha \in X$ for each $\alpha \in I$; next assume that $\overline{G} \neq \{0\}$. Then we can assume that $f_\alpha(G_\alpha) \neq \{0\}$ for each $\alpha \in I$. Hence according to 6.5 (i), for each $\alpha \in I$ the linearly ordered group $f_\alpha(G_\alpha)$ is isomorphic to some $G_{p(1)}$; let $p(0)$ be the minimal of the primes $p(1)$ with this property and let $\alpha(1)$ be the corresponding element of I.

Let $\bar{g} \in \overline{G}$. There is $\alpha(2) \in I$ with $\alpha(2) \geqslant \alpha(1)$ such that $\bar{g} \in f_{\alpha(2)}(G_{\alpha(2)}))$. Then $f_{\alpha(2)}(G_{\alpha(2)}) \supseteq f_{\alpha(1)}(G_{\alpha(1)}$. There is $p(2) \in P$ such that $f_{\alpha(2)}(G_{\alpha(2)})$ is isomorphic to $G_{p(2)}$. In view of the minimality of $p(1)$ we obtain that $p(1) \leqslant p(2)$. Next, there exists an isomorphism φ of $G_{p(1)}$ into $G_{p(2)}$. Thus according to 6.5 (ii) the relation $i(1) = i(2)$ is valid. This yields that $f_{\alpha(2)}(G_{\alpha(2)}) = f_{\alpha(1)}(G_{\alpha(1)})$. Hence $\overline{G} = f_{\alpha(1)}(G_{\alpha(1)})$.

Therefore $\overline{G}$ is isomorphic to $G_{p(1)}$. So we have

Lemma 6.6. *$X \in \mathscr{C}$ and $X \leqslant C^\ell$.*

For $p \in P$ we denote by X_p the collection of all $G \in C^\ell$ such that either $G = \{0\}$ or G is isomorphic to $G_{p(1)}$ for some $p(1) \geqslant p$. By the same method as above we can prove

Lemma 6.7. *Let $p \in P$. Then $X_p \in \mathscr{C}$ and $X_p \leqslant C^\ell$.*

Lemma 6.8. *Let $Y \in C^\ell$, $C_0 \neq Y < X$. Then there is $p \in P$ such that $Y = X_p$.*

Proof. According to the definition of X there is $p \in P$ such that $G_p \in Y$ and $G_{p(1)} \neq Y$ whenever $p(1) < p$. Then clearly $Y = X_p$. □

If $p(1) < p(2)$, then $X_{p(1)} > X_{p(2)}$. Thus 6.8 yields

Theorem 6.9. *There is no atom A in $\mathscr{C}$ with $A \leqslant X$.*

Corollary 6.10. *The interval $[C_0, C^\ell]$ of $\mathscr{C}$ fails to be atomic.*

In view of the above results the question arises whether the collection of atoms in $[C_0, C^\ell]$ is nonempty. The "natural" candidates of being atoms here seem to be the limit classes $P(Z)$, $P(Q)$ and $P(R)$, where Z, Q and R are additive groups of all integers, all rationals or all reals, respectively, with the natural linear order.

In what follows we perform the corresponding discussion for $P(Z)$, $P(Q)$ and $P(R)$. It will be shown that $P(Z)$ fails to be an atom of $[C_0, C^\ell]$; on the other hand, both $P(Q)$ and $P(R)$ are atoms of $[C_0, C^\ell]$.

In view of 2.3, $P(Z)$, $P(Q)$ and $P(R)$ are elements of $[C_0, C^\ell]$.

Lemma 6.11. *Let ψ be a homomorphic mapping of Q into Q, $\varphi(Q) \neq \{0\}$. Then $\varphi(Q) = Q$.*

Proof. If $q_1, q_2 \in Q$, $n \in \mathbb{N}$ and $nq_1 = q_2$, then $n\varphi(q_1) = \varphi(q_2)$. Hence $\varphi(Q)$ is divisible. The only nonzero divisible subgroup of Q is Q; thus $\varphi(Q) = Q$. □

Lemma 6.12. *Let* (1) *be valid. Assume that $G_\alpha = Q$ for each $\alpha \in I$ and $\overline{G} \neq \{0\}$. Then $\overline{G}$ is isomorphic to Q.*

Proof. Since $\overline{G} \neq \{0\}$ we can assume that $f_\alpha(G_\alpha) \neq \{0\}$ for each $\alpha \in I$. Next, if $\alpha, \beta \in I$ and $\alpha < \beta$, then in view of 6.11 the relation $f_{\alpha\beta}(G_\alpha) = G_\beta$ is valid.

