Nikolaos S. Papageorgiou

On the solution set of nonconvex subdifferential evolution inclusions

Czechoslovak Mathematical Journal, Vol. 44 (1994), No. 3, 481–500

Persistent URL: `http://dml.cz/dmlcz/128477`

ON THE SOLUTION SET OF NONCONVEX SUBDIFFERENTIAL EVOLUTION INCLUSIONS

NIKOLAOS S. PAPAGEORGIOU, Melbourne

(Received July 23, 1992)

1. INTRODUCTION

The purpose of this work is to investigate the nonemptiness and density properties of the solution set of the evolution inclusion

$$\left\{\begin{array}{c} -\dot{x}(t) \in \partial\varphi\big(t, x(t)\big) + \operatorname{ext} F\big(t, x(t)\big) \text{ a.e.,} \\ x(0) = x_0. \end{array}\right\} \tag{1}$$

Here $\operatorname{ext} F(t, x)$ stands for the set of extreme points of the orientor field $F(t, x)$. This evolution inclusion is important in control theory, in connection with the "bang-bang" principle.

First we show that under certain continuity hypothesis on the orientor field $F(t, \cdot)$, the solution set of the multivalued Cauchy problem (1) is nonempty. Subsequently by strengthening the hypothesis on $F(t, \cdot)$ to Hausdorff Lipschitz continuity, we show that the solution set of (1) is a dense, G_δ-set (i.e. a residual set) in the solution of

$$\left\{\begin{array}{c} -\dot{x}(t) \in \partial\varphi\big(t, x(t)\big) + F\big(t, x(t)\big) \text{ a.e.,} \\ x(0) = x_0. \end{array}\right\} \tag{2}$$

Our work here extends Theorem 4.1 of DeBlasi-Pianigiani [9], who studied differential inclusions in reflexive spaces, with no subdifferential term present and with the orientor field being compact (see hypothesis H', p. 486) or jointly continuous and satisfying a Lipschitz condition involving the Kuratowski measure of noncompactness (see hypothesis K, p. 488). Their hypotheses, preclude the applicability of their work to partial differential equations with multivalued terms and in particular to distributed parameter control systems. However, their techniques and methods are very interesting and have inspired our approach in this paper. Our density result extends theorems 5.1 and 5.2 of [21], where the orientor field was of special type and instead of ext $F(t,x)$ versus $F(t,x)$ that is considered here, we had $F(t,x)$ versus $\overline{\text{conv}}F(t,x)$ (i.e. a standard "relaxation theorem"). In the last section, we use our result from this paper to establish a "bang-bang" type property for a class of nonlinear parabolic control systems with control constraints.

2. Mathematical preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{nonempty, closed, (convex)}\}$$
$$\text{and} \quad P_{(w)k(c)}(X) = \{A \subseteq X\colon \text{nonempty, (weakly-)compact, (convex)}\}.$$

A multifunction (set-valued function), is said to be "measurable" if and only if for all $x \in X$, the $\mathbb{R}_+$-valued function $\omega \mapsto d\big(x, F(\omega)\big) = \inf\{\|x - z\| \colon z \in F(\omega)\}$ is measurable. Next let $\mu(\cdot)$ be a finite measure defined on (Ω, Σ). By $S_F^p 1 \leqslant p \leqslant \infty$, we will denote the set of measurable selectors of $F(\cdot)$, that belong in the Lebesgue-Bochner space $L^p(X)$; i.e. $S_F^p = \{f \in L^p(X)\colon f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$. In general, this set may be empty. However, using Aumann's selection theorem (see for example Wagner [24], Theorem 5.10), we can easily check that for a measurable multifunction $F\colon \Omega \to P_f(X)$, the set S_F^p is nonempty if and only if $\omega \mapsto \inf\{\|x\| \colon x \in F(\omega)\} \in L_+^p$.

Let $\varphi\colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We will say that $\varphi(\cdot)$ is proper if it is not identically $+\infty$. Assume that $\varphi(\cdot)$ is proper, convex and lower semicontinuous (l.s.c.) (usually this family of $\overline{\mathbb{R}}$-valued functions is denoted by $\Gamma_0(X)$). By $\operatorname{dom}\varphi$, we will denote the effective domain of $\varphi(\cdot)$; i.e. $\operatorname{dom}\varphi = \{x \in X : \varphi(x) < +\infty\}$. The subdifferential of $\varphi(\cdot)$ at x, is the set $\partial\varphi(x) = \{x^* \in X^* \colon (x^*, y - x) \leqslant \varphi(y) - \varphi(x) \text{ for all } y \in \operatorname{dom}\varphi\}$ (in this definition by $(\cdot,\cdot)$ we denote the duality brackets for the pair (X, X^*)). It is well-known that if $\varphi(\cdot)$ is Gateaux differentiable at a point $x \in X$, then $\partial\varphi(x) = \{\varphi'(x)\}$. We say that $\varphi \in \Gamma_0(X)$ is of compact type, if for every $\lambda \in \mathbb{R}_+$, the level set $\{x \in X : \|x\|^2 + \varphi(x) \leqslant \lambda\}$ is compact.

Recall that on $P_f(X)$, we can define a generalized metric, known in the literature as Hausdorff metric, by setting for $A, B \in P_f(X)$

$$h(A,B) = \max\left[\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right]$$

where $d(a,B) = \inf\{\|a - b\| \colon b \in B\}$ and $d(b,A) = \inf\{\|b - a\| \colon a \in A\}$. It is well-known (see for example Klein-Thompson [15]), that the metric space $\big(P_f(X), h\big)$

is complete. A multifunction $F\colon X \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from X into the metric space $\big(P_f(X), h\big)$.

If Y, Z are Hausdorff topological spaces and $G\colon Y \to 2^Z \setminus \{\emptyset\}$, we say that $G(\cdot)$ is upper semicontinuous (u.s.c.), if for every $U \subseteq Z$ open, $G^+(U) = \{y \in Y : G(y) \subseteq U\}$ is open (see Klein-Thompson [15]).

Now let $T = [0, b]$ and H a separable Hilbert space. By a "strong solution" of (1) (resp. of (2)), we mean a function $x(\cdot) \in C(T, H)$ s.t. $x(\cdot)$ is absolutely continuous on any closed subinterval of $(0, b)$, $x(t) \in D(A)$ a.e. on $(0, b)$, $x(0) = x_0$ and $-\dot{x}(t) \in \partial\varphi\big(t, x(t)\big) + f(t)$ a.e. with $f \in S^2_{F(\cdot, x(\cdot))}$ (resp. $f \in S^2_{\operatorname{ext} F(\cdot, x(\cdot))}$). Recall that an absolutely continuous function $x\colon (0, b) \to X$ is strongly differentiable almost everywhere, so in the above inclusion $\dot{x}(\cdot)$ is the strong derivative of $x(\cdot)$.

Following Yotsutani [26], we will make the following hypothesis concerning $\varphi(t, x)$ and it will be valid throughout this work:

$H(\varphi)$: $\varphi\colon T \times H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function s.t.

(1) for every $t \in T$, $\varphi(t, \cdot)$ is proper, convex, l.s.c. (i.e. $\varphi(t, \cdot) \in \Gamma_0(H)$) and of compact type,

(2) for any positive integer r, there exists a constant $K_r > 0$, an absolutely continuous function $g_r\colon T \to \mathbb{R}$ with $\dot{g}_r \in L^\beta(T)$ and function of bounded variation $h_r\colon T \to \mathbb{R}$ s.t. if $t \in T$, $x \in \operatorname{dom}\varphi(t, \cdot)$ with $\|x\| \leqslant r$ and $s \in [t, b]$, then there exists $\hat{x} \in \operatorname{dom}\varphi(s, \cdot)$ satisfying

$$\|\hat{x} - x\| \leqslant |g_r(s) - g_r(t)|\big(|\varphi(t, x)| + K_r\big)^\alpha$$
$$\text{and } \varphi(s, \hat{x}) \leqslant \varphi(t, x) + |h_r(s) - h_r(t)|\big(\varphi(t, x) + K_r\big)$$

where $\alpha \in [0, 1]$ and $\beta = 2$ if $\alpha \in [0, 1/2]$ or $\beta = 1/(1 - \alpha)$ if $\alpha \in [\frac{1}{2}, 1]$.

