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$\mathcal{J}$-CLASSES IN THE DIRECT PRODUCT OF TWO SEMIGROUPS

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In [3] the mutual relation between a principal two-sided ideal $J(a,b)$ in the direct product of two semigroups and the direct product of two principal two-sided ideals $J(a) \times J(b)$ is investigated. In particular, some conditions are given under which $J(a,b) = J(a) \times J(b)$ holds.

The aim of the present paper is to study the mutual relation between a $\mathcal{J}$-class $J_{(a,b)}$ in $S_1 \times S_2$ and the direct product $J_a \times J_b$ of two $\mathcal{J}$-classes both in the general case and in the special case of maximal $\mathcal{J}$-classes. Finally, we give conditions under which $J_{(a,b)} = J_a \times J_b$.

All notions and notations which are not defined are meant as in [1].

1.

Theorem 1. *Let J_a be a $\mathcal{J}$-class in S_1, J_b a $\mathcal{J}$-class in S_2, $J_{(a,b)}$ a $\mathcal{J}$-class in $S_1 \times S_2$. Then*

1. $J_{(a,b)} \subseteq J_a \times J_b$;

2. if $J_{(a,b)} \subset J_a \times J_b$, then $J_a \times J_b$ is the union of at least two $\mathcal{J}$-classes in $S_1 \times S_2$.

Proof. 1. Let $(u,v) \in J_{(a,b)}$, then $J(u,v) = J(a,b)$. If $(u,v) = (a,b)$, then $J(u) = J(a)$ in S_1 and $J(v) = J(b)$ in S_2. If $(u,v) \neq (a,b)$, then $(u,v) \in [(S_1a \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)]$ and $(a,b) \in [(S_1u \times S_2v) \cup (uS_1 \times vS_2) \cup (S_1uS_1 \times S_2vS_2)]$. This implies that (u,v) belongs to at least one of the summands and (a,b) belongs to at least one of the summands. If e.g. $(u,v) \in (S_1a \times S_2b)$ and $(a,b) \in (S_1u \times S_2v)$ then $u \in S_1a$, $v \in S_2b$ and $a \in S_1u$, $b \in S_2v$. Hence we have $J(u) \subseteq J(a)$ and $J(a) \subseteq J(u)$, hence $J(a) = J(u)$, so $u \in J_a$. Similarly, we can show that $v \in J_b$, therefore $(u,v) \in J_a \times J_b$.

2. Let $(u,v) \in J_a \times J_b - J_{(a,b)}$. Then $u \in J_a$, $v \in J_b$, hence $J_u = J_a$ in S_1 and $J_v = J_b$ in S_2. Then $J_u \times J_v = J_a \times J_b$ and by 1, $J_{(u,v)} \subseteq J_u \times J_v = J_a \times J_b$. □

Corollary. *If $J_a = \{a\}$ in S_1, $J_b = \{b\}$ in S_2, then $J_{(a,b)} = J_a \times J_b$ in $S_1 \times S_2$.*

Definition 1 ([7]). A nonempty subset M of a semigroup S is said to be a two-sided antiideal of S, if $M \cap \{SM, MS, SMS\} = \emptyset$.

Theorem 2. *If $(a,b) \in S_1 \times S_2$ is a one-element two-sided antiideal in $S_1 \times S_2$, then $J_{(a,b)} = \{(a,b)\}$.*

Proof. Let $(a,b) \notin \{(S_1a \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)\}$. If $|J_{(a,b)}| > 1$, then there is at least one element $(u,v) \in J_{(a,b)}$ such that $(u,v)\#(a,b)$ and $J(u,v) = J(a,b)$, hence

$$\begin{aligned}&(u,v) \cup (S_1u \times S_2v) \cup (uS_1 \times vS_2) \cup (S_1uS_1 \times S_2vS_2)\\ &= (a,b) \cup (S_1a \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2).\end{aligned}$$

Consequently, $(u,v) \in \{(S_1a \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)\}$ and $(a,b) \in \{(S_1u \times S_2v) \cup (uS_1 \times vS_2) \cup (S_1uS_1 \times S_2vS_2)\}$.

If e.g. $(u,v) \in (S_1a \times S_2b)$ and $(a,b) \in (S_1u \times S_2v)$, then $(S_1u \times S_2v) \subseteq (S_1a \times S_2b)$, $(uS_1 \times vS_2) \subseteq (S_1aS_1 \times S_2bS_2)$,

$$(S_1uS_1 \times S_2vS_2) \subseteq (S_1aS_1 \times S_2bS_2) \tag{1}$$

and $(S_1a \times S_2b) \subseteq (S_1u \times S_2v)$, $(aS_1 \times bS_2) \subseteq (S_1uS_1 \times S_2vS_2)$,

$$(S_1aS_1 \times S_2bS_2) \subseteq (S_1uS_1 \times S_2vS_2). \tag{2}$$

From (1) we obtain

$$\begin{aligned}J(u,v) &= (u,v) \cup (S_1u \times S_2v) \cup (uS_1 \times vS_2) \cup (S_1uS_1 \times S_2vS_2)\\ &\subseteq (S_1a \times S_2b) \cup (S_1aS_1 \times S_2bS_2) \subseteq J(a,b).\end{aligned}$$

However, $J(a,b) = J(u,v)$, therefore $(a,b) \in J(u,v) \subseteq (S_1a \times S_2b) \cup (S_1aS_1 \times S_2bS_2)$, hence

$$(a,b) \in \{(S_1a \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)\},$$

which contradicts the hypothesis.

