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OSCILLATION OF NEUTRAL DIFFERENTIAL EQUATIONS WITH OSCILLATING COEFFICIENTS*

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1. INTRODUCTION

Consider the first-order neutral delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - P(t)x(t-\tau)\right] + Q(t)x(t-\sigma) = 0 \tag{1}$$

where

$$\tau \in (0,\infty),\ \sigma \in \mathbb{R}^+,\ P \in C\big([t_0,\infty),\mathbb{R}\big) \text{ and } Q \in C\big([t_0,\infty),\mathbb{R}^+\big). \tag{2}$$

The oscillatory behaviour of Eq. (1) has been investigated by many authors. For example see [1–3, 5, 7–10]. For a survey, see [4]. Most of the papers mentioned above, however, concern Eq. (1) under the hypothesis

$$\int_{t_0}^{\infty} Q(s)\,\mathrm{d}s = \infty \tag{3}$$

which has played a key role in the study of the oscillation of Eq. (1). Considerably less is known about the oscillatory behaviour of the solutions of Eq. (1) when (3) is not satisfied or $Q(t)$ is oscillating. In the first case, we refer to a single paper [9]. On the other hand, we have not yet seen any papers dealing with the latter case when $Q(t)$ is oscillating. Moreover, when $P(t)-1$ is allowed to oscillate the corresponding study of Eq. (1) is also relatively scarce in the literature. For example, the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[x(t) - \Big(\frac{1}{2} + \sin t\Big)x(t-2\pi)\Big] + t^{-\beta}x(t) = 0, \quad t \geqslant 1$$

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is such an equation, where $\beta > 1$. From Theorem 1 in Section 2 we will see that every solution of this equation is always oscillatory for any $\beta > 1$. This is really a surprising result. It will impel us to further study for the case when $P(t) - 1$ is oscillating.

In Section 2 we establish sufficient conditions for the oscillation of all solutions of Eq. (1) when $P(t) - 1$ is oscillating. Several concrete examples are also introduced. In Section 3 we list some oscillation results of Eq. (1) when $Q(t)$ is allowed to oscillate. These results are new.

Let $m = \max\{\tau, \sigma\}$. By a solution of Eq. (1) we mean a function $x \in C([t_1 - m, \infty), \mathbb{R})$, for some $t_1 \geqslant t_0$, such that $x(t) - P(t)x(t-\tau)$ is continuously differentiable on $[t_1, \infty)$ and such that Eq. (1) is satisfied for $t \geqslant t_1$.

Assume that (2) holds, $t_1 \geqslant t_0$ and let $\varphi \in C([t_1 - m, t_1], \mathbb{R})$ be a given initial function. Then we can easily see by the method of steps that Eq. (1) has a unique solution $x \in C([t_1 - m, \infty), \mathbb{R})$ such that

$$x(t) = \varphi(t) \quad \text{for } t_1 - m \leqslant t \leqslant t_1.$$

As usual, a solution of Eq. (1) is called nonoscillatory if it is eventually positive or eventually negative. Otherwise, the solution is called oscillatory.

In the sequel, for the sake of convenience, we define

$$E[t_1, t_2] = \{t \colon t \in [t_1 + i\tau, t_2 + i\tau], \quad i = 0, 1, 2, \ldots\}$$

where $t_2 > t_1 \geqslant t_0$.

2. The case $P(t) - 1$ is oscillating

The main results in this section are the following two theorems.

Theorem 1. *Assume that* (2) *holds and that $P(t)$ is oscillatory. Further assume that there exist $t_2 > t_1 \geqslant t_0$ and $\alpha > 1$ such that*

$$P(t) \geqslant \alpha \quad \text{for } t \in E[t_1, t_2] \tag{4}$$

and that for any $\varepsilon > 0$

$$\int_{E[t_1+\sigma, t_2+\sigma]} Q(s) e^{\varepsilon s} \, ds = \infty. \tag{5}$$

Then every solution of Eq. (1) *oscillates.*

Theorem 2. *Assume that* (2), (4) *and* (5) *hold and that there exists a* $t^* \geqslant t_0$ *such that*

$$(6) \qquad P(t^* + i\tau) \leqslant 1 \quad \text{for } i = 0, 1, 2, \dots .$$

Then every solution of Eq. (1) *oscillates.*

Before we prove Theorems 1 and 2, let us first examine several examples.