Remarks. (a) This hypothesis is more general than the one used by Watanabe [25].

(b) If $\varphi(t, \cdot) = \varphi(\cdot) \in \Gamma_0(H)$ (i.e. there is no t-dependence) and $\varphi(\cdot)$ is of compact type, then it is clear that hypothesis $H(\varphi)$ above is automatically satisfied. So Theorem 3.1 of this paper also improves Theorem 3.1 of Kravvaritis-Papageorgiou [16].

In what follows by $S_e(x_0)$ (resp. $S(x_0)$) we will denote the set of strong solutions of (1) (resp. of (2)).

In this section we establish the nonemptiness of the solution set $S_e(x_0)$ ("extremal solutions"). To this end we will need the following hypothesis on the orientor field $F(t,x)$.

$H(F)$: $F\colon T \times H \to P_{wkc}(H)$ is a multifunction s.t.

(1) $t \to F(t,x)$ is measurable,

(2) $x \to F(t,x)$ is h-continuous,

(3) $|F(t,x)| = \sup\{\|y\|\colon y \in F(t,x)\} \leqslant \alpha(t) + \beta(t)\|x\|$ a.e. with $\alpha(\cdot), \beta(\cdot) \in L^2_+$.

H_0: $x_0 \in \operatorname{dom}\varphi(0,\cdot)$.

Let $L^1_w(H)$ denote the Lebesgue-Bochner space $L^1(H)$ equipped with the norm $\|x\|_w = \sup\limits_{0\leqslant s\leqslant t\leqslant b} \| \int_s^t x(\tau)\,\mathrm{d}\tau \|$ ("weak norm"). Also from Yotsutani [26], we know that given $f \in L^2(H)$, the Cauchy problem $-\dot{x}(t) \in \partial\varphi\big(t, x(t)\big) + f(t)$ a.e., $x(0) = x_0$, has a unique strong solution $p(f)(\cdot) \in C(T,H)$. The next proposition establishes the continuity of $p(\cdot)$ with respect to the weak norm on $L^2(H)$.

Proposition 3.1. *If $\{f_n, f\}_{n\geqslant 1} \subseteq L^2(H)$, $f_n \xrightarrow{\|\cdot\|_w} f$ and $\sup\limits_{n\geqslant 1} \|f_n\|_{L^2(H)} < \infty$, then $p(f_n) \to p(f)$ in $C(T,H)$.*

Proof. First we will show that $f_n \xrightarrow{w} f$ in $L^2(H)$. To this end let $s(\cdot) \in L^2(H)$ be a step function; i.e.

$$s(t) = \sum_{k=1}^{N} \chi_{(t_{k-1},t_k)}(t) v_k .$$

Then we have:

$$\begin{aligned} |(f_n - f, s)_{L^2(H)}| &\leqslant \sum_{k=1}^{N} \left\| \int_{t_k-1}^{t_k} \big(f_n(s) - f(s)\big)\,\mathrm{d}s \right\| \cdot \|v_k\| \\ &\leqslant \|f_n - f\|_w \cdot \sum_{k=1}^{N} \|v_k\| \to 0 \text{ as } N \to \infty. \end{aligned}$$

Because step functions are dense in $L^2(H)$, we conclude that $f_n \xrightarrow{w} f$ in $L^2(H)$.

For economy in the notation, let $x_n = p(f_n)$ and $x = p(f)$. Exploiting the monotonicity of the subdifferential operator, we get

$$\begin{aligned} &\big(-\dot{x}_n(t) + \dot{x}(t), x(t) - x_n(t)\big) \leqslant \big(f_n(t) - f(t), x(t) - x_n(t)\big) \text{ a.e.} \\ &\Rightarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x_n(t) - x(t)\|^2 \leqslant \|f_n(t) - f(t)\| \cdot \|x_n(t) - x(t)\| \text{ a.e.} \\ &\Rightarrow \frac{1}{2}\|x_n(t) - x(t)\|^2 \leqslant \int_0^t \|f_n(s) - f(s)\| \cdot \|x_n(s) - x(s)\,\mathrm{d}s\,. \end{aligned}$$

Applying Lemma A.5, p. 157 of Brezis [7], we get

$$\|x_n(t) - x(t)\| \leqslant \int_0^t \|f_n(s) - f(s)\, \mathrm{d}s\,.$$

Since $f_n \xrightarrow{w} f$ in $L^2(H)$, there exists $M_1 > 0$ s.t. $\|f_n\|_2, \|f\|_2 \leqslant M_1$. So we have for all $t \in T$ and all $n \geqslant 1$

$$\|x_n(t)\| \leqslant \|x\|_\infty + 2\sqrt{b}M_1 = M_2 < \infty.$$

Also from Yotsutani [26] (inequality (7.9), p. 645), we have that there exists $M_3 > 0$ such that for all $n \geqslant 1$ and all $t \in T$

$$\varphi\big(t, x_n(t)\big) \leqslant M_3$$

(the constant M_3 depends only on the variation of h ($\operatorname{var}(h)$) and on $\|\dot{g}\|_\beta$, M_1, x_0 and $\varphi(0, x_0)$). So for all $t \in T$

$$\begin{aligned}&\{x_n(t)\}_{n\geqslant 1} \subseteq \{z \in H\colon \|z\|^2 + \varphi(t,z) \leqslant M_2^2 + M_3 = M_4\}\\ \Rightarrow\ &\overline{\{x_n(t)\}}_{n\geqslant 1} \text{ is compact in } H \text{ (recall that } \varphi(t,\cdot) \text{ is of compact type).}\end{aligned}$$

Also if $s, t \in T$, $s \leqslant t$, we have

$$\begin{aligned}\|x_n(t) - x_n(s)\| &= \left\| \int_s^t \dot{x}_n(\tau)\, \mathrm{d}\tau \right\| \leqslant \int_s^t \|\dot{x}_n(\tau)\|\, \mathrm{d}\tau\\ &= \int_0^b \chi_{[s,t]}(\tau)\|\dot{x}_n(\tau)\|\, \mathrm{d}\tau \leqslant \sqrt{t-s}\left(\int_0^b \|\dot{x}_n(\tau)\|^2\, \mathrm{d}\tau\right)^{1/2}\\ &\quad\text{(Cauchy-Schwartz inequality).}\end{aligned}$$

But from Yotsutani [26] (see Lemma 6.11, p. 644), we have that

$$\sup_{n\geqslant 1} \|\dot{x}_n\|_{L^2(H)} = M_5 < \infty.$$

Hence we deduce that $\{x_n\}_{n\geqslant 1} \subseteq C(T,H)$ is equicontinuous.

Thus invoking the Arzela-Ascoli theorem, we have that $\{x_n\}_{n\geqslant 1}$ is relatively compact in $C(T,H)$. Furthermore since $\{\dot{x}_n\}_{n\geqslant 1}$ is $L^2(H)$-bounded, it is relatively sequentially weakly compact. So by passing to a subsequence if necessary, we may assume that $x_n \to y$ in $C(T,H)$, while $\dot{x}_n \xrightarrow{w} v$ in $L^2(H)$. It is easy to see that $v = \dot{y}$.

Let $\Phi\colon L^2(H) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\Phi(x) = \begin{cases} \int_0^b \varphi\big(t, x(t)\big)\, \mathrm{d}t & \text{if } \varphi\big(\cdot, x(\cdot)\big) \in L^1(T),\\ +\infty & \text{otherwise}\end{cases}$$

(note that from Lemma 3.4, p. 629 of Yotsutani [26], for every $x: T \to H$ measurable, $t \to \varphi(t, x(t))$ is measurable). It is well known (see for example Yotsutani [26], Lemma 4.4), that

$$\partial\Phi(x) = \{v \in L^2(H) : v(t) \in \partial\varphi(t, x(t)) \text{ a.e.}\}.$$

Then for every $n \geqslant 1$, we have

$$[x_n, -\dot{x}_n - f_n] \in \operatorname{Gr}\partial\Phi.$$

But recall that $\operatorname{Gr}\partial\Phi$ is demiclosed (i.e. sequentially closed in $L^2(H) \times L^2(H)_w$: see for example Brezis [7]). Since $[x_n, -\dot{x}_n - f_n] \overset{s\times w}{\to} [y, -\dot{y} - f]$ in $L^2(H) \times L^2(H)$, we have $[y, -\dot{y} - f] \in \operatorname{Gr}\Phi \Rightarrow -\dot{y}(t) \in \partial\varphi(t, y(t)) + f(t)$ a.e., $y(0) = x_0 \Rightarrow y = p(f) = x$. Hence every subsequence of $\{p(f_n)\}_{n\geqslant 1}$ has a further subsequence that converges to $p(f)$ in $C(T, H)$. Therefore $p(f_n) \to p(f)$ in $C(T, H)$. □

Using this continuity result we can have the following existence theorem for Cauchy problem (1).

