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ON ABELIAN GROUPS BY WHICH BALANCED EXTENSIONS OF A RATIONAL GROUP SPLIT II

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Let R be a proper subgroup of the additive group Q of the rational numbers that contains Z, and let $\mathbf{t}$ denote its type. In an earlier paper [BF] we proposed the problem of finding all torsion-free abelian groups A—which we called *R-groups*—such that

$$\operatorname{Bext}^1(A, R) = 0.$$

Here $\operatorname{Bext}^1(A, R)$ stands for the group of all balanced extensions of R by A. In [BF] those R-groups A were described whose elements had types $\leqslant \mathbf{t}$. This paper is a continuation of [BF] and is devoted to the discussion of the general case: here no assumption is made concerning the types of elements in A.

It should be pointed out rightaway that in view of the recent results contained in Fuchs-Magidor [FM], the main theorems in [BF] can be stated more generally, without putting the restriction that the R-groups with elements of types $\leqslant \mathbf{t}$ have cardinality $\leqslant \aleph_\omega$. In particular, Lemma 6.2 and Theorem 7.1 hold for all cardinals κ whenever V = L is assumed. In other words, V = L *implies that a torsion-free group of arbitrary cardinality whose non-zero elements have types* $\leqslant \mathbf{t}$ *is an R-group if and only if the group* $\breve{A}$ (see below) *is a Butler group.*

In the present paper, the subgroup $A^{**}(\mathbf{t})$ generated by all the elements in A whose types are either $> \mathbf{t}$ or are incomparable with $\mathbf{t}$ plays an important role. Assuming V = L, completely satisfactory results will be obtained for arbitrary R-groups A in case there is a separative chain from $A^{**}(\mathbf{t})$ to A, e.g., if $A^{**}(\mathbf{t})$ is countable or is a balanced subgroup in A. More specific results can be stated on groups in which the types of elements are comparable with $\mathbf{t}$, in particular, if all their types are $\geqslant \mathbf{t}$.

All groups in this paper are torsion-free abelian groups. As customary, $\chi(a), \mathbf{t}(a)$ will denote the characteristic and the type of an element a in a given group G. We shall also use the notation $\pi = \{\text{primes } p \mid pR = R\}$ and $\pi' = \{\text{primes } p \mid pR \neq R\}$.

Let $\breve{A} = A \otimes R_0$ be the localization of A at the collection of primes π'; here R_0 denotes the group of all rational numbers whose denominators are divisible only by primes in π. Lemma 1.6 in [BF] states:

1.1. Lemma. *A torsion-free group A is an R-group if and only if $\breve{A} = A \otimes R_0$ is an R-group.*

We shall write $A^{**}(\mathbf{t})$ for the subgroup of A which is generated by all the elements in A whose types are either $> \mathbf{t}$ or are incomparable with $\mathbf{t}$. Note that always $A^{**}(\mathbf{t}) \otimes R_0 = (A \otimes R_0)^{**}(\mathbf{t})$.

The following example shows that $A = A^{**}(\mathbf{t})$ is not sufficient to ensure that A is an R-group.

1.2. Example. Assume that R_i $(i = 1, 2, 3)$ are rational groups of types $\mathbf{t}_i > \mathbf{t}$, where $\mathbf{t}_1 \cap \mathbf{t}_2 = \mathbf{t}_1 \cap \mathbf{t}_3 = \mathbf{t}_2 \cap \mathbf{t}_3 = \mathbf{t}$. Let $C = R_1x_1 \oplus R_2x_2 \oplus R_3x_3$, and let B be the pure subgroup of C generated by the element $x = x_1 + x_2 + x_3$. Thus $\mathbf{t}(B) = \mathbf{t}$. Then the group $A = C/B$ satisfies $A = A^{**}(\mathbf{t})$. It is readily checked that $0 \to B \to C \to A \to 0$ is a balanced-projective resolution of A which is not splitting. Hence A is not an R-group.

The following lemma is most useful.