Example 1. Consider the neutral delay differential equation

$$(7) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big[x(t) - \Big(\frac{1}{2} + \sin t\Big)x(t - 2\pi)\Big] + t^{-\beta}x(t - \sigma) = 0$$

where $\sigma \geqslant 0$, $\beta \in \mathbb{R}$. Clearly, $P(t) = \frac{1}{2} + \sin t$ is oscillatory and if $t_1 = \frac{1}{4}\pi$, $t_2 = \frac{3}{4}\pi$ and $\alpha = \frac{1}{2}(1 + \sqrt{2})$, then the condition (4) is satisfied. It is easy to see that for any $\varepsilon > 0$, $\beta \in \mathbb{R}$

$$\int_{E[t_1+\sigma, t_2+\sigma]} s^{-\beta}\mathrm{e}^{\varepsilon s}\,\mathrm{d}s = \infty.$$

Hence the condition (5) is satisfied. Therefore, by Theorem 1 every solution of Eq. (7) oscillates.

Example 2. The neutral delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[x(t) - t(\sin t)x(t - 4\pi)\Big] + t^{-\beta}(\ln^{-\gamma} t)x(t) = 0, \quad t \geqslant 2$$

where $\beta, \gamma \in \mathbb{R}$, satisfies all hypotheses of Theorem 1. Therefore every solution of this equation oscillates.

Example 3. The neutral delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[x(t) - \mathrm{e}^{\sin 2t}x(t - \pi)\Big] + t^{-\beta}x(t) = 0$$

where $\beta \in \mathbb{R}$, satisfies all hypotheses of Theorem 2. Therefore every solution of this equation oscillates.

Example 4. The neutral delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[x(t) - (2 + \sin t)x(t - 2\pi)\Big] + t^{-\beta}(1 + \sin t)x(t - \pi) = 0$$

satisfies all hypotheses of Theorem 2 for every $\beta \in \mathbb{R}$. Therefore every solution of this equation oscillates.

Although not all of the above four examples satisfy the condition (3) when $\beta > 1$, their solutions are still oscillatory. Further, we should note that none of the above four examples satisfies the oscillation conditions for Eq. (1) in the literature.

Now we start to prove Theorems 1 and 2. To this end, let us first state and prove a lemma which will enable us to complete the proof of Theorem 2.

Lemma 1. *Assume that* (2) *and* (6) *hold and that* $Q(t)$ *does not eventually equal zero identically. Let* $x(t)$ *be an eventually positive solution of Eq.* (1) *and set*

$$y(t) = x(t) - P(t)x(t-\tau). \tag{8}$$

Then we eventually have

$$y(t) > 0. \tag{9}$$

Proof. Let $T_1 \geqslant t_0$ be such that $x(t-m) > 0$ for $t \geqslant T_1$. Then by (1) and (8) we have

$$y'(t) = -Q(t)x(t-\sigma) \leqslant, \not\equiv 0, \quad \text{for } t \geqslant T_1$$

which implies that $y(t)$ is nonincreasing on $[T_1, \infty)$ and does not equal a constant eventually. Hence, if (9) does not hold, then eventually $y(t) < 0$. Therefore, there exist $T_2 > T_1$ and $M > 0$ such that

$$y(t) \leqslant -M \quad \text{for } t \geqslant T_2.$$

That is,

$$x(t) \leqslant -M + P(t)x(t-\tau), \quad t \geqslant T_2. \tag{10}$$

Now we choose a positive integer n^* to be such that $t^* + n^*\tau \geqslant T_2$. Then by (6) and (10) we get

$$x(t^* + n^*\tau + j\tau) \leqslant -M + x\big(t^* + n^*\tau + (j-1)\tau\big), \quad j = 0, 1, 2, \ldots,$$

which yields

$$x(t^* + n^*\tau + j\tau) \leqslant -M(j+1) + x\big(t^* + (n^*-1)\tau\big) \to -\infty \quad \text{as } j \to \infty,$$

which contradicts the fact that $x(t)$ is eventually positive. The proof is complete. □

Remark 1. Lemma 1 improves Lemma 1 in [2] by removing the hypothesis that $P(t)$ is bounded and by relaxing the hypothesis that $Q(t) \geqslant q > 0$ for $t \geqslant t_0$.