Theorem 3.1. *If hypotheses $H(\varphi)$, $H(F)$ and H_0 hold, then the solution set $S_e(x_0)$ of (1) is nonempty.*

Proof. First we will establish an a priori uniform bound for the elements in $S(x_0) \subseteq C(T, H)$. So let $x(\cdot) \in S(x_0)$ and let $y(\cdot) \in C(T, H)$ be the unique solution of the evolution inclusion

$$\left\{\begin{array}{c} -\dot{y}(t) \in \partial\varphi(t, y(t)) \text{ a.e.}, \\ y(0) = x_0. \end{array}\right\}$$

The existence of $y(\cdot)$ is guaranteed by the result of Yotsutani [26]. As we did in the proof of Proposition 3.1, by exploiting the monotonicity of the subdifferential operator and using Lemma A.5, p. 157 of Brezis [7], we get

$$\|x(t) - y(t)\| \leqslant \int_0^t \|f(s)\| \,\mathrm{d}s \leqslant \int_0^t \big(\alpha(s) + \beta(s)\|x(s)\|\big) \,\mathrm{d}s$$

where $f \in S^2_{F(\cdot, x(\cdot))}$ and $-\dot{x}(t) \in \partial\varphi(t, x(t)) + f(t)$ a.e. Hence we have

$$\|x(t)\| \leqslant \|y\|_{C(T,H)} + \int_0^t \big(\alpha(s) + \beta(s)\|x(s)\|\big) \,\mathrm{d}s\,, \quad t \in T.$$

Applying Gronwall's inequality, we deduce that there exists $M_1 > 0$ s.t. if $x(\cdot) \in S(x_0)$ and $t \in T$, then

$$\|x(t)\| \leqslant M_1.$$

Hence without any loss of generality, we may assume that $|F(t,x)| \leqslant \sup\{\|y\|: y \in F(t,x)\} \leqslant \psi(t)$ a.e. with $\psi(\cdot) \in L^2_+$ (just consider instead $\hat{F}(t,x) = F(t, p_{M_1}(x))$, with $p_{M_1}(\cdot)$ being the M_1-radial projection and use hypothesis $H(F)(3)$). Set $B(\psi) = \{u \in L^2(H): \|u(t)\| \leqslant \psi(t) \text{ a.e.}\}$. Let $K = p(B(\psi))$, where $p(\cdot)$ is the solution map as in Proposition 3.1. We claim that K is relatively compact in $C(T, H)$. To this end let $x(\cdot) \in K$ and $t < t'$. We have

$$\|x(t') - x(t)\| \leqslant \int_t^{t'} \|\dot{x}(s)\| \, ds \leqslant \Big(\int_0^b \chi_{[t,t']}(s)^2 \, ds\Big)^{1/2} \Big(\int_0^b \|\dot{x}(s)\|^2 \, ds\Big)^{1/2} \leqslant (t'-t)^{1/2} M_2$$

since from Yotsutani [26] (Lemma 6.11), we know that there exists $M_2 > 0$ s.t. for all $x \in K$ $\|\dot{x}\|_{L^2(H)} \leqslant M_2$. Hence we have established that K is equicontinuous.

Next let $K(t) = \{x(t): x(\cdot) \in K\}$. Recall (see Yotsutani [26] (inequality (7.9), p. 645)) that there exists $M_3 > 0$ depending only on the total variation of h, on $\|\dot{g}\|_{L^\beta}$, on $\|\psi\|_2$, on M_1, on x_0 and on $\varphi(0, x_0)$, and

$$\varphi(t, x(t)) \leqslant M_3.$$

Hence $K(t) \subseteq \{x \in H: \|x\|^2 + \varphi(t, x(t)) \leqslant M_1^2 + M_3 = M_4\}$ and the latter is relatively compact in H, since by hypothesis $H(\varphi)$, $\varphi(t, \cdot)$ is of compact type. Therefore $\overline{K(t)}$ is compact in H. Thus by the Arzela-Ascoli theorem, we get that K is compact in $C(T, H)$. Set $\hat{K} = \overline{\operatorname{conv}} K$. From Mazur's theorem, we know that $\hat{K} \in P_{kc}(C(T,H))$. Observe that $S(x_0) \subseteq \hat{K}$.

Now let $R: \hat{K} \to P_{wkc}(L^1(H))$ be defined by

$$R(y) = S^1_{F(\cdot, y(\cdot))}.$$

Apply Theorem 1.1 of Tolstonogov [23], to get $g: \hat{K} \to L^1_w(H)$ a continuous map such that $g(y) \in \operatorname{ext} R(y) = S^1_{\operatorname{ext} F(\cdot, y(\cdot))}$ (see Benamara [6]) for all $y \in \hat{K}$ (recall that $L^1_w(H)$ denotes the Lebesgue-Bochner space $L^1(H)$ furnished with the norm $\|h\|_w = \sup\limits_{0 \leqslant t \leqslant t' \leqslant b} \|\int_t^{t'} h(s) \, ds\|$). Then consider the map $v: \hat{K} \to \hat{K}$ defined by $v = p o g$. Using Proposition 3.1 above, we get that $v(\cdot)$ is continuous. Apply Schauder's fixed point theorem to get $y \in \hat{K}$ s.t. $v(y) = y$. Clearly then $y \in S_e(x_0) \Rightarrow S_e(x_0) \neq \emptyset$. $\square$

Remark. Note that $\operatorname{ext} F(t,x)$ need not be a closed set and $x \to \operatorname{ext} F(t,x)$ is not necessarily a lower semicontinuous multifunction. Hence Theorem 3.1 improves the corresponding existence results of Kravvaritis-Papageorgiou [16] and Papageorgiou [21].

In this section we show that the set $S_e(x_0)$ is residual in $S(x_0)$; i.e. it is a dense, G_δ-subset of $S(x_0)$. Recall (see Papageorgiou [21]) that $S(x_0) \in P_k(C(T,H))$.

Our approach uses the Choquet function of the orientor field $F(t,x)$. This method was used recently by DeBlasi-Pianigiani [9], who showed that the Choquet theory of extreme points of compact convex sets, is the right tool in the study of nonconvex differential inclusions.

So let $\{x_k^*\}_{k\geqslant 1} \subseteq H$, $\|x_k^*\| = 1$ be a sequence which is dense in the unit sphere of H. Following Choquet [8] and DeBlasi-Pianigiani [9], we define a function $\gamma_F\colon T \times H \times H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ by

$$\gamma_F(t,x,v) = \begin{cases} \displaystyle\sum_{k=1}^{n} \frac{(x_k^*, v)^2}{2^k} & v \in F(t,x), \\ +\infty & v \notin F(t,x). \end{cases}$$

Let $\mathrm{Aff}(X) = \{$the set of all continuous affine functions $a\colon H \to \mathbb{R}\}$. Let $\hat{\gamma}_F\colon T \times H \times H \to \mathbb{R} \cup \{-\infty\}$ be defined by

$$\hat{\gamma}_F(t,x,v) = \inf\{a(v)\colon a \in \mathrm{Aff}(X) \text{ and } a(z) \geqslant \gamma_F(t,x,z) \text{ for all } z \in F(t,x)\}.$$

(As always, $\inf \emptyset = -\infty$).

Then the Choquet function $\delta_F\colon T \times H \times H \to \mathbb{R} \cup \{-\infty\}$ corresponding to the orientor field $F(t,x)$, is defined by

$$\delta_F(t,x,v) = \hat{\gamma}_F(t,x,v) - \gamma_F(t,x,v).$$

Then next proposition establishes the properties of $\delta_F(t,x,v)$ that we will need in the sequel (see also DeBlassi-Pianigiani [9], Proposition 2.1, p. 473).