1.3. Lemma. *In any R-group A with $\breve{A} = A$, $A^{**}(\mathbf{t})$ is a pure subgroup.*

Proof. Suppose that $A = \breve{A}$ is a torsion-free group, and $a \notin A^{**}(\mathbf{t})$ is an element in the purification of $A^{**}(\mathbf{t})$ in A. Then necessarily $\mathbf{t}(a) \leqslant \mathbf{t}$. Evidently, a can be chosen such that we can find a prime p and an element $r \in R$ with p not dividing either a or r. Since A/pA is an elementary p-group, we have $A/pA = \langle a + pA \rangle \oplus A'/pA$ for some subgroup A' of A of index p that contains $A^{**}(\mathbf{t})$. Define

$$G = \langle R \oplus A', r/p + a \rangle .$$

This group fits into the exact sequence $0 \to R \to G \xrightarrow{\varphi} A \to 0$ where φ acts as the identity on A' and maps the additional generator of G onto a. Note that $G^{**}(\mathbf{t}) = A^{**}(\mathbf{t})$, while the elements of A' not in $A^{**}(\mathbf{t})$ retain their types in G. The generator $r/p + a$ has the same type as a. Hence it follows that, under φ, each element of A has a preimage of the same type, so the sequence is balanced. By way of contradiction, suppose that it splits, say, $G = R \oplus C$ for some subgroup C of G. As $A^{**}(\mathbf{t})$ is generated by elements of types $> \mathbf{t}$ and of types incomparable with $\mathbf{t}$, we have $\mathrm{Hom}(A^{**}(\mathbf{t}), R) = 0$. This shows that $G^{**}(\mathbf{t})$ is fully invariant in G, consequently, $G^{**}(\mathbf{t}) = G^{**}(\mathbf{t}) \cap R \oplus G^{**}(\mathbf{t}) \cap C$. Hence $G^{**}(\mathbf{t}) \leqslant C$ follows. But then also $a \in C$, whence $r/p + a \in G$ implies $r/p \in R$, a contradiction. We conclude that A can not be an R-group. □

We can now prove another relevant property. Note that by the preceding lemma the factor group $A/A^{**}(\mathbf{t})$ is torsion-free whenever $A = \breve{A}$.

1.4. Lemma. *If $A = \breve{A}$ is an R-group, then all non-zero elements of the factor group $A/A^{**}(\mathbf{t})$ have types $\leqslant \mathbf{t}$.*

Proof. If not, then there exists a pure subgroup B of A that contains $A^{**}(\mathbf{t})$, but not as a summand, and $B/A^{**}(\mathbf{t})$ is of rank 1 but not of type $\leqslant \mathbf{t}$. There is an element $a \in B \setminus A^{**}(\mathbf{t})$ such that for some prime $p \in \pi', p$ does not divide a but divides the coset $a + A^{**}(\mathbf{t})$; note that a has type $\leqslant \mathbf{t}$.

Define the subgroup A' and the group G in the same way as it was done in the proof of (1.3). As above, it follows that $A^{**}(\mathbf{t}) = G^{**}(\mathbf{t})$, and the additional generator $r/p + a$ has the same type as a. If $G = R \oplus C$ for some subgroup C of G, then as above we obtain $G^{**}(\mathbf{t}) \leqslant C$. Hence $G/G^{**}(\mathbf{t}) = (R + G^{**}(\mathbf{t}))/G^{**}(\mathbf{t}) \oplus C/G^{**}(\mathbf{t})$. Type consideration shows that in this factor group, $B/G^{**}(\mathbf{t})$ is fully invariant, so it is contained in the second summand, i.e. $B \leqslant C$. The proof can now be completed by arguing similarly as above that $r/p + a \in G$ implies $r/p \in R$, a contradiction. □

1.5. Lemma. *If $A = \breve{A}$ is an R-group, then the factor group $A/A^{**}(\mathbf{t})$ is again an R-group.*

Proof. The (pure) exact sequence $0 \to A^{**}(\mathbf{t}) \to A \to A/A^{**}(\mathbf{t}) \to 0$ induces the exact sequence $\mathrm{Hom}(A^{**}(\mathbf{t}), R) \to \mathrm{Ext}^1(A/A^{**}(\mathbf{t}), R) \to \mathrm{Ext}^1(A, R)$. Here Hom vanishes, since $A^{**}(\mathbf{t})$ is generated by elements of types larger than or incomparable with $\mathbf{t}$. Consequently, the map between the two Ext groups is monic. It follows that its restriction $\mathrm{Bext}^1(A/A^{**}(\mathbf{t}), R) \to \mathrm{Bext}^1(A, R) = 0$ is likewise monic [DHR, 4.1]. But this means that $A/A^{**}(\mathbf{t})$ too is an R-group, as asserted. □