In the case that $(u,v) \in (aS_1 \times bS_2)$ and $(a,b) \in (uS_1 \times vS_2)$, or any other possibility, we proceed analogously. □

Corollary. *If $J_{(a,b)} = J_a \times J_b$, then either*
1. *$J_a = \{a\}$ and $J_b = \{b\}$ or*
2. *no element in $J_a \times J_b$ is a two-sided intiideal in $S_1 \times S_2$.*

The following example indicates that 2 in Corollary represents only a necessary condition.

E x a m p l e 1. Let $S_1 = \{a_1, a_2, a_3, a_4\}$, $S_2 = \{b_1, b_2, b_3, b_4\}$ be two semigroups, in which associative binary operations are given by means of multiplicative tables:

	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_2	a_2
a_3	a_1	a_2	a_3	a_4
a_4	a_1	a_2	a_3	a_4

	b_1	b_2	b_3	b_4
b_1	b_1	b_1	b_1	b_1
b_2	b_1	b_1	b_1	b_1
b_3	b_1	b_2	b_3	b_4
b_4	b_1	b_2	b_3	b_4

$J_{a_3} = \{a_3, a_4\}$ in S_1, $J_{b_2} = \{b_2\}$ in S_2. Then $J_{a_3} \times J_{b_2} = \{(a_3, b_2), (a_4, b_2)\}$.

We have $(a_3, b_2) \in (S_1 a_3 \times S_2 b_2)$, so (a_3, b_2) is not a two-sided antiideal in $S_1 \times S_2$. Similarly $(a_4, b_2) \in (S_1 a_4 \times S_2 b_2)$, so (a_4, b_2) is not a two-sided antiideal in $S_1 \times S_2$. Hence no element in $J_{a_3} \times J_{b_2}$ is a two-sided antiideal in $S_1 \times S_2$; however,

$$\begin{aligned} J(a_3, b_2) &= (S_1 a_3 \times S_2 b_2) \cup (S_1 a_3 S_1 \times S_2 b_2 S_2) \\ &= \{a_1, a_2, a_3\} \times \{b_1, b_2\} \cup \{a_1, a_2, a_3, a_4\} \times \{b_1\} \\ &= \{(a_1, b_1), (a_2, b_1), (a_3, b_1), (a_4, b_1), (a_1, b_2), (a_2, b_2), (a_3, b_2)\}, \\ J(a_4, b_2) &= (S_1 a_4 \times S_2 b_2) \cup (S_1 a_4 S_1 \times S_2 b_2 S_2) \\ &= \{a_1, a_2, a_4\} \times \{b_1, b_2\} \cup \{a_1, a_2, a_3, a_4\} \times \{b_1\} \\ &= \{(a_1, b_1), (a_2, b_1), (a_3, b_1), (a_4, b_1), (a_1, b_2), (a_2, b_2), (a_4, b_2)\}. \end{aligned}$$

We have $J(a_3, b_2) \# J(a_4, b_2)$, $(a_3, b_2) \notin J(a_4, b_2)$, $(a_4, b_2) \notin J(a_3, b_2)$. So $J_{(a_3, b_2)} = \{(a_3, b_2)\}$, $J_{(a_4, b_2)} = \{(a_4, b_2)\}$, but none of them is a two-sided antiideal in $S_1 \times S_2$.

Lemma 1. *Let $J_a \times J_b$ contain more than one element. If (a, b) is in any two components of $\{(S_1 a \times S_2 b), (a S_1 \times b S_2), (S_1 a S_1 \times S_2 b S_2)\}$, then $(a, b) \in (S_1 a S_1 \times S_2 b S_2)$.*

P r o o f. It is sufficient to show that $(a, b) \in (S_1 a \times S_2 b) \cap (a S_1 \times b S_2)$ implies $(a, b) \in (S_1 a S_1 \times S_2 b S_2)$. Let $(a \in S_1 a \wedge a \in a S_1)$ and $(b \in S_2 b \wedge b \in b S_2)$. As $a \in S_1 a$, we have $a S_1 \subseteq S_1 a S_1$ and because $a \in a S_1 \subseteq S_1 a S_1$, then $a \in S_1 a S_1$. Similarly we can show that $b \in (S_2 b S_2)$, so $(a, b) \in (S_1 a S_1 \times S_2 b S_2)$. □

Theorem 3. *If $(a, b) \in (S_1 a S_1 \times S_2 b S_2)$, then $J_{(a,b)} = J_a \times J_b$.*

P r o o f. If $(a, b) \in (S_1 a S_1 \times S_2 b S_2)$, then $J(a) = S_1 a S_1$, $J(b) = S_2 b S_2$. $J_a \subseteq J(a)$ in S_1, $J_b \subseteq J(b)$ in S_2. If $(c, d) \in J_a \times J_b$ then $(c, d) \in (S_1 a S_1 \times S_2 b S_2)$. It implies $J(c, d) \subseteq (S_1 a S_1 \times S_2 b S_2) \subseteq J(a, b)$. Since it can be verified that $S_1 c S_1 = S_1 a S_1$ and $S_2 d S_2 = S_2 b S_2$, then $J(a) \times J(b) = (S_1 a S_1 \times S_2 b S_2) = (S_1 c S_1 \times S_2 d S_2) = J(c) \times J(d)$, so $(a, b) \in (S_1 c S_1 \times S_2 d S_2)$. Hence we have $J(a, b) \subseteq (S_1 c S_1 \times S_2 d S_2) \subseteq J(c, d)$.