Proposition 4.1. *If hypothesis $H(F)$ holds, then*
(i) $(t,x,v) \to \delta_F(t,x,v)$ *is measurable,*
(ii) $(x,v) \to \delta_F(t,x,v)$ *is u.s.c.,*
(iii) $v \to \delta_F(t,x,v)$ *is concave and is strictly concave on* $F(t,x)$,
(iv) $0 \leqslant \delta_F(t,x,v) \leqslant 4\alpha(t)^2 + 4\beta(t)^2\|x\|^2$ *a.e. for all* $(t,x,v) \in \mathrm{Gr}\, F$,
(v) $\delta_F(t,x,v) = 0$ *if and only if* $v \in \mathrm{ext}\, F(t,x)$.

Proof. (i) From Theorem 3.3 of Papageorgiou [18], we know that because of hypotheses $H(F)(1)$ and (2), $(t,x) \to F(t,x)$ is measurable. Hence $\mathrm{Gr}\, F = \{(t,x,z) \in T \times H \times H\colon z \in F(t,x)\} \in B(T) \times B(H) \times B(H)$, with $B(T)$ (resp.

$B(H)$) being the Borel σ-field of T (resp. of H). Using this fact and the definition of $\gamma_F(t,x,v)$, it is clear that $(t,x,v) \to \gamma_F(t,x,v)$ is measurable. Furthermore because of the closedness and the convexity of the set $F(t,x)$, it is evident that $\gamma_F(t,x,\cdot)$ is a l.s.c. and convex function. Hence using Lemma 2.1 (i) of Hiai-Umegaki [13], we get that for every $x^* \in H$

$$(t,x) \to \eta_F(x^*)(t,x) = \sup\left[\gamma_F(t,x,z) - (x^*,z) \colon z \in F(t,x)\right]$$

is a measurable function.

Now observe that

$$\hat{\gamma}_F(t,x,v) = \inf\left[(x^*,v) + \eta_F(x^*)(t,x) \colon x^* \in H\right].$$

Fix $(t,x) \in T \times H$. Then the set $F(t,x)$ equipped with the weak topology is a compact metrizable set. Let $\theta_{x^*}(t,x)(\cdot) = \gamma_F(t,x,\cdot) - (x^*,\cdot)$. Our claim is that if $x_n^* \xrightarrow{s} x^*$ in H, then $\theta_{x_n^*}(t,x)(\cdot) \xrightarrow{e} \theta_{x^*}(t,x)(\cdot)$, where by $\xrightarrow{e}$ we denote convergence in the epigraphical sense (see Attouch [1], p. 39). To this end, first note that for every $z \in H$, we have

$$(3) \qquad \theta_{x_n^*}(t,x)(z) = \gamma_F(t,x,z) - (x_n^*,z) \to \gamma_F(t,x,z) - (x^*,z) = \theta_{x^*}(t,x)(z).$$

Also if $z_n \xrightarrow{w} z$ in $F(t,x)$, then we have

$$\begin{aligned} \underline{\lim}\, \theta_{x_n^*}(t,x)(z_n) &= \underline{\lim}\left[\gamma_F(t,x,z_n) - (x_n^*,z_n)\right] \\ &\geqslant \underline{\lim}\, \gamma_F(t,x,z_n) - \lim(x_n^*,z_n). \end{aligned}$$

But $\gamma_F(t,x,\cdot)$ being l.s.c., convex, is weakly l.s.c. and so we have

$$\underline{\lim}\, \gamma_F(t,x,z_n) \geqslant \gamma_F(t,x,z).$$

Thus finally we have:

$$(4) \qquad \underline{\lim}\, \theta_{x_n^*}(t,x)(z_n) \geqslant \gamma_F(t,x,z) - (x^*,z) = \theta_{x^*}(t,x)(z).$$

Hence from (3) and (4) above and the properties of epigraphical convergence (see Attouch [1], p. 39), we have that

$$\theta_{x_n^*}(t,x)(\cdot) \xrightarrow{e} \theta_{x^*}(t,x)(\cdot) \text{ as } n \to \infty.$$

Then invoking Theorem 2.11, p. 132 of Attouch [1], we get

$$\eta(x_n^*)(t,x) \to \eta(x^*)(t,x) \text{ as } n \to \infty.$$

Therefore if $\{x_m\}_{m\geqslant 1}$ is dense in H, we can write that

$$\begin{gathered}\hat{\gamma}_F(t,x,v) = \inf_{m\geqslant 1}\left[(x_m^*,v) + \eta(x_m^*)(t,x)\right] \\ \Rightarrow (t,x,v) \to \gamma_F(t,x,v) \text{ is measurable} \\ (t,x,v) \to \delta_F(t,x,v) \text{ is measurable.}\end{gathered}$$

(ii) Since by hypothesis $H(F)$, $F(t,\cdot)$ is h-continuous, from the definition of $\gamma_F(t,x,v)$ we can easily check that $\gamma_F(t,\cdot,\cdot)$ is continuous on $\operatorname{Gr} F(t,\cdot)$ for the relative $H \times H_w$-topology (here H_w denotes the Hilbert space H equipped with the weak topology) and also is l.s.c. on $H \times H$. Next recall that

$$\hat{\gamma}_F(t,x,v) = \inf\left[(x^*,v) + \eta_F(x^*)(t,x) \colon x^* \in H\right].$$

So to establish the upper semicontinuity of $\hat{\gamma}_F(t,\cdot,\cdot)$, it suffices to show that $\eta_F(x^*)(t,\cdot)$ is u.s.c. To this end we need to show that for every $\lambda \in \mathbb{R}$, the set

$$U_\lambda = \{x \in H \colon \eta_F(x^*)(t,x) \geqslant \lambda\}$$

is closed in H. So let $x_n \in U_\lambda$ and assume $x_n \xrightarrow{s} x$. Then there exist $z_n \in F(t,x_n)$ such that

$$\lambda \leqslant \gamma_F(t,x_n,z_n) - (x^*,z_n).$$

Since $F(t,\cdot)$ is h-continuous, its support function $x \to \sigma(z^*, F(t,x)) = \sup\left[(z^*,z) \colon z \in F(t,x)\right]$ is continuous for every $z^* \in H$ (just recall by Hörmander's formula we have $h(F(t,x),F(t,y)) = \sup\left[|\sigma(z^*,F(t,x)) - \sigma(z^*,F(t,y))| \colon \|z^*\| \leqslant 1\right]$) and so since $F(\cdot,\cdot)$ is assumed to be $P_{wkc}(H)$-valued, we have that $F(t,\cdot)$ is u.s.c. as a multifunction from H into H_w (see Aubin-Ekeland [3], Theorem 10, p. 128). Thus Theorem 7.4.2, p. 90 of Klein-Thompson [15] tells us that $\overline{\bigcup_{n\geqslant 1} F(t,x_n)}^w \in P_{wk}(H)$. Since $\{z_n\}_{n\geqslant 1} \subseteq \overline{\bigcup_{n\geqslant 1} F(t,x_n)}^w$, by passing to a subsequence if necessary, we may assume that $z_n \xrightarrow{w} z \in F(t,x)$ (since $F(t,\cdot)$ is u.s.c. from H into H_w). Then recalling that on $\operatorname{Gr} F(t,\cdot)$, $\gamma_F(t,\cdot,\cdot)$ is continuous from $H \times H_w$ into $\mathbb{R}$, we have

$$\begin{gathered}\lim \gamma_F(t,x_n,z_n) = \gamma_F(t,x,z) \\ \Rightarrow \lambda \leqslant \gamma_F(t,x,z) - (x^*,z) \\ \Rightarrow \lambda \leqslant \eta(x^*)(t,x) \\ \Rightarrow x \in U_\lambda.\end{gathered}$$

So indeed the set U_λ is closed and so we have established the upper semicontinuity of $\eta_F(x^*)(t,\cdot)$. Therefore $\hat{\gamma}_F(t,\cdot,\cdot)$ is u.s.c. on $H \times H$ and then so is $(x,v) \to \delta_F(t,x,v)$.