Proof of Theorem 1. Assume, by way of contradiction, that Eq. (1) has an eventually positive solution $x(t)$. Thus, there exists a $T^* \geqslant t_2$ such that $x(t-m) > 0$ for $t \geqslant T^*$. Set $y(t)$ as in (8). Then by (1) we see that $y(t)$ is nonincreasing for $t \geqslant T^*$. Since $P(t)$ is oscillatory, $y(t)$ must be positive on $[T^*, \infty)$, i.e.,

$$x(t) > P(t)x(t-\tau) \quad \text{for } t \geqslant T^*. \tag{11}$$

Now we choose a positive integer N such that $t_1 + N\tau \geqslant T^*$ and $t_2 + N\tau \geqslant \tau$. Then by (4) and (11) we get

$$x(t) > \alpha x(t-\tau) \quad \text{for } t \in E[t_1 + N\tau, t_2 + N\tau]. \tag{12}$$

Let

$$M = \min\left\{x(t)\colon t \in \left[t_1 + (N-1)\tau, t_2 + (N-1)\tau\right]\right\}.$$

Clearly, M is a positive constant. As $\alpha > 1$, it follows that there is an $\varepsilon > 0$ such that

$$\alpha \geqslant \mathrm{e}^{\varepsilon(t_2+N\tau)}.$$

Hence, we have

$$x(t) \geqslant M\mathrm{e}^{\varepsilon(t_2+N\tau)} \geqslant M\mathrm{e}^{\varepsilon t}, \quad t \in [t_1 + N\tau, t_2 + N\tau].$$

For $t \in \left[t_1 + (N+1)\tau, t_2 + (N+1)\tau\right]$ we have

$$x(t) \geqslant M\alpha^2 \geqslant M\mathrm{e}^{2\varepsilon(t_2+N\tau)} \geqslant M\mathrm{e}^{\varepsilon(t_2+(N+1)\tau)} \geqslant M\mathrm{e}^{\varepsilon t}.$$

Similarly, we can obtain in general that

$$x(t) \geqslant M\mathrm{e}^{\varepsilon t} \quad \text{for } t \in \left[t_1 + (N+i)\tau, t_2 + (N+i)\tau\right],\ i = 0, 1, 2, \dots$$

and so

$$x(t) \geqslant M\mathrm{e}^{\varepsilon t} \quad \text{for } t \in E[t_1 + N\tau, t_2 + N\tau],$$

which yields

$$x(t-\sigma) \geqslant M\mathrm{e}^{-\varepsilon\sigma}\mathrm{e}^{\varepsilon t}, \quad t \in E[t_1 + \sigma + N\tau, t_2 + \sigma + N\tau]. \tag{13}$$

Integrating (1) from $t_1 + \sigma + N\tau$ to t and noting (13), we have

$$\begin{aligned}
-y(t) + y(t_1 + \sigma + N\tau) &= \int_{t_1+\sigma+N\tau}^{t} Q(s)x(s-\sigma)\,\mathrm{d}s \\
&\geqslant \int_{[t_1+\sigma+N\tau,t]\cap E[t_1+\sigma+N\tau,t_2+\sigma+N\tau]} Q(s)x(s-\sigma)\,\mathrm{d}s \\
&\geqslant M\mathrm{e}^{-\varepsilon\sigma} \int_{[t_1+\sigma+N\tau,t]\cap E[t_1+\sigma+N\tau,t_2+\sigma+N\tau]} Q(s)\mathrm{e}^{\varepsilon s}\,\mathrm{d}s\,.
\end{aligned}$$

In view of (5) we have

$$\lim_{t\to\infty}\int_{[t_1+\sigma+N\tau,t]\cap E[t_1+\sigma+N\tau,t_2+\sigma+N\tau]} Q(s)\mathrm{e}^{\varepsilon s}\,\mathrm{d}s = \infty$$

and so $y(t) \to -\infty$ as $t \to \infty$, which is a contradiction. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Let $x(t)$ be an eventually positive solution of Eq. (1) and let $y(t)$ be defined by (8). Then by Lemma 1 we have eventually

$$y(t) > 0.$$

The rest of the proof is similar to that of Theorem 1. Here we omit it. □

3. The case $Q(t)$ may be oscillating

In this section we will present several sufficient conditions for the oscillation of all solutions of Eq. (1) when $Q(t)$ is allowed to oscillate. To this end, we need to establish the following three lemmas which are very interesting by themselves.