Let $x \in \overline{G}$ and let $\alpha \in I$. There exists $\gamma \in I$ such that $\gamma \geqslant \alpha$ and $x \in f_\gamma(G_\gamma)$. Thus $x = f_\gamma(x_1)$ for some $x_1 \in G_\gamma$. There is $x_2 \in G_\alpha$ with $x_1 = f_{\alpha\gamma}(x_2)$. Hence $f_\alpha(x_2) = x$ and so f_α is an epimorphism. We obtain that f_α is an isomorphism of G_α onto $\overline{G}$. Therefore $\overline{R}$ is isomorphic to Q. □

Proposition 6.13. *$P(Q)$ is an atom of $\mathscr{C}$ and $P(Q) \in [C_0, C^\ell]$.*

Proof. Since $Q \in C^\ell$, in view of $C^\ell \in \mathscr{C}$ we obtain that $P(Q) \in [C_0, C^\ell]$. The fact that $P(Q)$ is an atom of $\mathscr{C}$ is a consequence of 6.12. □

Now we shall apply a similar argument for R. A homomorphism of a linearly ordered group G_1 into a linearly ordered group G_2 is said to be complete if, whenever $\{x_i\}_{i \in I} \subseteq G_1$ and $\bigvee_{i \in I} x_i$ exists in G_1, then $V_{i \in I}\varphi(x_i) = \varphi(\bigvee_{i \in I} x_i)$ holds in G_2. This condition is equivalent to the corresponding dual one.

Lemma 6.14. *Let φ be a homomorphism of R into R. Then φ is a complete homomorphism.*

P r o o f. If $\varphi(R) = \{0\}$, then the assertion is trivial. Suppose that $\varphi(R) \neq \{0\}$. Then φ is a monomorphism. Let $x, x_i \in R$ $(i \in I)$, $x = \bigvee_{i \in I} x_i$. Then $\varphi(x_i) \leqslant \varphi(x)$ for each $i \in I$. By way of contradiction, suppose that the relation $\bigvee_{i \in I} \varphi(x_i) = \varphi(x)$ does not hold. Then there is $y \in R$ such that $\varphi(x_i) < y < \varphi(x)$ for each $i \in I$. Put $\varphi(x) - y = z$.

Since $\varphi(R) \neq \{0\}$ there is $t \in R$ with $\varphi(t) > 0$. For each positive integer m we have $\varphi(\frac{1}{m}t) = \frac{1}{m}\varphi(t)$. If $z \leqslant \frac{1}{m}\varphi(t)$ for each $m \in \mathbb{N}$, then $mz \leqslant \varphi(t)$ for each positive integer m, which is impossible. Thus there is $0 < z_1 \in R$ such that $\varphi(z_1) < z$. Therefore $\varphi(x) > \varphi(x) - \varphi(z_1) > \varphi(x) - z = y > \varphi(x_i)$ for each $i \in I$; hence $x > x - z_1 > x_i$ for each $i \in I$, which is a contradiction. □

Lemma 6.15. *Let φ be a homomorphism of R into R, $\varphi(R) \neq \{0\}$. Then $\varphi(R) = R$.*

P r o o f. Let $0 < y \in R$. By the same method as in the proof of 6.14 we obtain that there is $0 < x \in R$ and that $\varphi(x) \leqslant y$. Thus there exist $x_n \in R$ $(n \in \mathbb{N})$ such that the sequence $\{x_n\}$ is increasing, upper bounded and $\bigvee_{n \in \mathbb{N}} \varphi(x_n) = y$. Then according to 6.14, $\varphi(\bigvee_{n \in \mathbb{N}} x_n) = y$. Therefore $\varphi(R) = R$. □

Lemma 6.16. *Let* (1) *be valid. Assume that $G_\alpha = R$ for each $\alpha \in I$ and $\overline{G} \in \{0\}$. Then $\overline{G}$ is isomorphic to R.*