(iii) Recall that $\gamma_F(t,x,\cdot)$ is convex. Also since

$$\hat{\gamma}_F(t,x,v) = \inf\left[(x^*,v) + \eta_F(x^*)(t,x) \colon x^* \in H\right],$$

we have that $\hat{\gamma}_F(t,x,\cdot)$ being the lower envelope of affine functions is concave. Therefore $v \to \delta_F(t,x,v)$ is concave. Strict concavity on $F(t,x)$ follows from the fact that the sequence $\{x_k^*\}_{k\geqslant 1}$ separates points in H.

(iv) First note that from the definition of the function $\eta_F(x^*)(t,x)$, we have:

$$\begin{aligned}
|\eta_F(x^*)(t,x)| &= \left|\sup\left[\gamma_F(t,x,z) - (x^*,z) \colon z \in F(t,x)\right]\right| \\
&= \left|\sup\left[\sum_{k=1}^{\infty} \frac{(x_k^*,z)^2}{2^k} - (x^*,z) \colon z \in F(t,x)\right]\right| \\
&\leqslant \sum_{k=1}^{\infty} \frac{\sigma(x_k^*,F(t,x))^2}{2^k} + \sigma\big(x^*,F(t,x)\big) \\
\leqslant \sum_{k=1}^{\infty} \frac{|F(t,x)|^2 \cdot \|x_k^*\|}{2^k} &+ |F(t,x)| \cdot \|x^*\| = |F(t,x)|^2 + |F(t,x)| \cdot \|x^*\|
\end{aligned}$$

where $|F(t,x)| = \sup\{\|z\| \colon z \in F(t,x)\}$. So we have:

$$\begin{aligned}
|\hat{\gamma}_F(t,x,v)| &= \left|\inf\left[(x^*,v) + \eta_F(x^*)(t,x) \colon x^* \in H\right]\right| \\
&\leqslant \inf\left[\|x^*\|\big(\|v\| + |F(t,x)|\big) + |F(t,x)|^2 \colon x^* \in H\right] \\
&= |F(t,x)|^2.
\end{aligned}$$

On the other hand, it is clear from the definition of $\gamma_F(t,x,v)$ that for all $v \in F(t,x)$ we have

$$|\gamma_F(t,x,v)| \leqslant |F(t,x)|^2.$$

So finally, we have for $(t,x,v) \in \operatorname{Gr} F$

$$0 \leqslant \delta_F(t,x,v) \leqslant |\hat{\gamma}_F(t,x,v)| + |\gamma_F(t,x,v)| \leqslant 2|F(t,x)|^2 \leqslant 4\alpha(t)^2 + 4\beta(t)^2\|x\|^2 \text{ a.e.}$$

(v) From part (iii) we know that $\delta_F(t,x,\cdot)$ is strictly concave on $F(t,x)$ and recall that $0 \leqslant \delta_F$. So from these two facts, we get easily that if $\delta_F(t,x,v) = 0$, then $v \in \operatorname{ext} F(t,x)$. On the other hand from Bauer's minimum principle (see Holmes [14], Corollary 2, p. 75) and the definition of $\hat{\gamma}_F$, we have that if $v \in \operatorname{ext} F(t,x)$, then $\hat{\gamma}_F(t,x,v) = \gamma_F(t,x,v)$ and so $\delta_F(t,x,v) = 0$. $\square$

Now to prove our Baire category type theorem, we will need the following stronger hypothesis on the orientor field $F(t,x)$.

$H(F)_1$: $F\colon T \times H \to P_{wkc}(H)$ is a multifunction s.t.

(1) $t \to F(t,x)$ is measurable,

(2) $h\big(F(t,x), F(t,y)\big) \leqslant k(t)\|x-y\|$ a.e. with $k(\cdot) \in L^1_+$,

(3) $|F(t,x)| \leqslant \alpha(t) + \beta(t)\|x\|$ a.e. with $\alpha, \beta \in L^2_+$.

The next result shows that $S_e(x_0)$ is residual in $S(x_0) \in P_k\big(C(T,H)\big)$.

Theorem 4.1. *If hypotheses $H(\varphi)$, $H(F)_1$ and H_0 hold, then $S_e(x_0)$ is a dense G_δ-subset of $S(x_0)$.*

Proof. Let $\lambda > 0$ and set

$$\Gamma_\lambda = \left\{x \in S(x_0)\colon \int_0^b \delta_F\big(t, x(t), -\dot{x}(t) - g(t)\big)\,\mathrm{d}t < \lambda\right\}$$

where $g \in L^2(H)$ is such that $g(t) \in \partial\varphi\big(t, x(t)\big)$ a.e. and $-\dot{x}(t) - g(t) \in F\big(t, x(t)\big)$ a.e. Our claim is that the set Γ_λ is open in $S(x_0)$. We will show that $S(x_0) \setminus \Gamma_\lambda$ is closed. So let $x_n \in S(x_0) \setminus \Gamma_\lambda$ and assume that $x_n \to x$ in $S(x_0)$. From Lemma 6.11 of Yotsutani [26] we know that $\{\dot{x}_n\}_{n\geqslant 1}$ is $L^2(H)$-bounded. This combined with hypothesis $H(F)_1(3)$ tells us that $\{g_n\}_{n\geqslant 1}$ is $L^2(H)$-bounded too. So by passing to a subsequence if necessary, we may assume that $\dot{x}_n \xrightarrow{w} z$ and $g_n \xrightarrow{w} g$ in $L^2(H)$. Clearly $z = \dot{x}$, while by using the integral functional $\Phi(\cdot)$ as in the proof of Proposition 3.1, and recalling that $\operatorname{Gr}\partial\Phi$ is demiclosed, we get that $g \in \partial\Phi(x) \Rightarrow g(t) \in \partial\varphi\big(t, x(t)\big)$ a.e. Also for every $h \in L^2(H)$ we have

$$(-\dot{x}_n - g_n, h)_{L^2(H)} \leqslant \sigma\big(h, S^1_{F(\cdot, x_n(\cdot))}\big), \quad n \geqslant 1.$$

From Theorem 4.5 of [19] we know that $\sigma\big(h, S^1_{F(\cdot,x_n(\cdot))}\big) \to \sigma\big(h, S^1_{F(\cdot,x(\cdot))}\big)$. Hence in the limit as $n \to \infty$, we get

$$(-\dot{x} - g, h)_{L^2(H)} \leqslant \sigma\big(h, S^1_{F(\cdot,x(\cdot))}\big).$$

Since $h \in L^2(H)$ was arbitrary and $S^1_{F(\cdot,x(\cdot))}$ is closed, convex, we deduce that

$$\begin{aligned} &-\dot{x} - g \in S^1_{F(\cdot,x(\cdot))} \\ &\Rightarrow -\dot{x}(t) - g(t) \in F\big(t, x(t)\big) \text{ a.e.} \end{aligned}$$

Furthermore because of Proposition 4.1 (i)–(iv), we can apply Theorem 2.1 of Balder [4] and get

$$\begin{aligned} \overline{\lim} \int_0^b \delta_F\big(t, x_n(t), -\dot{x}_n(t) - g_n(t)\big)\,\mathrm{d}t &\leqslant \int_0^b \delta_F\big(t, x(t), -\dot{x}(t) - g(t)\big)\,\mathrm{d}t \\ \Rightarrow \lambda &\leqslant \int_0^b \delta_F\big(t, x(t), -\dot{x}(t) - g(t)\big)\,\mathrm{d}t \end{aligned}$$

with $g \in L^2(H)$, $g(t) \in \partial\varphi(t, x(t))$ a.e. and $-\dot{x}(t) - g(t) \in F(t, x(t))$ a.e. So $x \in S(x_0)\Gamma_\lambda \Rightarrow \Gamma_\lambda$ is indeed open in $S(x_0)$.