The last relation with the previous one give $J(c,d) = J(a.b)$. We have proved that $J_a \times J_b \subseteq J_{(a,b)}$ and because in general $J_{(a,b)} \subseteq J_a \times J_b$ by Theorem 1, we conclude

$$J_{(a,b)} = J_a \times J_b.$$

□

It remains to find conditions under which $J_{(a,b)} = J_a \times J_b$ in the case that $J_a \times J_b$ contains more than one element and either

(i) $(a,b) \in (S_1 a \times S_2 b) \wedge (a,b) \notin [(aS_1 \times bS_2) \cup (S_1 a S_1 \times S_2 b S_2)]$

or

(ii) $(a,b) \in (aS_1 \times bS_2) \wedge (a,b) \notin [(S_1 a \times S_2 b) \cup (S_1 a S_1 \times S_2 b S_2)]$.

Lemma 2. *Let $J_a \times J_b$ contain more than one element, $(a,b) \in (S_1 a \times S_2 b) \wedge (a,b) \notin [(aS_1 \times bS_2) \cup (S_1 a S_1 \times S_2 b S_2)]$. Let $(a,b) \in J_a \times J_b$, $(a_1,b) \in J_a \times J_b$, $J_{(a,b)} \neq J_{(a_1,b)}$. Then neither $J(a_1,b) \subset J(a,b)$ nor $J(a,b) \subset J(a_1,b)$.*

P r o o f. Suppose that $J(a_1,b) \subset J(a,b)$. We will show that $(a,b) \notin J(a_1,b)$. If $(a,b) \in J(a_1,b)$, then $J(a,b) \subseteq J(a_1,b)$. The last relation with our assumption give $J(a_1,b) = J(a,b)$, which contradicts the hypothesis, hence $(a,b) \notin J(a_1,b)$, so $(a,b) \notin [(S_1 a_1 \times S_2 b) \cup (S_1 a_1 S_1 \times S_2 b S_2)]$. Consequently, $(a,b) \notin (S_1 a_1 \times S_2 b) \wedge (a,b) \notin (S_1 a_1 S_1 \times S_2 b S_2)$. It implies $a \notin S_1 a_1$, since $b \in S_2 b$. From the assumption of Lemma 2 we have: I. $(a,b) \notin (S_1 a S_1 \times S_2 b S_2)$, and from the relation above we have: II. $(a,b) \notin (S_1 a_1 S_1 \times S_2 b S_2)$. From I and II we get the following possibilities:

I. 1. $a \notin S_1 a S_1 \wedge b \notin S_2 b S_2$,	II. 1′. $a \notin S_1 a_1 S_1 \wedge b \notin S_2 b S_2$,
2. $a \in S_1 a S_1 \wedge b \notin S_2 b S_2$,	2′. $a \in S_1 a_1 S_1 \wedge b \notin S_2 b S_2$,
3. $a \notin S_1 a S_1 \wedge b \in S_2 b S_2$,	3′. $a \notin S_1 a_1 S_1 \wedge b \in S_2 b S_2$.

Since we have supposed $J(a_1,b) \subset J(a,b)$, we have $(a_1,b) \in [(S_1 a \times S_2 b) \cup (S_1 a S_1 \times S_2 b S_2)]$, so $(a_1 b)$ belongs to at least one of the two summands. In both cases we get $J(a_1) \subseteq J(a)$. We shall show that if we combine any possibility of I with any possibility of II, then we find that some of them cannot occur and in the remaining cases $J(a_1) \subset J(a)$ holds.

$(1,1')$: $a \notin S_1 a S_1 \wedge a \notin S_1 a_1 S_1$. Then $a \notin S_1 a_1 \wedge a \notin S_1 a_1 S_1$ implies $a \notin (S_1 a_1 \cup S_1 a_1 S_1) = J(a_1)$, therefore $J(a_1) \subset J(a)$.

$(1,2')$: $a \notin S_1 a S_1 \wedge a \in S_1 a_1 S_1$. This cannot occur, since $a \in S_1 a_1 S_1$ implies $a \in S_1 a S_1$, and this contradicts the hypothesis.

$(1,3')$: $a \notin S_1 a S_1 \wedge a \notin S_1 a_1 S_1$. Then similarly as in $(1,1')$ we get $J(a_1) \subset J(a)$.

$(2,1')$: $a \in S_1 a S_1 \wedge a \notin S_1 a_1 S_1$. Then $a \notin S_1 a_1 \wedge a \notin S_1 a_1 S_1$ implies $J(a_1) \subset J(a)$.

$(2.2')$: $a \in S_1aS_1 \wedge a \in S_1a_1S_1$. It implies $a_1 \in S_1a_1S_1 \wedge a \in S_1a_1S_1$. Then $S_1aS_1 = S_1a_1S_1$ and from $S_1a_1 \subset S_1a$ (since $J(a_1) \subseteq J(a)$ and $a \notin S_1a_1$) we get $S_1a_1 \cup S_1a_1S_1 \subset S_1a \cup S_1aS_1$, hence $J(a_1) \subset J(a)$.

$(2,3')$: $a \in S_1aS_1 \wedge a \notin S_1a_1S_1$. Then similarly as in $(2,1')$, $J(a_1) \subset J(a)$.