P r o o f. According to 2.3, $\overline{G}$ is linearly ordered. Next, in view of 6.3. $\overline{G}$ is archimedean. Without loss of generality we can assume that $\overline{G}$ is a subgroup of R. Next, we can suppose that $f_\alpha(G_\alpha) \neq \{0\}$ for each $\alpha \in I$. Thus according to 6.15, $f_\alpha(G_\alpha) = R$ for each $\alpha \in I$, whence $\overline{G} = R$. □

Proposition 6.17. *$P(R)$ is an atom of $\mathscr{C}$ and $P(R) \in [C_0, C^\ell]$.*

P r o o f. Since R belongs to C^ℓ we obtain that $P(R) \in [C_0, C^\ell]$. In view of 6.16, $P(R)$ is an atom of $\mathscr{C}$. □

Lemma 6.18. *$Q \in P(Z)$.*

P r o o f. Let I be the set of all positive integers; for $m(1)$ and $m(2)$ in I we put $m(1) \leqslant m(2)$ if either $m(1) = m(2)$ or $m(2)$ is divisible by $m(1)$.

Next, for each $\alpha \in I$ let G_α be the additive group of all rationals of the form $\frac{n}{m}$, where n runs over Z and $m = \alpha$; the group G_α is linearly ordered in the natural way. Thus G_α is isomorphic to Z for each $\alpha \in I$. For $\alpha, \beta \in I$ with $\alpha < \beta$ we have $G_\alpha \subset G_\beta$; let $\varphi_{\alpha,\beta}$ be the identical mapping of G_α into G_β. It is clear that $\{G_\alpha\}_{\alpha \in I} \longrightarrow \bar{G}$, where $\bar{G}$ is isomorphic to Q. Thus $Q \in P(Z)$. □

Corollary 6.19. *$P(Q) < P(Z)$; hence $P(Z)$ fails to be an atom in $\mathscr{C}$.*

Proof. In view of 6.18 we have $P(Q) \leqslant P(Z)$; according to 6.13 and 6.12 the relation $Z \notin P(Q)$ is valid. Hence $P(Q) < P(Z)$. Since $P(Q)$ is an atom in $\mathscr{C}$, $P(Z)$ cannot be an atom. □

The proof of the following result will be omitted.

Proposition 6.20. *Let $\mathscr{V}$ be a variety of lattice ordered groups. Then the class $\mathscr{V} \cap C^\ell$ belongs to $\mathscr{C}$.*

The natural question arises how large is the interval $[C_0, C^\ell]$ of $\mathscr{C}$. We will show that there exists an injective mapping of the class of all infinite cardinals into $[C_0, C^\ell]$; hence $[C_0, C^\ell]$ is a proper class.

Lemma 6.21. *For each infinite cardinal β there exists a linearly ordered group G_β such that*

(i) *G_β is a simple lattice ordered group, and*

(ii) $\operatorname{card} G_\beta \geqslant \beta$.

Proof. This is a consequence of results of Chehata [1] and Dlab [3] (cf. also [16], pp. 90–91). □

Let $G \in C^\ell$. For $H \in C^\ell$ consider the following condition:

(G) Let $0 \neq h \in H$. Then there is a subgroup H_1 of H with $h \in H_1$ such that H_1 is isomorphic to G.

Lemma 6.22. *Let G be a simple linearly ordered group, $G \neq \{0\}$. Let* (1) *be valid and suppose that each G_α ($\alpha \in I$) satisfies the condition (G). Then $\bar{G}$ satisfies the condition (G) as well.*

Proof. Let $0 \neq \bar{h} \in \bar{G}$. There exists $\alpha \in I$ and $h \in G_\alpha$ such that $f_\alpha(h) = \bar{h}$. Next, there is a subgroup H_1 of G_α with $h \in H_1$ such that H_1 is isomorphic to G. Put $f_\alpha(H_1) = H_2$. Then H_2 is a homomorphic image of H_1 and $0 \neq \bar{h} \in H_2$. Therefore, since H_2 is simple, H_2 is isomorphic to H_1 and so H_2 is isomorphic to G. □

Let $G \neq \{0\}$ be a simple linearly ordered group. Put $A = \{G\}$ and let A^* be as in 1.5.

Lemma 6.23. *Each nonzero element of A^* satisfies the condition* (G).