Next we will show that

$$S_e(x_0) = \bigcap_{n \geqslant 1} \Gamma_n$$

where $\Gamma_n = \Gamma_{\lambda_n}$, $\lambda_n = \frac{1}{n}$. Using Proposition 4.1 (v), we see that

$$S_e(x_0) \subseteq \bigcap_{n \geqslant 1} \Gamma_n.$$

On the other hand, if $x \in \bigcap_{n \geqslant 1} \Gamma_n$, then

$$0 \leqslant \int_0^b \delta_F\big(t, x(t), -\dot{x}(t) - g(t)\big)\, dt < \frac{1}{n} \quad \text{for all } n \geqslant 1$$
$$\Rightarrow \int_0^b \delta_F\big(t, x(t) - \dot{x}(t) - g(t)\big)\, dt = 0.$$

Since $0 \leqslant \delta_F$, we get that $\delta_F\big(t, x(t), -\dot{x}(t) - g(t)\big) = 0$ a.e. Hence once again Proposition 4.1 (v) tells us that

$$-\dot{x}(t) - g(t) \in \operatorname{ext} F\big(t, x(t)\big) \text{ a.e.}$$
$$\Rightarrow x \in S_e(x_0).$$

Thus we have shown that $S_e(x_0) = \bigcap_{n \geqslant 1} \Gamma_n$; i.e. $S_e(x_0)$ is a G_δ-subset of $S(x_0)$.

Next we are going to show that $\overline{S_e(x_0)}^{C(T,H)} = S(x_0)$. From [21] we know that $S(x_0) \in P_k\big(C(T, H)\big)$. Let $x(\cdot) \in S(x_0)$. So by definition there exists $f \in S^2_{F(\cdot, x(\cdot))}$ such that $-\dot{x}(t) \in \partial\varphi\big(t, x(t)\big) + f(t)$ a.e., $x(0) = x_0$. Let $\hat{K} = \overline{\operatorname{conv}} p\big(B(\psi)\big)$, as in the proof of Theorem 3.1, $\xi = |\hat{K}| = \sup\{\|x\|_\infty : x \in \hat{K}\}$ and consider the multifunction $L: \hat{K} \to 2^{L^1(H)}$ defined by

$$L(y) = \Big\{h \in S^1_{F(\cdot, y(\cdot))} : \|f(t) - h(t)\| < \frac{\varepsilon}{4b\xi} + k(t)\|x(t) - y(t)\| \text{ a.e.}\Big\}.$$

A simple application of Aumann's selection theorem tells us that $L(\cdot)$ has nonempty, closed and decomposable values (i.e. if $h_1, h_2 \in L(y)$ and $A \subseteq T$ is measurable, we have $\chi_A h_1 + \chi_{A^c} h_2 \in L(y)$), while from Proposition 2.3 of Fryszkowski [11], we have that $L(\cdot)$ is in addition l.s.c. Hence $y \to \overline{L(y)}$ is l.s.c. So applying Fryszkowski's continuous selection theorem [11], we get $v_\varepsilon : \hat{K} \to L^1(H)$ a continuous map such that for all $\hat{y} \in K$, $v_\varepsilon(y) \in \overline{L(y)}$. Apply Theorem 1.1 of Tolstonogov [23], to get $w_\varepsilon : \hat{K} \to L^1_w(H)$ a continuous map such that $w_\varepsilon(y) \in \operatorname{ext} R(y) = S^1_{\operatorname{ext} F(\cdot, y(\cdot))}$ and $\|v_\varepsilon(y) - w_\varepsilon(y)\|_w < \frac{\varepsilon}{2}$ for all $y \in \hat{K}$.

Now let $\varepsilon_n = \frac{1}{n}$ and set $v_{\varepsilon_n} = v_n$, $w_{\varepsilon_n} = w_n$. Let $y_n \in \hat{K}$ for which we have $y_n = (\mathrm{p} \circ w_n)(y_n)$. Existence of such points is guaranteed by Proposition 3.1 and Schauder's fixed point theorem. Since $\hat{K} \subseteq C(T, H)$ is compact (see the proof of Theorem 3.1), by passing to a subsequence if necessary, we may assume that $y_n \xrightarrow{s} y$ in $C(T, H)$. Then we have

$$\begin{aligned}
&\big(-\dot{y}_n(t) + \dot{x}(t), x(t) - y_n(t)\big) \leqslant \big(w_n(y_n)(t) - f(t), x(t) - y_n(t)\big) \text{ a.e.} \\
&\Rightarrow \frac{1}{2}\|y_n(t) - x(t)\|^2 \leqslant \int_0^t \big(w_n(y_n)(s) - f(s), x(s) - y_n(s)\big)\, \mathrm{d}s \\
&\qquad \leqslant \int_0^t \big(w_n(y_n)(s) - v_n(y_n)(s), x(s) - y_n(s)\big)\, \mathrm{d}s \\
&\qquad\quad + \int_0^t \big(v_n(y_n)(s) - f(s), x(s) - y_n(s)\big)\, \mathrm{d}s\,.
\end{aligned}$$

By construction we have that $w_n(y_n) - v_n(y_n) \xrightarrow{\|\cdot\|_w} 0$ in $L^2(H)$ and since $\sup_{n\geqslant 1} \|w_n(y_n) - v_n(y_n)\|_{L^2(H)} \leqslant 2\|\psi\|_2 < +\infty$, we have that $w_n(y_n) - v_n(y_n) \xrightarrow{w} 0$ in $L^2(H)$. So we get

$$\int_0^t \big(w_n(y_n)(s) - v_n(y_n)(s), x(s) - y_n(s)\big)\, \mathrm{d}s \to 0 \text{ as } n \to \infty.$$

On the other hand

$$\begin{aligned}
&\int_0^t \big(v_n(y_n)(s) - f(s), x(s) - y_n(s)\big)\, \mathrm{d}s \\
&\leqslant \int_0^t \|v_n(y_n)(s) - f(s)\| \cdot \|x(s) - y_n(s)\|\, \mathrm{d}s \\
&\leqslant \int_0^t \Big(\frac{1}{4b\xi n} + k(s)\|x(s) - y_n(s)\|\Big) \cdot \|x(s) - y_n(s)\|\, \mathrm{d}s \\
&\leqslant \frac{1}{2n} + \int_0^t k(s)\|x(s) - y_n(s)\|^2\, \mathrm{d}s\,.
\end{aligned}$$

So we get

$$\lim_{n\to\infty} \int_0^t \big(v_n(y_n)(s) - f(s), x(s) - y_n(s)\big)\, \mathrm{d}s \leqslant \int_0^t k(s)\|x(s) - y(s)\|^2\, \mathrm{d}s\,.$$

Therefore in the limit as $n \to \infty$, we get

$$\|x(t) - y(t)\|^2 \leqslant 2\int_0^t k(s)\|x(s) - y(s)\|^2\, \mathrm{d}s\,.$$

Invoking Gronwall's inequality, we get that $x = y$. Hence $y_n \to x$ in $C(T,H)$. Clearly $y_n \in S_e(x_0)$ and so we have that $\overline{S_e(x_0)}^{C(T,H)} = S(x_0)$. $\square$

R e m a r k. It is well-known that if instead the orientor field $F(r,x)$ satisfies hypothesis $H(F)$, then the density part of Theorem 4.1 is no longer true. There is a simple two dimensional counterexample due to Plis [22] illustrating this. So if instead we assume hypothesis $H(F)$, then we only have the following weaker version of Theorem 4.1:

Theorem 4.2. *If hypotheses $H(\varphi)$, $H(F)$ and H_0 hold, then $S_e(x_0)$ is a nonempty, G_δ-subset of $S(x_0)$.*

5. A special case

Let $K(t)$ be a moving set in H which satisfies the following hypothesis:
$H(K)$: $K: T \to P_{kc}(H)$ and there exists $v \in L^2_+$ such that for all $0 \leqslant t \leqslant t'$ we have

$$h(K(t'), K(t)) \leqslant \int_t^{t'} v(s)\,\mathrm{d}s\,.$$

Let $\varphi(t,x) = \delta_{K(t)}(x)$, where $\delta_{K(t)}(\cdot)$ is the indicator function of the set $K(t)$; i.e. $\delta_{K(t)}(x) = 0$ if $x \in K(t)$ and $\delta_{K(t)}(x) = +\infty$ if $x \notin K(t)$. Then it is easy to see that hypothesis $H(\varphi)$ is satisfied with $g_r(t) = V(t) = \int_0^t v(s)\,\mathrm{d}s$, $\dot{g}_r(t) = v(t)$, $\beta = 2$, $\alpha = 0$ and $K_r = 1$. Recalling $\partial\varphi(t,x) = \partial\delta_{K(t)}(x) = N_{K(t)}(x)$, the normal cone to the set $K(t)$ at x, Cauchy problems (1) and (2) take the following form:

$$\left\{\begin{array}{c} -\dot{x}(t) \in N_{K(t)}\big(x(t)\big) + \operatorname{ext} F\big(t, x(t)\big) \text{ a.e.,} \\ x(0) = x_0 \end{array}\right\} \tag{5}$$

and

$$\left\{\begin{array}{c} -\dot{x}(t) \in N_{K(t)}\big(x(t)\big) + F\big(t, x(t)\big) \text{ a.e.,} \\ x(0) = x_0. \end{array}\right\} \tag{6}$$

Problems of this form arise in theoretical mechanics in the study of elastoplastic systems (see Moreau [17]). When $K(t) = K$ (i.e. the set is time independent), then the resulting evolution inclusion is called "Differential Variational Inequality" and describes dynamical models of resource allocation in mathematical economics (see Aubin-Cellina [2], Henry [12], Papageorgiou [20]).