Consider the delay differential inequality

$$x'(t) + Q(t)x(t-\sigma) \leqslant 0, \quad t \in [a,b] \tag{14}$$

where $a < b$, $\sigma > 0$ and $Q \in C([a,b], \mathbb{R}^+)$.

Lemma 2. *Assume that $b - a \geqslant 2\sigma$ and*

$$\int_{b-\sigma}^{b} Q(s)\,\mathrm{d}s \geqslant 1. \tag{15}$$

Then (14) has no solutions $x(t)$ satisfying $x(t) > 0$ for $t \in [b-3\sigma, b]$.

Proof. If not, there is a solution $x(t)$ of (14) satisfying $x(t) > 0$ for $t \in [b-3\sigma, b]$. Then by (14) we have

$$x'(t) \leqslant -Q(t)x(t-\sigma) \leqslant 0, \quad t \in [b-2\sigma, b]$$

which implies that $x(t)$ is nonincreasing on $[b-2\sigma, b]$. Thus, integrating (14) from $b-\sigma$ to b, we have

$$x(b-\sigma) \geqslant x(b) + \int_{b-\sigma}^{b} Q(s)x(s-\sigma)\,\mathrm{d}s > x(b-\sigma)\int_{b-\sigma}^{b} Q(s)\,\mathrm{d}s$$

which yields

$$\int_{b-\sigma}^{b} Q(s)\,\mathrm{d}s < 1,$$

which contradicts (15) and so the proof is complete. □

Lemma 3. *Assume that there exist an integer $k \geqslant 3$ and $\beta \in [0,1)$ such that $b - k\sigma \geqslant a$ and*

$$\int_{t-\sigma}^{t} Q(s)\,\mathrm{d}s \geqslant \beta, \quad t \in \big[b-(k-2)\sigma, b\big]. \tag{16}$$

Let $x(t)$ be a solution of (14) *satisfying $x(t) > 0$ for $t \in \big[b-(k+1)\sigma, b\big]$. Then for $i = 0, 1, \ldots, k-3$ we have*

$$x(t) > \alpha_i x(t-\sigma) \quad \text{for } t \in \big[b-(k-2)\sigma, b-(i+1)\sigma\big] \tag{17}$$

where

$$\alpha_0 = \frac{\beta^2}{2(1-\beta)}, \ \alpha_i = \alpha_{i-1}^2 + \beta\alpha_{i-1} + \frac{1}{2}\beta^2, \quad i = 1, 2, \ldots, k-3. \tag{18}$$

Proof. If $\beta = 0$, then (17) naturally holds. Next, we assume $\beta \in (0,1)$. By (16), for any $t \in \big[b-(k-2)\sigma, b-\sigma\big]$ there exists $t^* \in \big[b-(k-2)\sigma, b\big]$ such that $t^* - \sigma \leqslant t < t^*$ and

$$\int_{t^*-\sigma}^{t^*} Q(s)\,\mathrm{d}s \geqslant \beta = \int_{t}^{t^*} Q(s)\,\mathrm{d}s. \tag{19}$$

From (14) we see that $x(t)$ is nonincreasing on $[b-k\sigma, b]$. Integrating (14) from t to t^* we have

$$x(t) \geqslant x(t^*) + \int_{t}^{t^*} Q(s)x(s-\sigma)\,\mathrm{d}s. \tag{20}$$

Noting the fact $b-(k-1)\sigma \leqslant t-\sigma \leqslant s-\sigma \leqslant t^*-\sigma \leqslant t \leqslant b-\sigma$ and integrating (14) from $s-\sigma$ to t we obtain

$$\begin{aligned} x(s-\sigma) &\geqslant x(t) + \int_{s-\sigma}^{t} Q(u)x(u-\sigma)\,\mathrm{d}u \\ &\geqslant x(t) + x(t-\sigma)\int_{s-\sigma}^{t} Q(u)\,\mathrm{d}u \\ &= x(t) + x(t-\sigma)\Big[\int_{s-\sigma}^{s} Q(u)\,\mathrm{d}u - \int_{t}^{s} Q(u)\,\mathrm{d}u\Big] \\ &\geqslant x(t) + x(t-\sigma)\Big[\beta - \int_{t}^{s} Q(u)\,\mathrm{d}u\Big]. \end{aligned}$$