$(3,1')$: $a \notin S_1aS_1 \wedge a \notin S_1a_1S_1$. Then similarly as in $(1,1')$, $J(a_1) \subset J(a)$.

$(3,2')$: $a \notin S_1aS_1 \wedge a \in S_1a_1S_1$. Similarly as in $(1,2')$ this cannot occur.

$(3,3')$: $a \notin S_1aS_1 \wedge a \notin S_1a_1S_1$. Then from $S_1a_1 \subset S_1a$ and from $J(a_1) \subseteq J(a)$ we get $J(a_1) \subset J(a)$.

Therefore, in all the cases that may occur we have $J(a_1) \subset J(a)$, but this is a contradiction because $a \in J_a$, $a_1 \in J_a$, so $J(a_1) = J(a)$. Hence our assumption $J(a_1, b) \subset J(a, b)$ cannot be fulfilled. In a similar way we could prove that $J(a, b) \subset J(a_1, b)$ cannot hold. □

Lemma 3. *Let $J_a \times J_b$ contain more than one element. Let $(a, b) \in (S_1a \times S_2b) \wedge (a, b) \notin [(aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)]$. Then $J_a \times J_b$ is the union of at least two different $\mathcal{J}$-classes iff at least for one of J_a, J_b the following holds: $S_1J_1 \subset S_1J_a$, $S_2J_2 \subset S_2J_b$ for every proper subset $J_1 \subset J_a$, $J_2 \subset J_b$.*

Proof. a. Let $J_a \times J_b$ be the union of at least two $\mathcal{J}$-classes. We will show that at least for one of the $\mathcal{J}$-classes J_a, J_b the inclusion $S_1J_1 \subset S_1J_a$, $S_2J_2 \subset S_2J_b$ holds, where J_1 is any proper subset of J_a, J_2 is any proper subset of J_b. Because $|J_1 \times J_b| > 1$, the following cases may occur: 1. $|J_a| > 1 \wedge |J_b| = 1$, 2. $|J_a| = 1 \wedge |J_b| > 1$, 3. $|J_a| > 1 \wedge |J_b| > 1$.

If 1 holds, then the $\mathcal{J}$-classes in $J_a \times J_b$ are of the form $J_{(a_i,b)}$, if 2 holds, then the $\mathcal{J}$-classes in $J_a \times J_b$ are of the form $J_{(a,b_i)}$, $i \in I$. If 3 holds, then we get the following possibilities:

(a) the $\mathcal{J}$-classes are of the form $J_{(a_i,b)}$, if $S_2b = S_2J_b$ and the case 1 occurs;

(b) the $\mathcal{J}$-classes are of the form $J_{(a,b_i)}$, if $S_1a = S_1J_a$ and the case 2 occurs;

(c) $S_1a \subset S_1J_a \wedge S_2b \subset S_2J_b$. Then there are at least two $\mathcal{J}$-classes of the form $J_{(a_i,b)}$ and at least two $\mathcal{J}$-classes of the form $J_{(a,b_i)}$, $i \in I$.

Let $J_{(a,b)}$, $J_{(a_1,b)}$ be any two $\mathcal{J}$-classes for $a \# a_1$, $J(a,b) \# J(a_1,b)$. Then $J(a,b) = (S_1a \times S_2b) \cup (S_1aS_1 \times S_2bS_2)$, $(a,b) \in (S_1a \times S_2b) \wedge (a,b) \notin [(aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)]$. Further, $J(a_1,b) = (S_1a_1 \times S_2b) \cup (S_1a_1S_1 \times S_2bS_2)$, $(a_1b) \in (S_1a_1 \times S_2b) \wedge (a_1,b) \notin [(a_1S_1 \times bS_2) \cup (S_1a_1S_1 \times S_2bS_2)]$.

We claim that $(a_1,b) \notin J(a,b)$. If $(a_1,b) \in J(a,b)$, then $J(a_1,b) \subseteq J(a,b)$. There are only two possibilities: either $J(a_1,b) = J(a,b)$, or $J(a_1,b) \subset J(a,b)$. The first possibility contradicts the fact $J_{(a_1,b)} \# J_{(a,b)}$. If the other possibility occurs, then by Lemma 2 it leads to a contradiction. Therefore, $(a_1,b) \notin [(S_1a \times S_2b) \cup (S_1aS_1 \times S_2bS_2)]$. So $(a_1,b) \notin (S_1a \times S_2b)$, hence $a_1 \notin S_1a$, as $b \in S_2b$. Similarly we can show that $(a,b) \notin J(a_1,b)$ and, moreover, $a \notin S_1a_1$.

Let $J_1 \subset J_a$ be any proper subset. Hence there exists at least one $a_i \in J_a$ such that $a_i \notin J_1$. Then $S_1 J_1 \subseteq S_1 J_a$. There are only two possibilities: either $S_1 J_1 = S_1 J_a$, or $S_1 J_1 \subset S_1 J_a$. If $S_1 J_1 = S_1 J_a$, then from the relation $c \in S_1 c$ for any $c \in J_a$ we get $J_a \subseteq S_1 J_a = S_1 J_1$. So any element of J_a is contained in $S_1 a_j$ for some $a_j \in J_1$, but this is a contradiction with the fact $a_i \notin S_1 a_j$ for $a_i \# a_j$. Therefore, the other possibilities occurs, namely $S_1 J_1 \subset S_1 J_a$, for any proper subset $J_1 \subset J_a$.