If by $S_e(x_0)$ (resp. $S(x_0)$), we denote the solution set (5) (resp. of (6)), then from Theorem 4.1, we get:

Theorem 5.1. *If hypotheses $H(K)$ and $H(F)_1$ hold and $x_0 \in K(0)$, then $S_e(x_0)$ is a dense G_δ-subset of $S(x_0)$.*

6. An application to control systems

In this section using Theorem 4.1, we obtain a "bang-bang" property for a class of nonlinear parabolic optimal control systems.

So let $Z \subseteq \mathbb{R}^N$ be a bounded domain with boundary $\Gamma = \partial Z$ and $p \geqslant 2$, $\beta > 0$.

$$(7)\qquad \left\{\begin{array}{r} \frac{\partial x}{\partial t} - \sum\limits_{k=1}^{N} \frac{\partial}{\partial z_k}\big(a(t,z)|\frac{\partial x}{\partial z_k}|^{p-2}\frac{\partial x}{\partial z_k}\big) + \beta x|x|^{p-2} = \\ = f\big(t,z,x(t,z)\big) + \big(b(t,z),u(t,z)\big) \text{ a.e.}, \\ x(0,z) = x_0(z),\ x\big|_{T\times\Gamma} = 0, \\ u(t,z) \in U(t,z) \text{ a.e.},\ u(\cdot,\cdot) \text{ is measurable.} \end{array}\right\}$$

In addition to (7), we also consider the following system in which admissible are only the "extremal" ("bang-bang") controls.

$$(8)\qquad \left\{\begin{array}{r} \frac{\partial x}{\partial t} - \sum\limits_{k=1}^{N} \frac{\partial}{\partial z_k}\big(a(t,z)|\frac{\partial x}{\partial z_k}|^{p-2}\frac{\partial x}{\partial z_k}\big) + \beta x|x|^{p-2} = \\ = f\big(t,z,x(t,z)\big) + \big(b(t,z),u(t,z)\big) \text{ a.e.}, \\ x(0,z) = x_0(z),\ x\big|_{T\times\Gamma} = 0, \\ u(t,z) \in \operatorname{ext} U(t,z) \text{ a.e.},\ u(\cdot,\cdot) \text{ is measurable.} \end{array}\right\}$$

We will need the following hypotheses on the data of (7):

$H(f)$: $f\colon T \times Z \times \mathbb{R} \to \mathbb{R}$ is a function s.t.

(1) $(t,z) \to f(t,z,x)$ is measurable,

(2) $|f(t,z,x) - f(t,z,y)| \leqslant k(t,z)|x-y|$ a.e. with $k(\cdot,\cdot) \in L^1(T \times Z)$,

(3) $|f(t,z,x)| \leqslant \alpha(t,z) + \beta(t,z)|x|$ a.e. with $\alpha \in L^2(T \times Z)$, $\beta \in L^\infty(T, L^2(Z))$.

$H(a)$: $a\colon T \times Z \to \mathbb{R}$ is a function s.t. $0 < m_1 \leqslant a(t,z) \leqslant m_2$, $|a(t,z) - a(s,z)| \leqslant \eta(z)|t-s|$ a.e. with $\eta(\cdot) \in L^\infty(Z)$.

$H(b)$: $b \in L^\infty(T \times Z, \mathbb{R}^l)$.

$H(U)$: $U\colon T \times Z \to P_{kc}(\mathbb{R}^l)$ is a measurable multifunction s.t. $|U(t,z)| = \sup\{\|u\|: u \in U(t,z)\} \leqslant \theta(t)$ a.e. with $\theta(\cdot) \in L^2(Z)$.

In the partial differential equation $\big(b(t,z),u(t,z)\big)$ denotes the inner product in $\mathbb{R}^l$ of the vectors $b(t,z)$ and $u(t,z)$.

As before by $S(x_0)$ (resp. $S_e(x_0)$) we denote the set of admissible state trajectories of (7) (resp. of (8)). We have that $S_e(x_0) \subseteq S(x_0) \subseteq C(T, L^2(Z))$.

Theorem 6.1. *If hypotheses $H(f)$, $H(a)$, $H(b)$, $H(U)$ hold and $x_0(\cdot) \in W_0^{1,p}(Z)$, then $S_e(x_0)$ is a dense G_δ-subset of $S(x_0)$.*

Proof. Let $H = L^2(Z)$ and define $\varphi\colon T \times H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ by

$$\varphi(t,x) = \begin{cases} \frac{1}{p} \sum\limits_{k=1}^{N} \int_Z a(t,z) |\frac{\partial x}{\partial z_k}|^p + \frac{\beta}{p} \int_Z |x|^p \, dz & \text{if } x \in W_0^{1,p}(Z), \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly $\varphi(t,\cdot) \in \Gamma_0(H)$ and note that

$$\{x \in H \colon \|x\|_2^2 + \varphi(t,x) \leqslant \lambda\}$$

is bounded in $W_0^{1,p}(Z)$. Since $W_0^{1,p}(Z)$ embeds into $L^2(Z)$ compactly (Sobolev's embedding theorem), we see that $\varphi(t,\cdot)$ is of compact type. Furthermore using hypothesis $H(a)$ and recalling that $\left(\int_Z \sum\limits_{k=1}^{N} |\frac{\partial x}{\partial z_k}|^p \, dz\right)^{1/p}$ is an equivalent norm on $W_0^{1,p}(Z)$, we get that $\varphi(t,x)$ satisfies hypothesis $H(\varphi)$. Then as in Barbu [5], we can check that

$$\partial\varphi(t,x) = -\frac{1}{p} \sum_{k=1}^{N} \frac{\partial}{\partial z_k} \left(a(t,z) \left| \frac{\partial x}{\partial z} \right|^{p-2} \frac{\partial x}{\partial z_k} \right) + \beta x |x|^{p-2} = L_p^\beta(x)$$

with $x \in D_p = \{y \in W_0^{1,p}(Z) \colon L_p^\beta(y) \in L^2(Z) = H\}$.

Also let $\hat{U}(t) = \{v \in L^2(Z, \mathbb{R}^l) \colon v(z) \in U(t,z) \text{ a.e.}\}$. Because of hypothesis $H(U)$, $\hat{U}\colon T \to P_{wkc}(L^2(Z,\mathbb{R}^l))$ is measurable and $|\hat{U}(t)| = \sup\{\|v\|_2 \colon v \in \hat{U}(t)\} \leqslant \theta(t)$ a.e. with $\theta(\cdot) \in L_+^2$. From Benamara [6] we know that $\operatorname{ext} \hat{U}(t) = \{v \in L^2(Z, \mathbb{R}^l) \colon v(z) \in \operatorname{ext} U(t,z) \text{ a.e.}\}$.