The last two lemmas imply that for an arbitrary R-group A satisfying $A = \breve{A}$, the factor group $A/A^{**}(\mathbf{t})$ is an R-group with elements of types $\leqslant \mathbf{t}$. The structure of these groups was investigated in [BF], and from the results proved there we can conclude

2.1. Theorem. *Let A be an R-group. Under the hypothesis $V = L$, the localization of the factor group $A/A^{**}(\mathbf{t})$ at the primes in π' is a Butler group with element of types $\leqslant \mathbf{t}$.*

We turn our attention to the subgroup $A^{**}(\mathbf{t})$. In a simplified form, we focus on torsion-free groups A such that $A = A^{**}(\mathbf{t})$. Consider a balanced-projective resolution $0 \to B \to C \to A \to 0$ of A, where C is completely decomposable. It is readily seen that in a fixed direct decomposition of C, the direct sum of the rank one summands of types $> \mathbf{t}$ and of types incomparable with $\mathbf{t}$ maps surjectively upon A. Hence we may as well assume that all the rank one summands of C have either types $> \mathbf{t}$ or types incomparable with $\mathbf{t}$.

The following result contains a characterization of R-groups $A = A^{**}(\mathbf{t})$ in terms of their balanced-projective resolutions.

2.2. Theorem. *Let A be a torsion-free group such that $A = A^{**}(\mathbf{t})$. Then A is an R-group if and only if $Hom(B, R) = 0$ holds for some balanced-projective resolution $0 \to B \to C \to A \to 0$ of A where C is completely decomposable with rank 1 summands of types larger than or incomparable with $\mathbf{t}$.*

P r o o f. To start with, observe that the analogue of the well-known Schanuel's lemma, applied to balanced-projective resolutions, implies that the stated condition does not depend on the chosen resolution. In fact, if $0 \to B' \to C' \to A \to 0$ is another balanced-projective resolution of A of the stated kind, then $B \oplus C' \cong B' \oplus C$ and hence $\operatorname{Hom}(B, R) \cong \operatorname{Hom}(B', R)$ follows.

For the proof, note that the given balanced-exact sequence implies the exactness of the induced sequence $0 = \operatorname{Hom}(C, R) \to \operatorname{Hom}(B, R) \to \operatorname{Bext}^1(A, R) \to \operatorname{Bext}^1(C, R) = 0$. Hence the claim is evident. □

(1.2) exhibits a typical example where $\operatorname{Hom}(B, R) \neq 0$.

3. R-GROUPS A WITH SEPARATIVE CHAINS FROM $A^{**}(\mathbf{t})$

In view of the last results our main concern is to find out whether or not the subgroup $A^{**}(\mathbf{t})$ of an R-group A is an R-group, and how the group A is put together from $A^{**}(\mathbf{t})$ and $A/A^{**}(\mathbf{t})$. Under an additional hypothesis on $A^{**}(\mathbf{t})$ (which is satisfied in several special cases) more complete results can be established.

A pure subgroup B of a torsion-free group G is called *separative* if for each $g \in G$ there is a countable subset $\{b_n \mid n < \omega\}$ in B such that $\{\chi(g+b_n) \mid n < \omega\}$ is a cofinal subset of $\{\chi(g+b) \mid b \in B\}$. A *separative chain* from B to G is a continuous well-ordered ascending chain $B = G_0 < G_1 < \ldots < G_\nu < \ldots$ of separative subgroups with union G where all the factor groups $G_{\nu+1}/G_\nu$ are countable. If $B = 0$, we simply talk of *a separative chain* in G.

Note that—assuming V = L—such a separative chain exists from any countable subgroup B of A to the group A itself. In fact, by [DHR], a countable subgroup B can be embedded in a balanced subgroup C of A which has cardinality $\leqslant \aleph_1$. By [FM], under the assumption of V = L, the group A/C admits a separative chain; evidently, this lifts to a separative chain from C to A. Since countable subgroups are separative, this chain can be refined by including B in it, so as to get a separative chain from B up to A.