b. As $J_a \times J_b$ contains more than one element, at least one of J_a, J_b contains more than one element. Let J_a contain more than one element and let $S_1 J_1 \subset S_1 J_a$ for every proper subset $J_1 \subset J_a$. Denote $S_1 J_a = L$. Then for any $x \in L$ there is $a_1 \in J_a$ such that $x \in S_1 a_1$. By the hypothesis $S_1 a \subset S_1 J_a = L$. Hence there is $y \in L$ such that $y \notin S_1 a$, but $y \in S_1 c$ for some $c \in J_a$, $c \# a$. We shall show that $c \notin S_1 a$. If $c \in S_1 a$, then $S_1 c \subseteq S_1 a$ and because $y \in S_1 c \subseteq S_1 a$, so $y \in S_1 a$ and this is a contradiction. We also show that $a \notin S_1 c$. If $a \in S_1 c$, then $S_1 a \subseteq S_1 c$. Hence we have $L = S_1 J_a = S_1 J_1$ where $J_1 = J_a - \{a\}$, but this is a contradiction with our assumption that $S_1 J_1 \subset S_1 J_a = L$ for every proper subset $J_1 \subset J_a$, so $c \notin S_1 a$, $a \notin S_1 c$.

Consider principal two-sided ideals $J(a, b)$ and $J(c, b)$ in $S_1 \times S_2$ with $a \in J_a$, $c \in J_a$. $J(a, b) = (S_1 a \times S_2 b) \cup (S_1 a S_1 \times S_2 b S_2)$, $J(c, b) = (S_1 c \times S_2 b) \cup (S_1 c S_1 \times S_2 b S_2)$. We show that $J(a, b) \# J(c, b)$. Indeed, $(a, b) \in J(a, b)$, but $(a, b) \notin J(c, b)$, since $(a, b) \notin (S_1 c \times S_2 b)$ as $a \notin S_1 c$. If $(a, b) \in (S_1 c S_1 \times S_2 b S_2)$, then $a \in S_1 c S_1$, $b \in S_2 b S_2$. Consequently $a \in S_1 c S_1$ implies $a \in S_1 a S_1$, hence $(a, b) \in (S_1 a S_1 \times S_2 b S_2)$ and this contradicts the fact that $(a, b) \notin (a S_1 \times b S_2) \cup (S_1 a S_1 \times S_2 b S_2)$, which is contained in Lemma 3. Similarly $(c, b) \in J(c, b)$, but $(c, b) \notin J(a, b)$, since $(c, b) \notin (S_1 a \times S_2 b)$ because $c \notin S_1 a$, $(c, b) \notin (S_1 a S_1 \times S_2 b S_2)$, because if $(c, b) \in (S_1 a S_1 \times S_2 b S_2)$,then $c \in S_1 a S_1$, $b \in S_2 b S_2$. However, $c \in S_1 a S_1$ implies $a \in S_1 a S_1$ and then $(a, b) \in (S_1 a S_1 \times S_2 b S_2)$ and it is a contradiction again. Therefore, for $(a, b) \in J_a \times J_b$, $(c, b) \in J_a \times J_b$, $(a, b) \# (c, b)$ we get $J(a, b) \# J(c, b)$, so $J_{(a,b)} \subset J_a \times J_b$, $J_{(c,b)} \subset J_a \times J_b$. Hence, $J_a \times J_b$ is the union of at least two $\mathcal{J}$-classes. □

Lemma 4. *Let $J_a \times J_b$ contain more then one element. Let $(a, b) \in (a S_1 \times b S_2) \wedge (a, b) \notin [(S_1 a \times S_2 b) \cup (S_1 a S_1 \times S_2 b S_2)]$. Then $J_a \times J_b$ is the union of at least two different $\mathcal{J}$-classes iff at least for one of J_a, J_b the following holds: $J_1 S_1 \subset J_a S_1$, $J_2 S_2 \subset J_b S_2$ for any proper subset $J_1 \subset J_a$, $J_2 \subset J_b$, respectively.*

Proof. The proof is similar to that of Lemma 3. □

From Lemma 3 we get

Theorem 4. *Let $J_a \times J_b$ contain more than one element. Let $(a, b) \in (S_1 a \times S_2 b) \wedge (a, b) \notin [(a S_1 \times b S_2) \cup (S_1 a S_1 \times S_2 b S_2)]$. Then $J_a \times J_b = J_{(a,b)}$ iff $S_1 a = S_1 J_a$ and $S_2 b = S_2 J_b$.*

Analogously from Lemma 4 we can obtain

Theorem 5. *Let $J_a \times J_b$ contain more than one element. Let $(a,b) \in (aS_1 \times bS_2) \wedge (a,b) \notin [(S_1 a \times S_2 b) \cup (S_1 a S_1 \times S_2 b S_2)]$. Then $J_a \times J_b = J_{(a,b)}$ iff $aS_1 = J_a S_1$ and $bS_2 = J_b S_2$.*

Remark 2. It is known (see [2]) that in the case of $\mathscr{L}$-classes ($\mathscr{R}$-classes) the situation is as follows: If $|L_a \times L_b| > 1$, then $L_a \times L_b$ is the union of at least two $\mathscr{L}$-classes iff $|L_a| > 1$ and $L_b = \{b\}$, $b \notin S_2 b$, or $L_a = \{a\}$, $a \notin S_1 a$ and $|L_b| > 1$ and any $\mathscr{L}$-class in $L_a \times L_b$ is one-element. If $|L_a| > 1$ and $|L_b| > 1$ then $L_a \times L_b = L_{(a,b)}$.