Directly substituting this into the right side of (20) we get

$$
\begin{aligned}
x(t) &\geqslant x(t^*) + \int_t^{t^*} Q(s)\Big[x(t) + x(t-\sigma)\Big(\beta - \int_t^s Q(u)\,\mathrm{d}u\Big)\Big]\,\mathrm{d}s \\
&= x(t^*) + \beta x(t) + \Big[\beta^2 - \int_t^{t^*}\int_t^s Q(u)Q(s)\,\mathrm{d}u\,\mathrm{d}s\Big]x(t-\sigma). \qquad (21)
\end{aligned}
$$

Interchanging the order of integration in (21) we have

$$
\int_t^{t^*}\int_t^s Q(u)Q(s)\,\mathrm{d}u\,\mathrm{d}s = \int_t^{t^*}\int_u^{t^*} Q(s)Q(u)\,\mathrm{d}s\,\mathrm{d}u = \int_t^{t^*}\int_s^{t^*} Q(u)Q(s)\,\mathrm{d}u\,\mathrm{d}s
$$

and hence

$$
\begin{aligned}
\int_t^{t^*}\int_t^s Q(u)Q(s)\,\mathrm{d}u\,\mathrm{d}s &= \frac{1}{2}\Big[\int_t^{t^*}\int_t^s Q(u)Q(s)\,\mathrm{d}u\,\mathrm{d}s + \int_t^{t^*}\int_s^{t^*} Q(u)Q(s)\,\mathrm{d}u\,\mathrm{d}s\Big] \\
&= \frac{1}{2}\Big(\int_t^{t^*} Q(u)\,\mathrm{d}u\Big)^2 = \frac{1}{2}\beta^2.
\end{aligned}
$$

Substituting this into (21), we get

$$
x(t) \geqslant x(t^*) + \beta x(t) + \frac{1}{2}\beta^2 x(t-\sigma). \qquad (22)
$$

Noting that $x(t^*) > 0$, we have

$$
x(t) > \frac{\beta^2}{2(1-\beta)}x(t-\sigma) = \alpha_0 x(t-\sigma), \quad t \in \big[b-(k-2)\sigma, b-\sigma\big] \qquad (23)
$$

which shows that (17) holds for $i = 0$.

For $t \in \big[b-(k-2)\sigma, b-2\sigma\big]$, by (16) there exists $t^* \in \big[b-(k-2)\sigma, b-\sigma\big]$ such that $t^* - \sigma \leqslant t < t^*$ and (19) holds. By (23) we have

$$
x(t^*) > \alpha_0 x(t^* - \sigma). \qquad (24)
$$

Next, using the same method as in the preceding proof we see that (22) still holds. Since $x(t)$ is nonincreasing on $[b-k\sigma, b]$, it follows by (23) and (24) that

$$
x(t^*) > \alpha_0 x(t^*-\sigma) \geqslant \alpha_0 x(t) > \alpha_0^2 x(t-\sigma).
$$

Substituting this and (23) into (22), we obtain

$$
\begin{aligned}
x(t) &> \alpha_0^2 x(t-\sigma) + \alpha_0\beta x(t-\sigma) + \frac{1}{2}\beta^2 x(t-\sigma) \\
&= \alpha_1 x(t-\sigma),
\end{aligned}
$$

which shows that (17) holds for $i = 1$. Finally, using a simple induction we see that (17) holds for every $i = 0, 1, \ldots, k-3$. The proof is complete. □

Lemma 4. *In addition to the assumptions of Lemma 3, let us assume that for some $i \in \{0, 1, \ldots, k-3\}$ and some $\bar{t} \in \big[b-(k-2)\sigma, b-(i+1)\sigma\big]$,*

$$\int_{\bar{t}-\sigma}^{\bar{t}} Q(s)\, ds \geqslant 1-\alpha_i. \tag{25}$$

Then (14) has no solution $x(t)$ satisfying $x(t) > 0$ for $t \in \big[b-(k+1)\sigma, b\big]$.