3.1. Lemma. *Let $A = \breve{A}$ be an R-group such that there is a separative chain from $A^{**}(\mathbf{t})$ to A. Then every subgroup in this separative chain, including $A^{**}(\mathbf{t})$ itself, is an R-group.*

Proof. Considering that the elements of A not in $A^{**}(\mathbf{t})$ are all of types $\leqslant \mathbf{t}$, we can argue as in [BF, 3.3–3.5]. We conclude that all members of this separative chain are R-groups. In particular, $A^{**}(\mathbf{t})$ is an R-group. □

By a *regular* subgroup of a torsion-free group A we mean a subgroup K such that each element $x \in K$ has the same type in K as in A; see Bican [B]. A pure subgroup B of a torsion-free group A is said to be *prebalanced* in A if for every rank 1 pure subgroup C/B of A/B there is a finite rank Butler group G such that $C = B + G$; cf. [FV].

3.2. Lemma. *Let A be a torsion-free group, and C a separative subgroup of corank 1 in A. If C is not prebalanced in A, then there is a regular subgroup K of A containing C such that either $A/K \simeq Z(p^\infty)$ for some prime p, or $A/K \simeq \oplus Z(p_n)$ for infinitely many different primes p_n.*

Proof. Set $B = C + \langle b \rangle_*$ with some $b \in A \setminus C$. By separativeness, there is a countable set $x_1, x_2, \ldots, x_m, \ldots$ of elements of A not in B such that for every

$x \in A \setminus B$ we can find an index m satisfying both $x_m \in x + B$ and $\chi(x) \leqslant \chi(x_m)$. If C is not prebalanced in A, then for each $m, C_m = B + \langle x_1 \rangle_* + \ldots + \langle x_m \rangle_*$ is a proper subgroup of A and A/C_m is an infinite torsion group, say $A/C_m \simeq \oplus Z(p_n^{k_n})$ for different primes p_n where each exponent k_n is either a positive integer or ∞.

If some k_n is ∞, then it suffices to imbed C_m in a subgroup K such that $A/K \simeq Z(p^\infty)$, and we are done.

If all the k_n are finite, then for every m we pick a prime p_m such that the primes $p_1, \ldots, p_m, \ldots$ are all distinct and there is a subgroup $K_m \geqslant C_m$ with $A/K_m \simeq Z(p_m)$. Setting $K = \bigcap_m K_m$, it remains to show that K is regular in A.

Given $x \in K \setminus C$, clearly, $x_m \in x + B$ and $\chi(x) \leqslant \chi(x_m)$ for an integer m. Then $\langle x_m \rangle_* \leqslant \bigcap_{m \leqslant n} K_n = K'$ implies $\langle x \rangle_* \leqslant K'$. Manifestly, $A/K \leqslant A/K_1 \oplus \ldots \oplus A/K_{m-1} \oplus A/K'$. This shows that the p-height of x in K' is the same as in A for all primes p with possible exception of the primes $p_1, \ldots, p_{m-1}$ where the heights might increase by 1. Consequently, the characteristics of x in K and in A can differ only at a finite set of primes where both are finite. □

We shall now concentrate on R-groups A with $pA = A$ for all $p \in \pi$.

3.3. Lemma. *Let A be an R-group such that $pA = A$ for all $p \in \pi$, and suppose that $A^{**}(\mathbf{t})$ is a separative subgroup in A. If $A^{**}(\mathbf{t})$ is not prebalanced in A, then there is a pure subgroup B of A which contains $A^{**}(\mathbf{t})$ with $B/A^{**}(\mathbf{t})$ of rank one such that B is not an R-group.*

Proof. Evidently, $A/A^{**}(\mathbf{t})$ has a rank one pure subgroup $B/A^{**}(\mathbf{t})$ such that $A^{**}(\mathbf{t})$ is not prebalanced in B. Hence the preceding lemma yields the existence of a regular subgroup K of B that contains $A^{**}(\mathbf{t})$ and B/K is as stated there. In the first alternative of (3.2), $p \in \pi'$, since for every $p \in \pi$ all pure subgroups of A are p-divisible. We infer that—no matter what B/K is like—there is a group R^* of type $> \mathbf{t}$ such that $R < R^* < Q$ and there is an isomorphism ϱ: $B/K \cong R^*/R$. (1.4) implies that $B/A^{**}(\mathbf{t})$ can be embedded in R^*.