Proof. This is a consequence of 6.22 and the definition of A^* (cf. Section 1). □

Let us denote by M the class of all infinite cardinals. We define by transfinite induction an injective mapping φ of M into C^ℓ as follows. For $\beta = \aleph_0$ let $\varphi(\beta) = G_\beta$ be as in 6.21. Suppose that $\aleph_0 < \beta \in M$ and that we have defined $\varphi(\beta(1))$ for each $\beta(1) < \beta$. There exists a cardinal β' with $\beta' > \operatorname{card} G_{\beta(1)}$ for each $\beta(1) < \beta$. According to 6.21 there exists a simple linearly ordered group G such that $\operatorname{card} G > \beta'$. We put $\varphi(\beta) = G$. Then φ is injective.

Next, for each $\beta \in M$ we set $\psi(\beta) = \{\varphi(\beta)\}^*$.

Proposition 6.24. *Let* $\beta_1, \beta_2 \in M, \beta_1 < \beta_2$. *Then* $\psi(\beta_1) \neq \psi(\beta_2)$.

Proof. Assume that $\psi(\beta_1) = \psi(\beta_2)$. Then $G_{\beta(1)} \in \psi(\beta_1)$ implies that $G_{\beta(1)} \in \psi(\beta_2)$. There exists $0 \neq x \in G_{\beta(1)}$. In view of 6.23 there exists a subgroup H of $G_{\beta(1)}$ such that $x \in H$ and H is isomorphic to $G_{\beta(2)}$. Hence $\operatorname{card} H = \operatorname{card} G_{\beta(2)} > \operatorname{card} G_{\beta(1)}$, which is a contradiction. □

References

[1] *C.G. Chehata*: An algebraically simple ordered group. Proc. London Math. Soc. *2* (1952), 183–197.

[2] *C.G. Chehata, R. Wiegandt*: Radical theory for fully ordered groups. Mathematica – Rév. d'Anal. Numér. Théor. Approx. *20* (1979), no. 43, 143–157.

[3] *V Dlab*: On a family of simple ordered groups. J. Austral. Math. Soc. *8* (1968), 591–608.

[4] *B.J. Gardner*: Some aspects of radical theory for fully ordered groups. Comment. Math. Univ. Carol. *26* (1985), 821–837.

[5] *D. Gluschankof*: Cyclic ordered groups and MV-algebras. Czechoslovak Math. J. *43(118)* (1993), 249–263.

[6] *G. Grätzer*: Universal Algebra. Princeton, 1968.

[7] *E. Halušková*: Direct limits of monounary algebras (to be submitted).

[8] *G. Higman, A.H. Stone*: On inverse systems with trivial limits. J. London Math. Soc. *29* (1954), 233–236.

[9] *P.D. Hill*: Relation of a direct limit group to associated vector group. Pacif. J. Math. *10* (1960), 1309–1312.

[10] *P.D. Hill*: Note on a direct limit group. Amer. Math. Monthly *67* (1960), 998–1000.

[11] *P.D. Hill*: Limits of sequences of finitely generated abelain groups. Proc. Amer. Math. Soc. *12* (1961), 946–950.

[12] *J. Jakubík*: On the lattice of radical classes of linearly ordered groups. Studia scient. math. Hungar *19* (1981), 76–86.

[13] *J. Jakubík*: Retracts of abelian cyclically ordered groups. Archivum mathem. *25* (1989), 13–18.

[14] *J. Jakubík, G. Pringerová*: Representations of cyclically ordered groups. Časopis. pěst. matem. *113* (1988), 184–196.

[15] *J. Jakubík, G. Pringerová*: Radical classes of cyclically ordered groups. Math. Slovaca *38* (1988), 255–268.

[16] *A.G. Kurosh*: Group theory, third edition. Moskva, 1967. (In Russian.)

[17] *V.M. Kopytov*: Lattice ordered groups. Moskva, 1984. (In Russian.)

[18] *L. Rieger*: On ordered and cyclically ordered groups, I, II, III. Věstník Král. čes. spol. nauk (1946), 1–31 (In Czech.); (1947), 1–33; (1948), 1–26.

[19] *S. Swierczkowski*: On cyclically ordered groups. Fund. Math. *47* (1959), 161–166.

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