Proof. Assume, by way of contradiction, that (14) has a solution $x(t)$ satisfying $x(t) > 0$ for $t \in \big[b-(k+1)\sigma, b\big]$. Then by Lemma 3 we have

$$x(t) > \alpha_i x(t-\sigma), \quad t \in \big[b-(k-2)\sigma, b-(i+1)\sigma\big]. \tag{26}$$

It is easy to see by (14) that $x(t)$ is nonincreasing on $[b-k\sigma, b]$. Integrating (1) from $t-\sigma$ to $t \in \big[b-(k-2)\sigma, b-(i+1)\sigma\big]$ and noting the monotonicity of $x(t)$ we have

$$\begin{aligned} x(t-\sigma) &\geqslant x(t) + \int_{t-\sigma}^{t} Q(s)x(s-\sigma)\, \mathrm{d}s \\ &\geqslant x(t) + x(t-\sigma)\int_{t-\sigma}^{t} Q(s)\, \mathrm{d}s \end{aligned}$$

which, together with (26), yields

$$\int_{t-\sigma}^{t} Q(s)\, \mathrm{d}s < 1-\alpha_i, \quad t \in \big[b-(k-2)\sigma, b-(i+1)\sigma\big],$$

which contradicts (25) and so the proof is complete. □

Theorem 3. *Assume that*

$$\tau, \sigma \in (0, \infty) \quad \text{and} \quad P, Q \in C\big([t_0, \infty), \mathbb{R}\big). \tag{27}$$

Further assume that there exists a real sequence $\{b_n\}$ such that $b_n \to \infty$ as $n \to \infty$ and

$$Q(t) \geqslant 0 \quad \text{for } t \in [b_n - 3\sigma, b_n], \tag{28}$$

(29) *$P(t)$ has zeros on $[b_n - \sigma, b_n]$ and $P(t) \geqslant 0$ for $t \in [b_n - 3\sigma, b_n - \sigma]$.*

If

$$\int_{b_n-\sigma}^{b_n} Q(s)\, \mathrm{d}s \geqslant 1 \tag{30}$$

then Eq. (1) *has no solutions which are positive on* $[b_n - 3\sigma - m, b_n]$, *where* $m = \max\{\tau, \sigma\}$. *Consequently, every solution of Eq.* (1) *oscillates.*

Proof. Assume, by way of contradiction, that Eq. (1) has a solution $x(t)$ satisfying $x(t) > 0$ for $t \in [b_n - 3\sigma - m, b_n]$. Set $y(t)$ as in (8). Then by (1) and (28) we see that $y(t)$ is nonincreasing on $[b_n - 3\sigma, b_n]$ which, together with (29), yields

$$y(t) > 0 \quad \text{for } t \in [b_n - 3\sigma, b_n - \sigma]. \tag{31}$$

In view of (29) we have

$$x(t) = y(t) + P(t)x(t-\tau) \geqslant y(t), \quad t \in [b_n - 3\sigma, b_n - \sigma]$$

and so

$$x(t-\sigma) \geqslant y(t-\sigma) \quad \text{for } t \in [b_n - 2\sigma, b_n].$$

Substituting this into (1) we have

$$y'(t) + Q(t)y(t-\sigma) \leqslant 0, \quad t \in [b_n - 2\sigma, b_n]. \tag{32}$$

In view of Lemma 2, (30) implies that (32) has no solution which is positive on $[b_n - 3\sigma, b_n - \sigma]$. This contradicts (31) and the proof of the theorem is complete. □

Theorem 4. *Assume that* (27) *holds and that there are an integer* $k \geqslant 3$ *and a real sequence* $\{b_n\}$ *such that* $b_n \to \infty$ *as* $n \to \infty$ *and*

$$Q(t) \geqslant 0, \quad t \in \big[b_n - (k+1)\sigma, b_n + \sigma\big] \tag{33}$$

and

$$P(t) \text{ has zeros on } [b_n, b_n + \sigma] \text{ and } P(t) \geqslant 0 \quad \text{for } t \in \big[b_n - (k+1)\sigma, b_n\big]. \tag{34}$$