We now form the following diagram where α is the natural map, δ is the natural map $B/A^{**}(\mathbf{t}) \to B/K$ followed by ϱ, while G and H are obtained successively via pullbacks:

$$
\begin{array}{ccccccccc}
 & & & & & & A^{**}(\mathbf{t}) & & \\
 & & & & & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & R & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & R & \longrightarrow & G & \longrightarrow & B/A^{**}(\mathbf{t}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & R & \longrightarrow & R^* & \xrightarrow{\gamma} & R^*/R & \longrightarrow & 0\,.
\end{array}
$$

It is important to choose γ in such a way that the middle row fails to split. Such a choice is possible, since $\mathrm{Hom}(B/A^{**}(\mathbf{t}), R^*)$ is countable, while the automorphism group of the infinite torsion group R^*/R has the power of the continuum.

If J is a rank one pure subgroup of B, then by the regularity of $K = \mathrm{Ker}\,\delta\alpha$ in B the group $\delta\alpha J$ is finite, and so [BF, 4.2, 4.3] guarantees the balancedness of the top row. To complete the proof, we must show that the top row does not split. By way of contradiction, assume it does, i.e. some $\varphi\colon B \to R^*$ satisfies $\gamma\varphi = \delta\alpha$. But $\gamma\varphi\beta = \delta\alpha\beta = 0$ (for the inclusion map $\beta\colon A^{**}(\mathbf{t}) \to B$) implies that φ maps $\beta A^{**}(\mathbf{t})$ into R. Thus $\varphi\beta A^{**}(\mathbf{t}) = 0$. Therefore, φ induces a map $\psi\colon B/A^{**}(\mathbf{t}) \to R^*$ such that $\varphi = \psi\alpha$. We conclude that $\gamma\psi\alpha = \gamma\varphi = \delta\alpha$ whence $\gamma\psi = \delta$, α being an epimorphism. This shows that the middle row splits, contradicting the choice of the homomorphism γ. □

3.4. Lemma. *Let A be an R-group satisfying $A = \check{A}$. If there is a separative chain from $A^{**}(\mathbf{t})$ to A, then $A^{**}(\mathbf{t})$ is a prebalanced subgroup.*

P r o o f. This is an immediate consequence of (3.1) and (3.3). □

We have arrived at one of the main results in this paper.

3.5. Theorem. *($V = L$) Let A be a group such that $A = \check{A}$ and there is a separative chain from $A^{**}(\mathbf{t})$ to A. A is an R-group if and only if*

(a) *$A^{**}(\mathbf{t})$ is an R-group;*

(b) *$A^{**}(\mathbf{t})$ is prebalanced in A;*

(c) *$A/A^{**}(\mathbf{t})$ is a Butler group with elements of types $\leqslant \mathbf{t}$ only.*

P r o o f. Suppose A is as stated and is an R-group. (a) and (b) follow from (3.1) and (3.4), respectively, while (c) is a consequence of (1.4), (1.5), and [BF, 7.1].

Conversely, assume that A satisfies (a)-(c). Condition (c) ensures the existence of a continuous well-ordered ascending chain $0 = A_0/A^{**}(\mathbf{t}) < A_1/A^{**}(\mathbf{t}) < \ldots < A_\nu/A^{**}(\mathbf{t}) < \ldots$ with rank 1 quotients $A_{\nu+1}/A_\nu$ and with union $A/A^{**}(\mathbf{t})$ such that, for each ν, $A_{\nu+1} = A_\nu + G_\nu$ where G_ν is a finite rank Butler group. (b) implies that all the subgroups in the chain $A_0 = A^{**}(\mathbf{t}) < A_1 < \ldots < A_\nu < \ldots$ with union A are prebalanced in A.

Let $0 \to R \to G \xrightarrow{\varphi} A \to 0$ be a balanced-exact sequence. We want to define a map $\psi \colon G \to R$ which is the identity on R. Setting $G_\nu = \varphi^{-1}(A_\nu)$, we obtain a chain $A^{**}(\mathbf{t}) = G_0 < G_1 < \ldots < G_\nu < \ldots$ with union G where the subgroups G_ν are prebalanced in G. Therefore, for each ν, we can write $G_{\nu+1} = G_\nu + J_1 + \ldots + J_k$ for a finite number of rank one subgroups J_i of G which are evidently all of types $\leqslant \mathbf{t}$. [BF, 2.1] guarantees that G_ν is $\mathbf{t}$-cobalanced in $G_{\nu+1}$. Condition (a) implies that there is a map $\psi_0 \colon G_0 \to R$ which is the identity on R. In view of the $\mathbf{t}$-cobalancedness of the group G_ν in $G_{\nu+1}$, we can extend $\psi_0 \colon G_0 \to R$ successively by a straightforward induction to a map $\psi_\nu \colon G_\nu \to R$. The union of all these maps yields a desired homomorphism $\psi \colon G \to R$. Hence A is an R-group, indeed. □

Let us point out that pure subgroups of R-groups need not be R-groups, not even in the finite rank case. The following example is a finite rank Butler group which is not an R-group, but it is a subgroup of a completely decomposable group (which is trivially an R-group).