In the cases of $\mathscr{J}$-classes the situation is different, as we can see from the following example.

Example 2. Let $S_1 = \{a_1, a_2, a_3, a_4\}$ and let an associative binary operation be given by means of the following table:

	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_2	a_2
a_3	a_1	a_2	a_3	a_4
a_4	a_1	a_2	a_3	a_4

$J_{a_3} = \{a_3, a_4\}$, $S_1 a_3 = \{a_1, a_2, a_3\}$, $a_3 S_1 = S_1$, $S_1 a_3 S_1 = S_1$.

$S_2 = A \cup B \cup \{0\}$, where A is the infinite cyclic group generated by an element $\{a\}$, $B = \{\dots b_{-2}, b_{-1}, b_0, b_1, b_2, \dots\}$, $\{0\}$ is zero in S_2. An associative binary operation is defined as follows: $a^i \cdot b_j = b_{i+j}$, $b_j \cdot a^i = b_i \cdot b_j = 0$.

$$\begin{aligned} &S_2 a^i = A \cup \{0\},\ a^i S_2 = S_2,\ S_2 a^i S_2 = S_2,\ J(a^i) = S_2,\ J_{a^i} = A. \\ &S_2 b_i = B \cup \{0\},\ b_i S_2 = 0,\ S_2 b_i S_2 = 0,\ J(b_i) = B \cup \{0\}, \\ &\ J_{b_i} = B,\ J(0) = \{0\},\ J_0 = \{0\}. \end{aligned}$$

Let us consider the direct product $S_1 \times S_2$, J_{a_3} in S_1, J_{b_i} in S_2. Then $J_{a_3} \times J_{b_i} = \{a_3, a_4\} \times B$. Consider the principal two-sided ideals $J(a_3, b_i)$ and $J(a_4, b_i)$ in $S_1 \times S_2$. We have

$$\begin{aligned} J(a_3, b_i) &= (a_3, b_i) \cup (S_1 a_3 \times S_2 b_i) \cup (a_3 S_1 \times b_i S_2) \cup (S_1 a_3 S_1 \times S_2 b_i S_2) \\ &= (a_3, b_i) \cup \{a_1, a_2, a_3\} \times \{B \cup 0\} \cup (S_1 \times \{0\}) \cup (S_1 \times \{0\}) \end{aligned}$$

$$
\begin{aligned}
&= \{a_1, a_2, a_3\} \times \{B \cup \{0\}\} \cup (S_1 \times \{0\}) \\
&= \{a_1, a_2, a_3\} \times \{B \cup \{0\}\} \cup \{(a_4, 0)\} \\
&= \{a_1, a_2, a_3\} \times B \cup (S_1 \times \{0\}), \\
J(a_4, b_i) &= (a_4, b_i) \cup (S_1 a_4 \times S_2 b_i) \cup (a_4 S_1 \times b_i S_2) \cup (S_1 a_4 S_1 \times S_2 b_i S_2) \\
&= (a_4, b_i) \cup \{a_1, a_2, a_4\} \times \{B \cup \{0\}\} \cup (S_1 \times \{0\}) \\
&= \{a_1, a_2, a_4\} \times \{B \cup \{0\}\} \cup (S_1 \times \{0\}) \\
&= \{a_1, a_2, a_4\} \times \{B\} \cup \{(a_3, 0)\} \\
&= \{a_1, a_2, a_4\} \times \{B\} \cup (S_1 \times \{0\}).
\end{aligned}
$$

It is evident that $J(a_3, b_i) \neq J(a_4, b_i)$, because $J(a_3, b_i)$ contains elements of the form $\{(a_3, b_i)\}$ that do not belong to $J(a_4, b_i)$, and conversely $J(a_4, b_i)$ contains elements of the form $\{(a_4, b_i)\}$ that do not belong to $J(a_3, b_i)$. Hence $J_{a_3} \times J_{b_i} = \{a_3, a_4\} \times \{B\}$ is decomposed into two $\mathcal{J}$-classes, namely $J_{(a_3,b_i)}$, $J_{(a_4,b_i)}$, and each of them contains infinite number of elements, but none of them is a two-sided antiideal in $S_1 \times S_2$.

2.

In this part we shall investigate the mutual relation between $J_{(a,b)}$ and $J_a \times J_b$ in $S_1 \times S_2$ provided J_a is a maximal $\mathcal{J}$-class in S_1, J_b is a maximal $\mathcal{J}$-class in S_2.

Remark 3. If J_a is a maximal $\mathcal{J}$-class in S_1, then $M_a = S - J_a$ is a maximal two-sided ideal in S and conversely ([4]).

For the factor semigroup S/M_a exactly one of the following two possibilities occurs ([6]):

1. $(S/M_a)^2 = \bar{0}$ and S/M_a is a two-element semigroup, $J_a = \{a\}$, $a \in S - S^2$;

2. $S/M_a = \bar{S}$ is a 0-simple semigroup and for every nonzero element $\bar{a} \in \bar{S}$ we have $\bar{S}\bar{a}\bar{S} = \bar{S}$, hence $a \in SaS$ for $a \in J_a = S - M_a$.