Further assume that there exists $\beta \in [0, 1)$ *such that*

$$\int_{t-\sigma}^{t} Q(s)\,\mathrm{d}s \geqslant \beta \quad \text{for } t \in \big[b_n - (k-2)\sigma, b_n\big]. \tag{35}$$

If for some $i \in \{0, 1, \ldots, k-3\}$ *and some* $\bar{t}_n \in \big[b_n - (k-2)\sigma, b_n - (i+1)\sigma\big]$,

$$\int_{\bar{t}_n - \sigma}^{\bar{t}_n} Q(s)\,\mathrm{d}s \geqslant 1 - \alpha_i \tag{36}$$

where α_i is defined by (18), *then every solution of Eq.* (1) *has zero on* $\left[b_n-(k+1)\sigma-m, b_n+\sigma\right]$. *Consequently, every solution of Eq.* (1) *oscillates.*

Proof. Assume, by way of contradiction, that Eq. (1) has a solution $x(t)$ satisfying $x(t)>0$ for $t\in\left[b_n-(k+1)\sigma-m, b_n-\sigma\right]$. Let $y(t)$ be defined by (8). Then by (1) and (33) we see that $y(t)$ is nonincreasing on $\left[b_n-(k+1)\sigma, b_n+\sigma\right]$, which together with (34) yields

$$y(t)>0 \quad \text{for } t\in\left[b_n-(k+1)\sigma, b_n\right]. \tag{37}$$

By (34) we obtain

$$x(t)=y(t)+P(t)x(t-\tau)\geqslant y(t), \quad t\in\left[b_n-(k+1)\sigma, b_n\right]$$

and so

$$x(t-\sigma)\geqslant y(t-\sigma), \quad t\in[b_n-k\sigma, b_n+\sigma].$$

Substituting this into (1) we get

$$y'(t)+Q(t)y(t-\sigma)\leqslant 0, \quad t\in[b_n-k\sigma, b_n+\sigma]. \tag{38}$$

In view of Lemma 4, the hypotheses of Theorem 4 imply that (38) has no solution which is positive on $\left[b_n-(k+1)\sigma, b_n\right]$. This contradicts (37) and the proof is complete. □

Remark 2. When $P(t)\equiv 0$, Theorem 3 contains Theorem 2.2.1 in [6], while Theorem 4 is still new even if $P(t)\equiv 0$.

Example 5. The neutral delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-\alpha(\sin t)x(t-\pi)\right]+2(\sin t)x\left(t-\frac{\pi}{3}\right)=0$$

where $\alpha\geqslant 0$ satisfies all hypotheses of Theorem 3 with $b_n=(2n+1)\pi$. Therefore, every solution of this equation oscillates.

Example 6. Consider the neutral delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-\left(\sin\frac{t}{4}\right)x(t-\pi)\right]+\alpha\left(\sin\frac{t}{5}\right)x(t-\pi)=0$$

where $\alpha>0.220489$. Let $b_n=40n\pi+4\pi$. If $\alpha\geqslant 0.4$, then the hypotheses of Theorem 3 are satisfied. If $0.220489<\alpha<0.4$, then taking $k=3$ we see that (33) and (34) hold and that for $t \in [b_n - \pi, b_n]$,

$$\begin{aligned}
\int_{t-\pi}^{t} \alpha \sin \frac{s}{5}\, \mathrm{d}s &= 5\alpha\Big(\cos \frac{t-\pi}{5} - \cos \frac{t}{5}\Big) \\
&= 10\alpha \sin \Big(\frac{t}{5} - \frac{\pi}{10}\Big) \sin \frac{\pi}{10} \\
&\geqslant 10\alpha \sin \frac{7\pi}{10} \sin \frac{\pi}{10} \\
&= 5\alpha\Big(\cos \frac{\pi}{5} - \cos \frac{2\pi}{5}\Big) \\
&\doteq 2.5\alpha = \beta
\end{aligned}$$

and

$$\int_{b_n-2\pi}^{b_n-\pi} \alpha \sin \frac{s}{5}\, \mathrm{d}s = 10\alpha \cos \frac{2\pi}{5} > 3\alpha \geqslant 1 - \frac{\beta^2}{2(1-\beta)}.$$

So all hypotheses of Theorem 4 are satisfied when $0.220489 < \alpha < 0.4$. Therefore, by Theorem 3 and Theorem 4 every solution of this equation oscillates.

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