3.6. Example. Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be incomparable types (both $> \mathbf{t}$) such that $\mathbf{t}_1 \cap \mathbf{t}_2 = \mathbf{t}$. Assume that all of $\mathbf{t}, \mathbf{t}_1, \mathbf{t}_2$ are 0 at the prime p. Let R_i be subgroups of Q which are of rank 1 and of type $\mathbf{t}_i$, and let $a_i \in R_i, a \in R$ be elements not divisible by p. Then R is a subgroup of the group

$$G = \langle R \oplus R_1 \oplus R_2, (a + a_1 + a_2)/p \rangle ,$$

but R is not a summand of G. In fact, the argument used in (1.2) shows that a complementary summand would contain the fully invariant subgroup $R_1 \oplus R_2$ of G, and then the additional generator $(a + a_1 + a_2)/p$ could not belong to the direct sum. It is readily seen that G is a balanced extension of R by $A = \langle R_1 \oplus R_2, (a_1 + a_2)/p \rangle$. Therefore, $G \in \mathrm{Bext}^1(A, R)$, but G is not splitting.

Let $\{x_1, x_2, \ldots, x_n\}$ be a maximal independent set in a torsion-free group G of finite rank n. By the *inner type* $\mathrm{IT}(G)$ of G we mean the type $\mathbf{t}(x_1) \cap \ldots \cap \mathbf{t}(x_n)$. It is not difficult to see that $\mathrm{IT}(G)$ does not depend on the particular choice of the maximal independent system: it is an invariant of G; see e.g. Mutzbauer [M]. For the group A in (3.6) we have $\mathrm{IT}(A) = \mathbf{t}$.

We can easily characterize the R-groups A of rank 2. As we know, it is enough to consider those satisfying $pA = A$ for all $p \in \pi$.

Manifestly, if A is decomposable, then it is a direct sum of two rank 1 groups, and as such it is an R-group.

For indecomposable groups of rank 2, we have

3.7. Lemma. *Let A be an indecomposable torsion-free group of rank 2 such that $pA = A$ for all $p \in \pi$. It is an R-group if and only if*

(i) *in case the rank of $A^{**}(\mathbf{t})$ is $\leqslant 1$: A is a Butler group;*

(ii) *in case $A = A^{**}(\mathbf{t})$: A satisfies $IT(A) > \mathbf{t}$.*

Proof. If $A^{**}(\mathbf{t}) = 0$, then we are in the situation of [BF, 4.5], and the claim in (i) follows. If $A^{**}(\mathbf{t})$ has rank one, then by (3.5) A is a prebalanced extension rank one group by a rank one group, so it is a Butler group.

Next suppose that $A = A^{**}(\mathbf{t})$. Choose a balanced-projective resolution $0 \to B \to C \xrightarrow{\varphi} A \to 0$ of A such that $C = \oplus_{i \in I} C_i$, where rk $C_i = 1$, and $\{\varphi C_i \mid i \in I\}$ is precisely the list of all pure rank one subgroups of A of types $> \mathbf{t}$. The index set I is countable, so $B = \operatorname{Ker} \varphi$ is a countable Butler group, generated by certain rank one pure subgroups of C. By the choice of the resolution, a rank one pure subgroup X cannot be equal to any of the C_i, therefore $\mathbf{t}(X) = \mathrm{IT}(A)$. Consequently, B is an $\mathrm{IT}(A)$-homogeneous countable Butler group, and as such it is completely decomposable (see Bican [B]). We conclude that Hom $(B, R) = 0$ if and only if $\mathrm{IT}(A) > \mathbf{t}$. Hence, by (2.2), A is an R-group exactly if $\mathrm{IT}(A) > \mathbf{t}$. □

4. R-GROUPS WITH TYPES COMPARABLE WITH $\mathbf{t}$

We can get more information about R-groups in case the types of their elements are comparable with $\mathbf{t}$.