Let $\hat{f}\colon T \times H \to H$ be the Nemitsky operator corresponding to $f(t,z,x)$ (i.e. $\hat{f}(t,x)(\cdot) = f(t,\cdot,x(\cdot))$). It is well-known that $\hat{f}(t,\cdot)$ is continuous, while if $h \in H$, then $\left(\hat{f}(t,x), h\right)_{L^2(Z)} = \int_Z f(t,z,x(z)) h(z) \, dz$. So by Fubini's theorem, $t \to \left(\hat{f}(t,x), h\right)_{L^2(Z)}$ is measurable $\Rightarrow t \to \hat{f}(t,x)$ is weakly measurable and since $L^2(Z)$ is separable, by the Pettis measurability theorem (see for example Diestel-Uhl [10], p. 40), we get that $t \to \hat{f}(t,x)$ is measurable from T into $L^2(Z) = H$. Then define $F\colon T \times H \to P_{wkc}(H)$ by

$$F(t,x) = \hat{f}(t,x) + \bigcup_{u \in \hat{U}(t)} \left(\hat{b}(t)(\cdot), u(\cdot)\right)$$

where $\hat{b}(t)(\cdot) = b(t,\cdot) \in L^\infty(Z,\mathbb{R}^l)$. As for $\hat{f}(t,x)$, via the Pettis measurability theorem, we can check that $t \to \hat{b}(t)$ is measurable from T into $L^2(Z,\mathbb{R}^l)$. So $(t,u) \to w(t)u = \big(\hat{b}(t)(\cdot), u(\cdot)\big)$ from $T \times L^2(Z,\mathbb{R}^l)$ into H is clearly measurable in t, continuous in u, hence jointly measurable. Since $\hat{U}(\cdot)$ is measurable, we can find $\hat{u}_n\colon T \to L^2(Z,\mathbb{R}^l) n \geqslant 1$ measurable maps s.t. $\hat{U}(t) = \overline{\{\hat{u}_n(t)\}}_{n\geqslant 1}$ (see Wagner [24], Theorem 4.2). Then for any $h \in H$ we have

$$\sigma\big(h, F(t,x)\big) = \big(\hat{f}(t,x), h\big)_{L^2(Z)} + \sup_{n\geqslant 1} \big(w(t)\hat{u}_n(t), h\big)_{L^2(Z)}$$
$$\Rightarrow t \to \sigma\big(h, F(t,x)\big) \text{ is measurable.}$$

But note that $\operatorname{Gr} F(\cdot,x) = \bigcap_{m\geqslant 1} \{z \in H\colon (h_m, z) \leqslant \sigma\big(h_m, F(t,x)\big)\}$, where $\{h_m\}_{m\geqslant 1}$ is dense in H. So $\operatorname{Gr} F(\cdot,x) \in B(T) \times B(H) \Rightarrow t \to F(t,x)$ is measurable (see Wagner [24]). Also note that because of hypothesis $H(f)$

$$h\big(F(t,x), F(t,y)\big) \leqslant \hat{k}(t)\|x-y\|_2$$

with $\hat{k}(t) = k(t,\cdot)$. Finally we also have

$$|F(t,x)| \leqslant \hat{\alpha}(t) + \hat{\beta}(t)\|x\| \text{ a.e.}$$

with $\hat{\alpha}(t) = \|\alpha(t,\cdot)\|_{L^2(Z)}, \hat{\beta}(t) = \|\beta(t,\cdot)\|_{L^2(Z)} \in L^2_+$. So we have satisfied hypothesis $H(F)_1$. Using Aumann's selection theorem (see Wagner [24], Theorem 5.10), we can easily check that (7) is equivalent to the following evolution inclusion (deparametrized (control-free) system):

$$\left\{\begin{array}{c} -\dot{x}(t) \in \partial\varphi\big(t, x(t)\big) + F\big(t, x(t)\big) \text{ a.e.}, \\ x(0) = \hat{x}_0 = x_0(\cdot). \end{array}\right\}$$

Similarly the equivalent evolution inclusion formulation of (8) is the following:

$$\left\{\begin{array}{c} -\dot{x}(t) \in \partial\varphi\big(t, x(t)\big) + \operatorname{ext} F\big(t, x(t)\big) \text{ a.e.}, \\ x(0) = \hat{x}_0 = x_0(\cdot). \end{array}\right\}$$

Applying Theorem 4.1, we get that $S_e(x_0)$ is a dense G_δ-subset of $S(x_0)$. □

Remark. If for almost all (t,z), $U(t,z)$ has a finite set of extreme points, then Theorem 6.1 above tells us that we can approximate any trajectory of (7) in the $C\big(T, L^2(Z)\big)$-norm, with states generated by admissible controls that take only a finite number of values.

References

[1] *H. Attouch*: Variantional Convergence for Functionals and Operators. Pitman, London, 1984.

[2] *J.-P. Aubin and A. Cellina*: Differential Inclusions. Springer, Berlin, 1984.

[3] *J.-P. Aubin and J. Ekeland*: Applied Nonlinear Analysis. Wiley, New York, 1983.

[4] *E. Balder*: Necessary and sufficient conditions for L_1-strong-weak lower semicontinuity of integral functionals. Nonl. Anal. — TMA *11* (1987), 1399-1404.

[5] *V. Barbu*: Nonlinear Semigroups and Differential Equations in Banach Spaces. Noordhoff International Publishing, Leyden, The Netherlands, 1976.

[6] *M. Benamara*: Points Extremaux Multi-applications et Fonctionelles Integrales. These du 3ème cycle, Université de Grenoble, 1975.

[7] *H. Brezis*: Operateurs Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert. North Holland, Amsterdam, 1973.

[8] *G. Choquet*: Lectures on Analysis, Vol. II. Benjamin, New York, 1969.

[9] *F. S. DeBlassi and G. Pianigiani*: Non-convex valued differential inclusions in Banach spaces. J. Math. Anal. Appl. *157* (1991), 469–494.

[10] *J. Diestel and J. Uhl*: Vector Measures, Math Surveys, Vol. 15. A.M.S., Providence, R.I., 1977.

[11] *A. Fryszkowski*: Continuous selections for a class of nonconvex multivalued maps. Studia Math *78* (1983), 163–174.

[12] *C. Henry*: Differential equations with discontinuous right-hand side for planning procedures. J. Econ. Theory *4* (1972), 545–551.

[13] *F. Hiai and H. Umegaki*: Integrals, conditional expectations and martingales of multivalued functions. J. Multiv. Anal. *7* (1977), 149–182.

[14] *R. Holmes*: Geometric Functional Analysis and its Applications. Springer, Berlin, 1975.

[15] *E. Klein and A. Thompson*: Theory of Correspondences. Wiley, New York, 1984.

[16] *D. Kravvaritis and N. S. Papageorgiou*: Multivalued perturbations of subdifferential type evolution equations in Hilbert spaces. J. Diff. Eqns. *76* (1988), 238–255.

[17] *J.-J. Moreau*: Evolution problem associated with a moving convex set in a Hilbert space. J. Diff. Eqns. *26* (1977), 347–374.

[18] *N. S. Papageorgiou*: On measurable multifunctions with applications to random multivalued equations. Math. Japonica *32* (1987), 437–464.

[19] *N. S. Papageorgiou*: Convergence theorems for Banach space valued integrable multifunctions. intern. J. Math and Math. Sci. *10* (1987), 433–442.

[20] *N. S. Papageorgiou*: Differential inclusions with state constraints. Proceedings of the Edinburgh Math. Soc. *32* (1989), 81–98.

[21] *N. S. Papageorgiou*: On evolution inclusions associated with time dependent convex subdifferentials. Comm. Math. Univ. Carol. *31* (1990), 517–527.

[22] *A. Plis*: Trajectories and quasi-trajectories of an orientor field. Bull. Acad. Polon. Sci. *10* (1962), 529–531.

[23] *A. Tolstonogov*: Extreme continuous selectors of multivalued maps and the bang-bang principle for evolution inclusions. Soviet Math. *317* (1991), 1–8.

[24] *D. Wagner*: Survey of measurable selection theorems. SIAM J. Cont. Optim. *15* (1977), 859–903.

[25] *J. Watanabe*: On certain nonlinear evolution equations. J. Math. Soc. Japan *25* (1973), 446–463.

[26] *S. Yotsutani*: Evolution equations associated with subdifferentials. J. Math. Soc. Japan *31* (1978), 623–646.

Author's address: Florida Institute of Technology, Department of Applied Mathematics, 150 West University Blvd., Melbourne, Florida 32901-6988, U.S.A.