Lemma 5 ([6]). *Let J_a be a maximal $\mathcal{J}$-class in a semigroup S and $|J_a| > 1$. Then $a \in SaS$.*

Theorem 6. *Let J_a be a maximal $\mathcal{J}$-class in S_1, J_b a maximal $\mathcal{J}$-class in S_2, and let $|J_a| > 1$ and $|J_b| > 1$. Then*

$$J_{(a,b)} = J_a \times J_b.$$

Proof. The statesment follows from Lemma 5 and Theorem 3. □

Corollary. *Let J_a be a maximal $\mathcal{J}$-class in S_1, J_b a maximal $\mathcal{J}$-class in S_2. If $J_a \times J_b$ is the union of at least two $\mathcal{J}$-classes in $S_1 \times S_2$, then either*

1. $|J_a| > 1$ *and* $J_b = \{b\}$, *or*
2. $J_a = a$ *and* $|J_b| > 1$.

Lemma 6. *Let J_a be a maximal $\mathcal{J}$-class in S_1, J_b a maximal $\mathcal{J}$-class in S_2 and let $J_{(a_1,b_1)} \subset J_a \times J_b$, $J_{(a_2,b_2)} \subset J_a \times J_b$, $J_{(a_1,b_1)} \neq J_{(a_2,b_2)}$. Then either*

1. $|J_a| > 1$, $J_b = \{b\}$, $b \in S_2 - S_2^2$, *or*
2. $J_a = \{a\}$, $a \in S_1 - S_1^2$, $|J_b| > 1$ *and*

$J_{(a_1,b_1)}$, $J_{(a_2,b_2)}$ *are uncomparable.*

P r o o f. From the Corollary of Theorem 6 we get that either 1. $|J_a| > 1$ and $J_b = \{b\}$, or 2. $J_a = \{a\}$ and $|J_b| > 1$. Let 1 hold. Then $b_1 = b_2 = b$. As both J_a and J_b are maximal $\mathcal{J}$-classes, then, since $|J_a| > 1$, we have $a \in S_1 a S_1$. However, $J_b = \{b\}$, therefore there are only two possibilities:

(i) $b \in SbS$,

(ii) $b \in S_2 - S_2^2$ by Remark 3.

If $b \in S_2 b S_2$, then by Theorem 3 we have $J_{(a,b)} = J_a \times J_b$, a contradiction to the hypothesis, therefore $b \in S_2 - S_2^2$ holds. Hence $b \notin (S_2 b \cup b S_2 \cup S_2 b S_2)$. It remains to show that $J(a_1, b)$, $J(a_2, b)$ are uncomparable. We have $(a_1, b) \in J(a_1, b)$ but $(a_1, b) \notin J(a_2, b)$ since $(a_1, b) \neq (a_2, b)$ as $a_1 \neq a_2$, and $(a_1, b) \notin [(S_1 a_2 \times S_2 b) \cup (a_2 S_1 \times b S_2) \cup (S_1 a_2 S_1 \times S_2 b S_2)]$ since $b \notin (S_2 b \cup b S_2 \cup S_2 b S_2)$. Similarly $(a_2, b) \in J(a_2, b)$, but $(a_2, b) \notin J(a_1, b)$. □

Theorem 7. *Let J_a be a maximal $\mathcal{J}$-class in S_1, J_b a maximal $\mathcal{J}$-class in S_2. Then either*

1. $J_a \times J_b$ *is a maximal $\mathcal{J}$-class in $S_1 \times S_2$ or*
2. $J_a \times J_b$ *is the union of at least two maximal $\mathcal{J}$-classes in $S_1 \times S_2$.*

P r o o f. With regard to Lemma 3 it is sufficient to show that if $J_{(a_1,b_1)} \subseteq J_a \times J_b$, then $J(a_1, b_1)$ is not contained as a proper subset in any principal ideal of $S_1 \times S_2$.

Suppose that there exists such an element $(u, v) \in S_1 \times S_2 - J_a \times J_b$ that $(a_1, b_1) \subset J(u, v)$. Then

$$\begin{aligned}&(a_1, b_1) \cup (S_1 a_1 \times S_2 b_1) \cup (a_1 S_1 \times b_1 S_2) \cup (S_1 a_1 S_1 \times S_2 b_1 S_2)\\ &\subset (u, v) \cup (S_1 u \times S_2 v) \cup (u S_1 \times v S_2) \cup (S_1 u S_1 \times S_2 v S_2).\end{aligned}$$

Since $(a_1, b_1) \neq (u, v)$, then

$$(a_1, b_1) \in [(S_1 u \times S_1 b) \cup (u S_1 \times v S_2) \cup (S_1 u S_1 \times S_2 v S_2)].$$

If e.g. $(a_1, b_1) \in (S_1u \times S_2v)$, then $a_1 \in S_1u$ and $b_1 \in S_2v$. Hence $J(a_1) \subseteq J(u)$ in S_1 and $J(b_1) \subseteq J(v)$ in S_2. If both $J(a_1) = J(u)$ and $J(b_1) = J(v)$, then $u \in J_{a_1}$ and $v \in J_b$ and $(u,v) \in J_a \times J_b$, a contradiction. Therefore either $J(a_1) \subset J(u)$, or $J(b_1) \subset J(v)$. It means that either J_a in S_1 or J_b in S_2 is not a maximal $\mathcal{J}$-class and this contradicts the hypothesis. For the remaining possibilities $(a_1, b_1) \in (uS_1 \times vS_2)$, $(a_1, b_1) \in (S_1uS_1 \times S_2vS_2)$, we could proceed analogously. □