We shall require a couple of lemmas. The following one generalizes a result which leads to Baer's famous ubiquitous lemma.

4.1. Lemma. *Let B be a pure subgroup of the torsion-free group A such that some $a \in A \setminus B$ satisfies $\mathbf{t}(a) = \mathbf{t}(a+b)$ for all $b \in B$. Then the set $\{\chi(a+b) \mid b \in B\}$ is directed upward.*

Proof. Given any $b_1, b_2 \in B$, there are relatively prime integers m, n such that $\chi(m(a + b_1)) = \chi(n(a + b_2))$. If the integers u, v satisfy $mu + nv = 1$, then the element $c = u(m(a + b_1)) + v(n(a + b_2)) = a + (mub_1 + nvb_2) \in a + B$ obviously satisfies $\chi(c) \geqslant \chi(m(a + b_1)) \geqslant \chi(a + b_1) \cup \chi(a + b_2)$. □

4.2. Lemma. *Let A be a torsion-free group such that the types of elements of A are all comparable with $\mathbf{t}$. Then*

(a) *for any $a \in A \setminus A^*(\mathbf{t})_*$, $\mathbf{t}(a+b) = \mathbf{t}(a)$ for all $b \in A^*(\mathbf{t})_*$;*

(b) *the exact sequence*

$$0 \to A^*(\mathbf{t})_* \to A \to A/A^*(\mathbf{t})_* \to 0$$

is balanced-exact if and only if $\mathbf{t}(a) = \mathbf{t}(a + A^*(\mathbf{t})_*)$ for every $a \in A \setminus A^*(\mathbf{t})_*$.

Proof. (a) By hypothesis, $\mathbf{t}(a) \leqslant \mathbf{t} \leqslant \mathbf{t}(b)$ holds whence $\mathbf{t}(a) \leqslant \mathbf{t}(a+b)$ is immediate. The case $\mathbf{t} < \mathbf{t}(a+b)$ is impossible, since then $a + b \in A^*(\mathbf{t})$ and $a \in A^*(\mathbf{t})$, a contradiction. Thus $\mathbf{t}(a+b) \leqslant \mathbf{t}$. This together with $\mathbf{t} \leqslant \mathbf{t}(b)$ gives $\mathbf{t}(a+b) \leqslant \mathbf{t}(a)$.

Claim (b) is an immediate consequence of (a). □

4.3. Lemma. *Let A be a torsion-free group the types of whose elements are comparable with $\mathbf{t}$. Then the subgroup $B = A^*(\mathbf{t})_*$ is separative in A.*

Proof. Let C/B be a rank one pure subgroup of A/B. By (4.2), every $c \in C \setminus B$ has the same type, so by (4.1) the set $\{\chi(c+b) \mid b \in B\}$ is directed upward. Using this fact, the characteristic of $c+B$ (which is always the supremum of characteristics of countably many elements in this coset) can easily be obtained as the union of an *ascending* chain of characteristics $\chi(c+b_i)$ for a suitable sequence $b_1, \ldots, b_i, \ldots \in B$.

We claim that $\{\chi(c+b_i) \mid i < \omega\}$ is cofinal in $\{\chi(c+b) \mid b \in B\}$. In fact, given any $b \in B$, by (4.2) the characteristics $\chi(c+b)$ and $\chi(c+b_1)$ differ only at a finite number of primes where they are both finite. Hence choosing i large enough, we can make $\chi(c+b_i)$ at least as large as $\chi(c+b)$ at these primes. □

4.4. Lemma. *Let A be an R-group all of whose elements have types comparable with $\mathbf{t}$. Then the subgroup $A^*(\mathbf{t})$ is an R-group.*

Proof. From (1.3) we know that $A^*(\mathbf{t}) = A^{**}(\mathbf{t})$ is a pure subgroup of the R-group A. We select a subgroup M of A which is maximally disjoint from B. Manifestly, M is pure in A.

Let $0 \to R \to H \xrightarrow{\beta} A^*(\mathbf{t}) \to 0$ be a balanced-exact sequence. We can form the following commutative diagram with exact rows (the bottom sequence exists as Ext^2 vanishes for abelian groups):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & H \oplus M & \xrightarrow{\beta\oplus 1} & A^*(\mathbf{t}) \oplus M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mu} & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & N & \xrightarrow{\sigma} & A & \longrightarrow & 0
\end{array}$$

where the right hand vertical map is the inclusion map.