Corollary. *Let J_a be a maximal $\mathcal{J}$-class in S_1 and $|J_a| > 1$, $J_b = \{b\}$, $b \in S - S^2$ a maximal $\mathcal{J}$-class in S_2. Then $J_a \times J_b$ is the union of maximal $\mathcal{J}$-classes in $S_1 \times S_2$ and each of them is one-element of the form $J_{(a_i,b)} = \{(a_i, b)\}$, $a_i \in J_a$.*

Theorem 8. *Let $u \in S_1$ be any element, $b \in S_2 - S_2^2$ ($a \in S_1 - S_1^2$, $v \in S_2$ any element). Then $J_{(u,b)} = \{(u,b)\}$ ($J_{(a,v)} = \{(a,v)\}$) is a maximal $\mathcal{J}$-class in $S_1 \times S_2$.*

Proof. Let $u \in S_1$ be any element, $b \in S_2 - S_2^2$. Then $b \notin (S_2b \cup bS_2 \cup S_2bS_2)$, hence b is an antiideal in S_2. Then $(u, b) \in S_1 \times S_2$ is an antiideal in $S_1 \times S_2$ and by Theorem 2 we have $J_{(u,b)} = \{(u,b)\}$. To prove that $J_{(u,b)}$ is maximal in $S_1 \times S_2$, it is sufficient to show that (u, b) is undecomposable in $S_1 \times S_2$. As $u \in S_1$, $b \in S_2$, then $(u,b) \in (S_1 \times S_2)$. But $b \in S_2 - S_2^2$, so $b \notin S_2^2$, and therefore $(u,b) \notin (S_1^2 \times S_2^2) = (S_1 \times S_2)^2$. This implies $(u,b) \in (S_1 \times S_2) - (S_1 \times S_2)^2$, hence $J_{(u,b)} = \{(u,b)\}$ is maximal. □

Theorem 9. *Let $J_{(a,b)}$ be any maximal $\mathcal{J}$-class in $S_1 \times S_2$. Then either*

1. $J_{(a,b)} = J_a \times J_b$, where J_a is a maximal $\mathcal{J}$-class in S_1, J_b a maximal $\mathcal{J}$-class in S_2, or

2. $J_{(a,b)} = \{(a,b)\}$, where $a \in S_1$ is any element, $b \in S_2 - S_2^2$, or $a \in S_1 - S_1^2$ and $b \in S_2$ is any element.

Proof. As $J_{(a,b)}$ is a maximal $\mathcal{J}$-class in $S_1 \times S_2$, then $S_1 \times S_2 - J_{(a,b)} = M_\alpha$ is a maximal ideal in $S_1 \times S_2$ and for the factor-semigroup $(S_1 \times S_2)/M_\alpha$ either

(a) $(S_1 \times S_2)/M_\alpha$ is a 0-simple semigroup and for $(a,b) \in (S_1 \times S_2) - M_\alpha = J_{(a,b)}$ we have $(a,b) \in (S_1 \times S_2)(a,b)(S_1 \times S_2)$, or

(b) $[(S_1 \times S_2)/M_\alpha]^2 = \bar{0}$ and $(S_1 \times S_2)/M_\alpha$ is a two-elements zero semigroup.

In the case (a) $(a,b) \in (S_1aS_1 \times S_2bS_2)$, so $a \in S_1aS_1$ and $b \in S_2bS_2$. Then $J_{(a,b)} = J_a \times J_b$ by Theorem 3. It remains to show that J_a is maximal in S_1, J_b is maximal in S_2. If J_a is not a maximal $\mathcal{J}$-class in S_1, then there is $u \in S_1 - J_a$ such that $J(a) \subset J(u)$. Then $J(a) = S_1aS_1 \subset (u \cup S_1u \cup uS_1 \cup S_1uS_1)$. It implies that $a \in (S_1u \cup S_1uS_1)$. If e.g. $a \in S_1u$, then $S_1aS_1 \subseteq S_1uS_1$. Further, $J(a,b) = (S_1aS_1 \times S_2bS_2) \subseteq (u,b) \cup (S_1u \times S_2b) \cup (uS_1 \times bS_1) \cup (S_1uS_1 \times S_2bS_2) = J(u,b)$. Now there are two possibilities: either $J(a,b) = J(u,b)$, or $J(a,b) \subset J(u,b)$.

If $J(a,b) = J(u,b)$, then $(u,b) \in J_{(a,b)} = J_a \times J_b$, therefore $u \in J_a$, which means $J(u) = J(a)$, a contradiction to $J(a) \subset J(u)$.

If $J(a,b) \subset J(u,b)$, then we have a contradiction to the hypothesis. Therefore J_a is a maximal $\mathcal{J}$-class in S_1. Similarly we can show that J_b is a maximal $\mathcal{J}$-class in S_2.

In the case (b) $(S_1 \times S_2) - M_\alpha = J_{(a,b)} = \{(a,b)\}$ and the element (a,b) is undecomposable in $S_1 \times S_2$, so $(a,b) \in (S_1 \times S_2) - (S_1 \times S_2)^2$. It means $(a,b) \notin (S_1 \times S_2)^2 = (S_1^2 \times S_2^2)$. Hence either $a \notin S_1^2$, or $b \notin S_2^2$. Therefore the $\mathcal{J}$-class $J_{(a,b)} = \{(a,b)\}$ is of the form: $a \in S_1$ is any element, $b \in S_2 - S_2^2$ or $a \in S_1 - S_1^2$, $b \in S_2$ is any element. □

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