To verify the balancedness of the bottom row, pick an $a \in A$. As $A^*(\mathbf{t}) \oplus M$ is an essential subgroup in A, without loss of generality we may assume $a = b + x$ ($b \in A^*(\mathbf{t}), x \in M$). By hypothesis, $\beta h = b$ for some $h \in H$ with $\mathbf{t}(h) = \mathbf{t}(b)$. Because of (4.2), $\mathbf{t}(x) = \mathbf{t}(a - b) = \mathbf{t}(a)$, thus for $y = \mu(h + x)$ we have $\mathbf{t}(y) \geqslant \mathbf{t}(h) \cap \mathbf{t}(x) = \mathbf{t}(b) \cap \mathbf{t}(x) = \mathbf{t}(x) = \mathbf{t}(a) = \mathbf{t}(\sigma y) \geqslant \mathbf{t}(y)$ (note that $\sigma y = \sigma\mu(h + x) = b + x = a$). Therefore, the bottom row splits. This implies the splitting of the top row, hence $A^*(\mathbf{t})$ is an R-group, indeed. □

We are ready to state the following theorem:

4.5. Theorem. *($V = L$) Let A be a torsion-free group such that all elements of A have types comparable with $\mathbf{t}$. Suppose there exists a separative chain from $A^*(\mathbf{t})$ to A. Then A is an R-group if and only if*

(i) *the sequence $0 \to A^*(\mathbf{t}) \to A \to A/A^*(\mathbf{t}) \to 0$ is balanced-exact, and*

(ii) *both $A^*(\mathbf{t})$ and $A/A^*(\mathbf{t})$ are R-groups.*

Proof. (i) and (ii) imply that in the induced exact sequence

$$\mathrm{Bext}^1(A/A^*(\mathbf{t}), R) \to \mathrm{Bext}^1(A, R) \to \mathrm{Bext}^1(A^*(\mathbf{t}), R)$$

the groups at both ends vanish. Consequently, A ought to be an R-group.

Conversely, assume that A is an R-group. Claim (ii) is contained in (1.5) and (4.4). To verify (i), note that by (4.3), $A^*(\mathbf{t})$ is a separative subgroup of A, and therefore so is C whenever $C/A^*(\mathbf{t})$ is a rank one pure subgroup of $A/A^*(\mathbf{t})$ (countable extensions of separative subgroups are again separative). By hypothesis, there exists a separative chain from $A^*(\mathbf{t})$ to A; it is readily seen that such a chain can be modified so as to include C (cf. proof of [BF, 7.1]). As in [BF, 3.5] we now conclude that C is an R-group. Hence (i) follows at once in view of (4.2). □

Note that in the preceding theorem we can add that the factor group $A/A^*(\mathbf{t})$ must have elements of types $\leqslant \mathbf{t}$. In fact, this is a consequence of (i).

5. R-groups with no elements of types $< \mathbf{t}$

In case none of the elements of A is of type $< \mathbf{t}$, a completely satisfactory result can be stated.

5.1. Theorem. *($V = L$) A torsion-free group $A = \breve{A}$ with no elements of types $< \mathbf{t}$ is an R-group if and only if*

$$A = A^{**}(\mathbf{t}) \oplus C$$

where

(i) $A^{**}(\mathbf{t})$ *is an R-group;*

(ii) C *is a* t*-homogeneous completely decomposable group.*

P r o o f. Let A have the stated structure. Since completely decomposable groups are trivially R-groups, (i) and (ii) imply that A itself is an R-group.

Conversely, assume A is an R-group. Since its elements are either of types $\geqslant \mathbf{t}$ or incomparable with $\mathbf{t}$, every element of A which is not in $A^{**}(\mathbf{t})$ must be of type exactly $\mathbf{t}$. From (1.4) and (1.5) we conclude that the factor group $A/A^{**}(\mathbf{t})$ is a $\mathbf{t}$-homogeneous R-group. In view of [BF] it is therefore a $\mathbf{t}$-homogeneous completely decomposable group whenever V = L. By Baer's ubiquitous lemma, $A^{**}(\mathbf{t})$ is balanced in A, so $A^{**}(\mathbf{t})$ is a summand of A. Consequently, it is again an R-group. □

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